

A THEORETICAL ANALYSIS OF THE TOWNSEND-HUXLEY SWARM TECHNIQUE FOR ELECTRONS

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Summary

The Townsend-Huxley swarm technique for electrons is analysed in detail. Particular emphasis is placed on boundary conditions, and those conditions appropriate to the Huxley-Crompton empirical relationship are derived and explained. Secondary emission at the electrodes due to both positive ion and photon bombardment is also taken into account. The principal result is an integral expression for the current ratio received at the circumferentially split anode. From this may be deduced the energy factor k_1 and the secondary emission coefficients. It is shown that it is possible to distinguish between positive ion and photon secondary emission and to determine the associated coefficients simultaneously in the stationary state. Highly accurate conditions for electrical breakdown are also obtained, secondary emission at the anode being taken into account.

I. INTRODUCTION

Using a modification due to Huxley (1940) of Townsend's technique for analysing the lateral diffusion of electrons in gases, Crompton and Jory (1962) showed that the ratio of electron drift velocity to the diffusion coefficient, and the associated energy factor k_1 , could be determined to a high degree of accuracy. Their results were, however, dependent on the use of an essentially empirical relationship (Huxley and Crompton 1955), while the range of E/p covered was such that the effects of ionization and excitation were unimportant.

The object of the present paper is to analyse the experimental method of Crompton and Jory in detail, particular emphasis being placed on boundary conditions. Those conditions appropriate to the Huxley-Crompton empirical relationship are derived and a physical explanation given. The emission of secondary electrons at the electrodes due to both photon and positive ion bombardment is also taken into account.

The principal result is an integral expression for the current ratio received at the circumferentially split anode. From this and the experimental measurements the energy factor k_1 may be deduced. In general, the integral does not reduce to a simple analytic form, but this is of little consequence since it may be readily computed. In addition to giving an expression from which k_1 may be determined, the results of this analysis have several other important applications. Highly accurate expressions for breakdown are obtained, while it is shown that the swarm technique, as employed by the previously mentioned authors, can be adapted to give secondary emission coefficients. Furthermore, and this is the most important application, it is possible

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using this technique to distinguish between photon and positive ion secondary electrons in the stationary state.

As already implied, only a theoretical treatment is given in the present paper. The application of the results obtained is given elsewhere, in particular for low values of E/p in the paper of Crompton and Jory (1962) and for high values of E/p in the paper by Crompton *et al.* (1965).

Throughout the present paper, continual reference is made to the second edition of Watson's text on Bessel functions (1944). For the sake of brevity, such references will be denoted by the initials WBF followed by the appropriate page number.

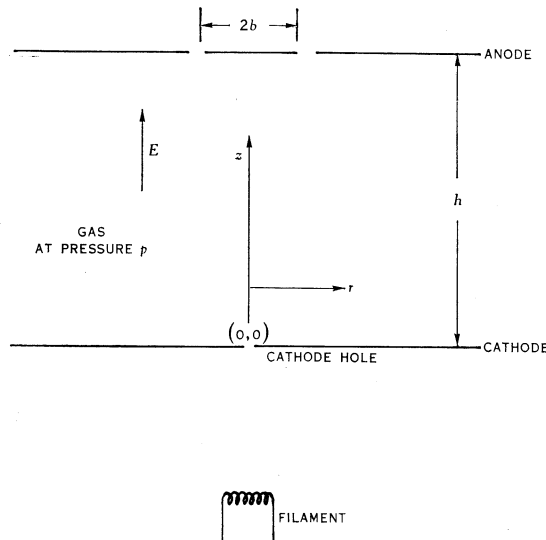


Fig. 1.—Experimental arrangement and geometry.

II. GENERAL THEORY

(a) Statement of the Problem

The essential elements of the experimental arrangement are shown in Figure 1, details being given in the paper by Crompton and Jory. For theoretical purposes, the diffusion chamber is bounded by two infinite plane electrodes separated by a distance h . The anode is circumferentially split at the radius b . Electrons are generated by a heated filament and enter the chamber through a small hole in the cathode; they then drift under the action of a uniform electric field towards the anode, diffusing radially and ionizing and exciting the gas in transit. Any positive ions formed drift to the cathode leading to the emission of secondary electrons, while further secondaries are produced by photon bombardment of both the anode and cathode.

The experimental results are expressed in terms of R , the ratio of the current received by the central anode disk to the total anode current, as a function of p and of E/p . The prime object of the theory is to present R in terms of the various parameters

of the problem, including boundary conditions, such that the energy factor k_1 may be deduced from this ratio.

(b) *General Solution of the Diffusion Equation*

The basic equations governing the drift and diffusion of the electrons are the continuity and flux equations

$$\operatorname{div} \mathbf{j} = \alpha' n, \quad (1)$$

$$\mathbf{j} = n\mu\mathbf{E} - D \operatorname{grad} n, \quad (2)$$

where \mathbf{j} = electron flux density,
 n = electron number density,
 α' = ionization coefficient,
 μ = electron mobility coefficient,
 D = electron diffusion coefficient,
 \mathbf{E} = electric field vector.

Since the problem is cylindrically symmetric, substitution of equation (2) in equation (1) gives for $\mathbf{E} = (0, 0, E)$:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial n}{\partial r} \right) + \frac{\partial^2 n}{\partial z^2} - 2\lambda \frac{\partial n}{\partial z} + 2\alpha\lambda n = 0, \quad (3)$$

where $\lambda = W/2D$,
 $W = \mu E$ = electron drift velocity,
 $\alpha = \alpha'/W$ [$= \alpha_T(1 - \alpha_T/2\lambda)$, α_T being Townsend's first ionization coefficient; see equation (95)].

Substitution will readily confirm that a general solution of (3) is

$$n = \int_0^\infty J_0(\xi r) \{A(\xi) \exp \beta_1 z + B(\xi) \exp \beta_2 z\} \xi \, d\xi, \quad (4)$$

where ξ^2 = a separation constant in equation (3),
 $\beta_1 = \lambda - (\xi^2 + \lambda'^2)^{\frac{1}{2}}$,
 $\beta_2 = \lambda + (\xi^2 + \lambda'^2)^{\frac{1}{2}}$,
 $\lambda'^2 = \lambda^2 - 2\alpha\lambda$ (or $\lambda - \lambda' = \alpha_T$).

In particular, it now follows from equations (2) and (4) that

$$j_z = D \int_0^\infty J_0(\xi r) \{\beta_2 A(\xi) \exp \beta_1 z + \beta_1 B(\xi) \exp \beta_2 z\} \xi \, d\xi. \quad (5)$$

$A(\xi)$ and $B(\xi)$ are to be determined from the imposition of boundary conditions at the cathode and anode of the diffusion chamber.

(c) *The Ratio R*

The ratio R is defined as

$$R = \frac{i_z(b, h)}{i_z(\infty, h)}, \quad (6)$$

where

$$i_z(r, h) = 2\pi \int_0^r j_z(r, h) r \, dr. \quad (7)$$

Substitution from equation (5) shows that

$$i_z(b, h) = 2\pi D \int_0^\infty J_1(\xi b) \psi(\xi, h) \, d(\xi b), \quad (8)$$

$$\text{where} \quad \psi(\xi, h) = \beta_2 A(\xi) \exp \beta_1 h + \beta_1 B(\xi) \exp \beta_2 h. \quad (9)$$

In principle, taking the limit $b \rightarrow \infty$ should give $i_z(\infty, h)$. However, there is a simpler method available for deriving this quantity. Since

$$i_z(\infty, h) = 2\pi \int_0^\infty j_z(r, h) r \, dr \quad (10)$$

$$= \lim_{\xi \rightarrow 0} 2\pi \int_0^\infty j_z(r, h) J_0(\xi r) r \, dr, \quad (11)$$

an application of the Hankel inversion theorem (WBF, p. 453) to equation (5) immediately gives

$$i_z(\infty, h) = 2\pi D \lim_{\xi \rightarrow 0} \psi(\xi, h) = 2\pi D \psi(0, h). \quad (12)$$

Therefore,

$$R = \int_0^\infty J_1(\xi b) \frac{\psi(\xi, h)}{\psi(0, h)} \, d(\xi b), \quad (13)$$

where $\psi(\xi, h)$ is given by equation (9).

(d) *Scaling Factors*

In the application of the final results of this analysis, the dependence of the various parameters of the problem on E/p is of major importance. Therefore, the scaling laws for the relevant quantities will be recorded. To understand these, however, the following definitions and relationships should be noted.

$$\left. \begin{aligned} \alpha' &\equiv n_g f_1(T_e); & \mu &\equiv e_e/m_e \nu; & D &\equiv \Lambda k T_e / m_e \nu; & \nu &\equiv n_g f_2(T_e); \\ k_1 &\equiv e_e D / \mu k T_g = e_e E / 2 \lambda k T_g \\ &= \Lambda k T_e / k T_g; \end{aligned} \right\} \quad (14)$$

where

T_e = electron temperature,

$f_1(T_e), f_2(T_e)$ = functions depending on T_e ,

T_g = gas temperature,

e_e = electronic charge,

- m_e = electronic mass,
 n_g = number density of gas atoms or molecules,
 ν = effective electron momentum-transfer collision frequency,
 k = Boltzmann's constant,
 k_1 = Townsend's energy factor,
 Λ = form factor of the order of unity, characteristic of the electron velocity distribution (see Huxley and Crompton 1962).

Subject to certain assumptions, which are briefly discussed in Section V, we have

$$\begin{aligned}
 T_e &\equiv T_e(E/n_g; T_g) \\
 &= T_e(E/p) \quad \text{for a given } T_g.
 \end{aligned} \tag{15}$$

Therefore, from the definitions of Section II(b), equations (14), and subsequent definitions still to be introduced, for a given gas at a given temperature the following parameters are functions of E/p only:

μp	α/p	λ/p	W	γ_i	Γ_i	ζ_c
Dp	α_T/p	λ'/p	\bar{c}	γ_p	Γ_p	ζ_a
k_1	η/p	λ_T/p	v_z			

(It is, of course, to be noted that the "secondary emission" coefficients, that is the γ 's and Γ 's, are also dependent on the nature of the electrode material.)

III. BOUNDARY CONDITIONS

For expression (4) to be the required solution it must satisfy the appropriate boundary conditions. The choice of $J_0(\xi r)$ as the relevant Bessel function, in the solution of equation (3), ensured that ξ^2 was a positive quantity and that n was everywhere finite, except perhaps at the cathode hole. Therefore, the only boundary conditions to be imposed are those at the electrodes. Since the effects of secondary emission at these electrodes have to be taken into account, such conditions will inevitably involve the particle flux density rather than the particle number density.

(a) General

The electrons in the diffusion chamber have random velocities. Hence, n and \mathbf{j} are variables which are essentially defined in terms of a velocity distribution function $f(\mathbf{c}, \mathbf{r})$. That is,

$$n = \int_{\mathbf{c}} f(\mathbf{c}, \mathbf{r}) \, d\mathbf{c}, \tag{16}$$

$$\mathbf{j} = \int_{\mathbf{c}} f(\mathbf{c}, \mathbf{r}) \mathbf{c} \, d\mathbf{c}, \tag{17}$$

where \mathbf{c} = particle velocity,
 $d\mathbf{c}$ = volume element of velocity space.

The integrals extend over all velocity space.

Referring $f(\mathbf{c}, \mathbf{r})$ to a system of spherical coordinates in velocity space, with the c_z axis parallel to the axial direction of the diffusion chamber, j_z may be redefined as follows:

$$j_z = j_{z+} + j_{z-}, \quad (18)$$

where

$$j_{z+} = \int_0^\infty \int_0^{2\pi} \int_0^{\frac{1}{2}\pi} f(\mathbf{c}, \mathbf{r}) c \cos \theta \, d\mathbf{c}, \quad (19)$$

$$j_{z-} = \int_0^\infty \int_0^{2\pi} \int_{\frac{1}{2}\pi}^\pi f(\mathbf{c}, \mathbf{r}) c \cos \theta \, d\mathbf{c}, \quad (20)$$

and

$$d\mathbf{c} = c^2 \sin \theta \, d\theta \, d\epsilon \, dc.$$

That is, at a given point, j_{z+} is the flux density associated with those particles moving towards the anode, and j_{z-} is the flux density associated with those moving towards the cathode.

To the same degree of approximation as equation (2), the velocity distribution function is given by (Huxley and Crompton 1962)

$$f(\mathbf{c}, \mathbf{r}) = f^0(c, \mathbf{r}) + \mathbf{v} \cdot \mathbf{c} f^1(c, \mathbf{r}), \quad (21)$$

where \mathbf{v} = electron mean mass velocity,

$$n\mathbf{v} = \mathbf{j}.$$

Therefore, it follows from equations (19) and (20) that

$$j_{z+} = \frac{1}{4}n\bar{c} + \frac{1}{2}j_z, \quad (22)$$

$$j_{z-} = -\frac{1}{4}n\bar{c} + \frac{1}{2}j_z, \quad (23)$$

with

$$n\bar{c} = \int_{\mathbf{c}} f^0(c, \mathbf{r}) c \, d\mathbf{c}. \quad (24)$$

The boundary conditions at the electrodes can now be expressed in terms of "reflection" coefficients and source functions. If the "reflection" coefficient at the cathode is ζ_c , then

$$j_{z+}(r, 0) = -\zeta_c j_{z-}(r, 0) + S_c(r, 0), \quad (25)$$

where $S_c(r, 0)$ = axial component of the source function, consisting of both primary and secondary electrons.

From use of (22) and (23) the boundary condition on j_z at the cathode is, therefore,

$$j_z(r, 0) = -\frac{1}{2}n(r, 0)\bar{c}a_c + \frac{2S_c(r, 0)}{1 + \zeta_c}, \quad (26)$$

where

$$a_c = (1 - \zeta_c)(1 + \zeta_c)^{-1}. \quad (27)$$

Similarly, it may be shown that at the anode

$$j_z(r, h) = \frac{1}{2}n(r, h)\bar{c}a_a - \frac{2S_a(r, h)}{1 + \zeta_a}. \quad (28)$$

The problem therefore reduces to one of giving expressions for the source functions and assigning values to the "reflection" coefficients ζ_c and ζ_a .

(b) *The Source Functions*

From the qualitative description given in Section II(a), we write

$$S_c(r, 0) = p(r, 0) + S_i(r, 0) + S_p(r, 0), \quad (29)$$

$$S_a(r, h) = S_p(r, h), \quad (30)$$

where $p(r, 0)$ = primary source axial flux density,
 $S_i(r, 0)$ = secondary source axial flux density due to bombardment of the cathode by positive ions,
 $S_p(r, 0)$ = secondary source axial flux density due to bombardment of the cathode by photons,
 $S_p(r, h)$ = secondary source axial flux density due to bombardment of the anode by photons.

It is to be noted that, as defined, the functions S account only for the axial components of flux at the electrode surfaces.

(i) *The Primary Source*.—The primary source consists of those electrons generated by the heated filament that enter the diffusion chamber through the small hole in the cathode. In order to avoid mathematical difficulties associated with discontinuous reflection coefficients, it will be assumed that this source may be approximated by a point function. That is,

$$p(r, 0) = p'(r, 0) \delta(r), \quad (31)$$

where
$$2\pi \int_0^\infty p'(r, 0) \delta(r) r dr = P, \quad (32)$$

P being the axial component of the total primary flux at the source hole.

(ii) *The Secondary Source* $S_i(r, 0)$.—Associated with ionization processes there is a continuity equation for the ions of the form

$$\text{div } \mathbf{j}_i = \alpha' n, \quad (33)$$

where \mathbf{j}_i = positive ion flux density.

Even for values of E/p approaching breakdown, radial diffusion of the ions will be small. Therefore, for the particular problem in hand, equation (33) is:

$$\frac{\partial j_{iz}}{\partial z} = \alpha' n. \quad (34)$$

Integrating this equation over z , subject to the boundary condition $j_{iz}(r, h) = 0$, this being consistent with the neglect of j_{ir} , we have

$$j_{iz}(r, 0) = -\alpha' \int_0^h n dz.$$

A secondary emission coefficient γ_{ic} is now defined such that

$$S_I(r, 0) = -\gamma_{ic} j_{iz}(r, 0) = \alpha' \gamma_{ic} \int_0^h n \, dz, \quad (35)$$

or, using equation (4),

$$S_I(r, 0) = \alpha W \gamma_{ic} \int_0^\infty J_0(\xi r) \{A(\xi) (\exp \beta_1 h - 1) \beta_1^{-1} + B(\xi) (\exp \beta_2 h - 1) \beta_2^{-1}\} \xi \, d\xi. \quad (36)$$

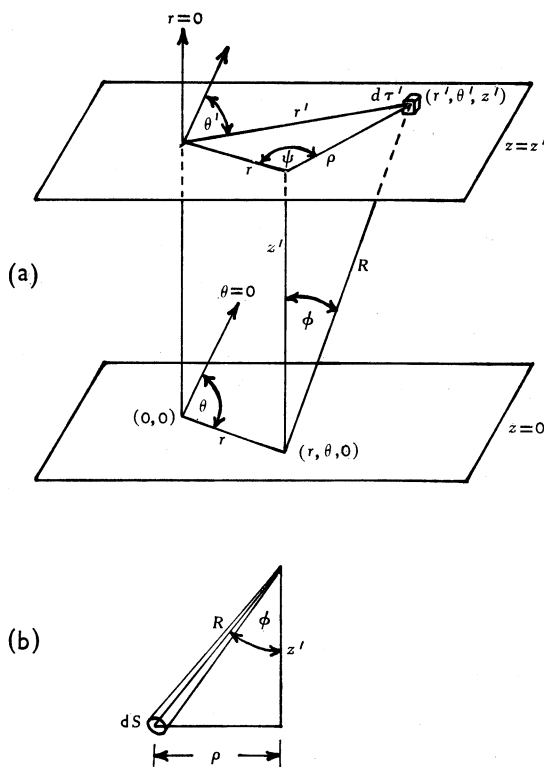


Fig. 2.—Geometry associated with cathode photo-emission.

(iii) *The Secondary Source* $S_p(r, 0)$.—Referring to Figure 2, the flux of photons δF , produced in a small volume element $d\tau'$ about the point (r', θ', z') , is given by

$$\delta F = \eta' n(r', \theta', z') d\tau', \quad (37)$$

where

$$\eta' = n_g f_3(T_e), \quad (38)$$

$f_3(T_e)$ being dependent on the electron temperature only.

The associated flux passing through an element of area dS , on the surface of a sphere of radius R centred at the point (r', θ', z') , is given by

$$\delta f_R dS = (\delta F / 4\pi R^2) dS. \quad (39)$$

If this element of area dS intersects the cathode at the point $(r, \theta, 0)$, then this same flux will also pass through an area dA , dA being the projection of dS on to the cathode. Therefore, since

$$dA \cos \phi = dS, \quad \cos \phi = z'/R, \quad (40)$$

the flux of photons per unit area of the cathode surface at the point $(r, \theta, 0)$, due to excitation collisions in the volume element $d\tau'$ about the point (r', θ', z') , is

$$\delta f_c = \frac{\eta' n(r', \theta', z') z'}{4\pi R^3} d\tau'. \quad (41)$$

Hence, the total flux density $f_c(r, \theta, 0)$ is given by

$$f_c(r, \theta, 0) = \int_{\tau'} \frac{\eta' n(r', \theta', z') z'}{4\pi R^3} d\tau', \quad (42)$$

the integration being over the entire volume of the diffusion chamber. As in the positive ion case, a secondary emission coefficient γ_{pc} is now defined such that

$$S_p(r, 0) = \gamma_{pc} f_c(r, \theta, 0), \quad (43)$$

it being noted that f_c is in fact independent of θ . But n is given by equation (4), and equation (43) becomes

$$S_p(r, 0) = \gamma_{pc} W \eta \int_{\tau'} \int_0^\infty \frac{J_0(\xi r') \{A(\xi) \exp \beta_1 z' + B(\xi) \exp \beta_2 z'\}}{4\pi R^3} z' \xi d\xi d\tau', \quad (44)$$

where, by analogy with the ionization coefficient α ,

$$\eta = \eta'/W.$$

In carrying out the integration over $d\tau'$, it is convenient to take as origin the point $(r, \theta, 0)$ and as integration variables ρ , ψ , and z' . Noting that

$$r'^2 = r^2 + \rho^2 - 2r\rho \cos \psi, \quad (45)$$

the integral in equation (44) may be re-written in the form

$$S_p(r, 0) = \gamma_{pc} W \eta \int_0^\infty \int_0^\pi K(r, 0; \xi; z') \{A(\xi) \exp \beta_1 z' + B(\xi) \exp \beta_2 z'\} dz' \xi d\xi, \quad (46)$$

$$\text{where} \quad K(r, 0; \xi; z') = \int_0^\infty \int_0^{2\pi} \frac{J_0\{\xi(r^2 + \rho^2 - 2r\rho \cos \psi)\}^\frac{1}{2}}{4\pi(\rho^2 + z'^2)^{3/2}} z' \rho d\rho d\psi. \quad (47)$$

The integration over ψ is a particular case of a Sonine-Gegenbauer integral (WBF, p. 367), and K reduces to

$$K(r, 0; \xi; z') = \frac{z' J_0(\xi r)}{2} \int_0^\infty \frac{J_0(\xi \rho)}{(\rho^2 + z'^2)^{3/2}} \rho d\rho. \quad (48)$$

This second integral is also known (WBF, p. 434), and we obtain

$$K(r, 0; \xi; z') = \frac{1}{2} J_0(\xi r) \exp(-\xi z'). \quad (49)$$

By substitution of this expression for K in equation (46) and integration over z' , the source function $S_p(r, 0)$ finally becomes

$$S_p(r, 0) = \frac{\gamma_{pc} \eta W}{2} \int_0^\infty J_0(\xi r) \left(\frac{A(\xi)}{\beta_1 - \xi} \{ \exp(\beta_1 - \xi)h - 1 \} + \frac{B(\xi)}{\beta_2 - \xi} \{ \exp(\beta_2 - \xi)h - 1 \} \right) \xi d\xi. \quad (50)$$

(iv) *The Secondary Source* $S_p(r, h)$.—This source function is similar to $S_p(r, 0)$ and may be derived in an identical manner. The result is

$$S_p(r, h) = \frac{\gamma_{pa} \eta W}{2} \int_0^\infty J_0(\xi r) \left(\frac{A(\xi)}{\beta_1 + \xi} \{ \exp \beta_1 h - \exp(-\xi h) \} + \frac{B(\xi)}{\beta_2 + \xi} \{ \exp \beta_2 h - \exp(-\xi h) \} \right) \xi d\xi, \quad (51)$$

where γ_{pa} = secondary emission coefficient characteristic of the anode material.

(c) The "Reflection" Coefficients

As defined by equation (25) and the corresponding equation for the anode surface, the "reflection" coefficients are no different from any of the other coefficients so far introduced. They are quantities that must be determined by experiment, the correct values being those which subsequently lead to consistent agreement between the experimental results and theoretical predictions. To be completely general, the final expressions derived in this analysis should be expressed in terms of the ζ 's. However, such expressions are extremely complex and, in order to simplify the algebra, certain experimental results will be used at this stage to determine the ζ 's explicitly.

Crompton and Jory showed that, in the absence of ionization and secondary processes, the ratio R is given to a remarkable degree of accuracy by the Huxley-Crompton empirical relationship

$$R = 1 - \frac{h}{(h^2 + b^2)^{\frac{1}{2}}} \exp[-\lambda \{ (h^2 + b^2)^{\frac{1}{2}} - h \}]. \quad (52)$$

Since this is essentially an empirical result, it will be used to determine the reflection coefficients. By use of a result given by Watson (WBF, p. 416), this expression for R may be rewritten in the integral form

$$R = \int_0^\infty J_1(\xi b) \exp[\{ \lambda - (\xi^2 + \lambda^2)^{\frac{1}{2}} \} h] d(\xi b). \quad (53)$$

Upon comparison of this with equation (13), it follows that

$$\frac{\psi(\xi, h)}{\psi(0, h)} = \exp \beta_1 h. \quad (54)$$

However, according to Section III(a), this result should be consistent with the boundary conditions

$$j_z(r, 0) = -\frac{1}{2} n(r, 0) \bar{c} a_c + \frac{2 p'(r, 0) \delta(r)}{1 + \zeta_c}, \quad (55)$$

$$j_z(r, h) = \frac{1}{2} n(r, 0) \bar{c} a_a. \quad (56)$$

By use of equations (4) and (5), these boundary conditions may be written in the form

$$\int_0^\infty J_0(\xi r) \{A(\xi) (\beta_2 + a_c \lambda_T) + B(\xi) (\beta_1 + a_c \lambda_T)\} \xi \, d\xi = \frac{2p'(r, 0) \delta(r)}{D(1 + \zeta_c)}, \quad (57)$$

$$\int_0^\infty J_0(\xi r) \{A(\xi) (\beta_2 - a_a \lambda_T) \exp \beta_1 h + B(\xi) (\beta_1 - a_a \lambda_T) \exp \beta_2 h\} \xi \, d\xi = 0, \quad (58)$$

where $\lambda_T = \bar{c}/2D. \quad (59)$

Upon use of the Hankel inversion theorem, two algebraic equations can now be obtained for $A(\xi)$ and $B(\xi)$. These may be readily solved and, from equations (9) and (13), the ratio R determined. The result is

$$R = \int_0^\infty J_1(\xi b) \frac{\psi(\xi, h)}{\psi(0, h)} \xi \, d\xi, \quad (60)$$

where

$$\frac{\psi(\xi, h)}{\psi(0, h)} = \frac{(\xi^2 + \lambda^2)^{\frac{1}{2}}}{\lambda} \frac{d(0) - c(0) \exp(-2\lambda h)}{d(\xi) - c(\xi) \exp\{-2(\xi^2 + \lambda^2)^{\frac{1}{2}} h\}} \exp \beta_1 h, \quad (61)$$

$$d(\xi) = \xi^2 + a_a a_c \lambda_T^2 + a_a \lambda_T \beta_2 - a_c \lambda_T \beta_1,$$

$$c(\xi) = \xi^2 + a_a a_c \lambda_T^2 + a_a \lambda_T \beta_1 - a_c \lambda_T \beta_2.$$

In the experiments of Crompton and Jory, λh is greater than 20, hence, to the degree of accuracy to which they worked,

$$\frac{\psi(\xi, h)}{\psi(0, h)} = \frac{(\xi^2 + \lambda^2)^{\frac{1}{2}}}{\lambda} \frac{d(0)}{d(\xi)} \exp \beta_1 h. \quad (62)$$

On comparison of this expression with equation (54), it follows that

$$d(0) (\xi^2 + \lambda^2)^{\frac{1}{2}} = \lambda d(\xi). \quad (63)$$

If we regard this now as an equation for the "reflection" coefficients, there is only one possible solution, namely

$$\left. \begin{aligned} a_c \lambda_T &= -\lambda, & \text{or } \zeta_c &= (\bar{c} + W)(\bar{c} - W)^{-\frac{1}{2}}, \\ a_a &\rightarrow \infty, & \text{or } \zeta_a &= -1. \end{aligned} \right\} \quad (64)$$

(The conditions $a_a \lambda_T = \lambda$, $a_c \rightarrow \infty$ will also satisfy equation (63). However, these conditions correspond to a positive density gradient at the anode and are not, therefore, realistic.)

In terms of the actual boundary conditions (55) and (56), these results correspond to

$$j_z(r, 0) = \frac{1}{2} n(r, 0) W + (1 - \lambda/\lambda_T) p(r, 0), \quad (65)$$

$$0 = n(r, h). \quad (66)$$

On the assumption that ionization and secondary effects in no way alter the "reflection" coefficients, the general boundary conditions, corresponding to equations

(64), therefore are

$$j_z(r, 0) = \frac{1}{2} n(r, 0) W + (1 - \lambda/\lambda_T) S_c(r, 0), \quad (67)$$

$$(2/\bar{c}) S_a(r, h) = n(r, h). \quad (68)$$

These are the boundary conditions appropriate to the problem in hand.

IV. SOLUTIONS

(a) Particle Flux and Particle Number Densities

From the expressions given for the source functions in Section III(b) and equations (4) and (5), the general boundary conditions (67) and (68) may be written in integral forms similar to equations (57) and (58). Therefore, by applying the Hankel inversion theorem, two algebraic equations for $A(\xi)$ and $B(\xi)$ may again be obtained. The solution of these is:

$$A(\xi) = - \frac{(1 - \lambda/\lambda_T)(P/2\pi D) X(\xi) \exp(-\beta_1 h)}{C(\xi) - D(\xi)}, \quad (69)$$

$$B(\xi) = \frac{(1 - \lambda/\lambda_T)(P/2\pi D) Y(\xi) \exp(-\beta_2 h)}{C(\xi) - D(\xi)}, \quad (70)$$

$$\text{where} \quad X(\xi) = 1 - \frac{\lambda'(\lambda - \lambda')}{(\lambda_T - \lambda)(\beta_2 + \xi)} \Gamma_{pa} [1 - \exp\{-(\xi + \beta_2)h\}], \quad (71)$$

$$Y(\xi) = 1 - \frac{\lambda'(\lambda - \lambda')}{(\lambda_T - \lambda)(\beta_1 + \xi)} \Gamma_{pa} [1 - \exp\{-(\xi + \beta_1)h\}], \quad (72)$$

$$C(\xi) = \exp(-\beta_1 h) X(\xi) \left(\lambda - \beta_2 + \frac{\lambda'(\lambda - \lambda')}{\beta_1} \Gamma_{ic} (\exp \beta_1 h - 1) \right. \\ \left. + \frac{\lambda'(\lambda - \lambda')}{\beta_1 - \xi} \Gamma_{pc} \{\exp(\beta_1 - \xi)h - 1\} \right), \quad (73)$$

$$D(\xi) = \exp(-\beta_2 h) Y(\xi) \left(\lambda - \beta_1 + \frac{\lambda'(\lambda - \lambda')}{\beta_2} \Gamma_{ic} (\exp \beta_2 h - 1) \right. \\ \left. + \frac{\lambda'(\lambda - \lambda')}{\beta_2 - \xi} \Gamma_{pc} \{\exp(\beta_2 - \xi)h - 1\} \right), \quad (74)$$

$$\Gamma_i = \frac{2\alpha\lambda}{\lambda'\lambda_T} \cdot \frac{\lambda_T - \lambda}{\lambda - \lambda'} \gamma_i, \quad (75)$$

$$\Gamma_p = \frac{\eta\lambda}{\lambda'\lambda_T} \cdot \frac{\lambda_T - \lambda}{\lambda - \lambda'} \gamma_p. \quad (76)$$

The additional subscripts a and c refer to the anode and cathode respectively.

(The formal difference of a factor of 2 between (75) and (76) arises from the fact that whereas all positive ions formed bombard the cathode, only one half of the photons produced do likewise, the other half bombarding the anode.)

Upon insertion of these expressions for $A(\xi)$ and $B(\xi)$ in equations (4) and (5), the solutions for n and j_z are essentially complete. Since, however, the principal

quantity of interest is R , there is no value in presenting either of these results in full, except in the case corresponding to a complete absence of ionization and excitation. This case is of particular interest, since it emphasizes the peculiar role of the "reflection" coefficients and the true physical nature of the boundary conditions. These are discussed in Section V.

For the case $\alpha = 0$, $\eta = 0$, it may be shown that

$$n = \left(1 - \frac{\lambda}{\lambda_T}\right) \frac{P}{2\pi D} \int_0^\infty J_0(\xi r) \frac{2 \exp \beta_1 z}{\beta_2 - \beta_1} \cdot \frac{1 - \exp(\beta_1 - \beta_2)(h - z)}{1 + \exp(\beta_1 - \beta_2)h} \xi \, d\xi, \quad (77)$$

where $\beta_1 = \lambda - (\xi^2 + \lambda^2)^{\frac{1}{2}}$, $\beta_2 = \lambda + (\xi^2 + \lambda^2)^{\frac{1}{2}}$,
 λ' being equal to λ .

Upon expansion of the denominator of the integrand in powers of $\exp(\beta_1 - \beta_2)h$, the resulting series may be integrated term by term (see WBF, p. 416). The result is

$$n = (1 - \lambda/\lambda_T)(P/2\pi D) e^{\lambda z} \sum_{N=-\infty}^{+\infty} (-1)^N \exp(-\lambda d_N)/d_N, \quad (78)$$

where $d_N^2 = r^2 + (z + 2Nh)^2$,

and N takes all integer values between plus and minus infinity, including $N = 0$.

From this expression for n , j_z and j_r may be readily obtained. However, the only quantity of real interest is $j_z(r, 0)$. From the boundary condition, equation (65), we have

$$j_z(r, 0) = \frac{1}{2} n(r, 0) W + (1 - \lambda/\lambda_T) p'(r, 0) \delta(r). \quad (79)$$

Upon integration of this equation over the cathode surface, or noting that $i_z(\infty, 0) = i_z(\infty, h)$ and using equation (12), it follows that

$$i_z(\infty, 0) = \left(1 - \frac{\lambda}{\lambda_T}\right) P \frac{1 - e^{-2\lambda h}}{1 + e^{-2\lambda h}} + \left(1 - \frac{\lambda}{\lambda_T}\right) P. \quad (80)$$

The physical significance of these results is readily understood. Referring to the series terms in equation (78), we see that this solution corresponds to a pole source ($N = 0$) of strength $(+1)$ at the origin, with image terms at $z = \pm 2Nh$ of strengths $(-1)^N$. The images at $\pm 2Nh$ ensure that (79) is satisfied, while complementary terms at $-2Nh$ and $2(N+1)h$ ensure that the boundary condition $n = 0$ is satisfied at the anode. Since n is not equal to zero at $z = 0$, equation (79) states that the true source of axial flux, as seen by the anode, consists of a point source plus a distributed source, and not a simple point source as originally postulated. For large λh these two sources are of equal strength, but as λh approaches zero the distributed source diminishes and the point source dominates.

(b) The Ratio R

It was shown in Section II(c) that this ratio is given by

$$R = \int_0^\infty J_1(\xi b) \frac{\psi(\xi, h)}{\psi(0, h)} \, d(\xi b),$$

where $\psi(\xi, h) = \beta_2 A(\xi) \exp \beta_1 h + \beta_1 B(\xi) \exp \beta_2 h$.

Upon use of expressions (69) and (70) for $A(\xi)$ and $B(\xi)$, this becomes

$$\psi(\xi, h) = -\left(1 - \frac{\lambda}{\lambda_T}\right) \frac{P}{2\pi D} \cdot \frac{\beta_2 X(\xi) - \beta_1 Y(\xi)}{C(\xi) - D(\xi)}. \quad (81)$$

For $\alpha = 0$, $\eta = 0$, this reduces to a relatively simple expression and, in particular,

$$\frac{\psi(\xi, h)}{\psi(0, h)} = \frac{\exp\{\lambda - (\xi^2 + \lambda^2)^{\frac{1}{2}}\}h + \exp[-\{\lambda + (\xi^2 + \lambda^2)^{\frac{1}{2}}\}h]}{1 + \exp\{-2(\xi^2 + \lambda^2)^{\frac{1}{2}}h\}}. \quad (82)$$

This is of course the semi-empirical result (54) from which the "reflection" coefficients were determined. Disregard of the factor $\exp\{-2(\xi^2 + \lambda^2)^{\frac{1}{2}}h\}$ corresponds to neglect of all image sources except that at $z = 2h$. As already remarked, this is a valid approximation in most practical applications. In the more general case a similar approximation is also valid, and this is considered in the next paragraph.

(i) *First Approximation*.—Although (81) could be written out exactly, there is no value in doing so since certain approximations are clearly justifiable. Since λh will be at least of the order of 10 in all practical cases, terms containing $\exp(-\beta_2 h)$ ($< e^{-\lambda h}$) may be ignored. In this case, (81) reduces to

$$\psi(\xi, h) = -\left(1 - \frac{\lambda}{\lambda_T}\right) \frac{P}{2\pi D} \frac{\{\beta_2 - \beta_1 Y(\xi)/X(\xi)\} \exp \beta_1 h}{\lambda - \beta_2 + \lambda'(\lambda - \lambda') M_1(\xi)}, \quad (83)$$

where
$$M_1(\xi) = \frac{(\exp \beta_1 h - 1) \Gamma_{ic}}{\beta_1} + \frac{\{\exp(\beta_1 - \xi)h - 1\} \Gamma_{pc}}{\beta_1 - \xi}$$

$$- \exp(\beta_1 h) \frac{Y(\xi)}{X(\xi)} \left(\frac{\Gamma_{ic}}{\beta_2} + \frac{\Gamma_{pc}}{\beta_2 - \xi} \exp(-\xi h) \right), \quad (84)$$

$$\begin{aligned} \frac{Y(\xi)}{X(\xi)} &= \left[1 - \frac{\lambda'(\lambda - \lambda')}{(\lambda_T - \lambda)(\beta_1 + \xi)} \Gamma_{pa} [1 - \exp\{-(\xi + \beta_1)h\}] \right] \\ &\times \left[1 - \frac{\lambda'(\lambda - \lambda')}{(\lambda_T - \lambda)(\beta_2 + \xi)} \Gamma_{pa} \right]^{-1}. \end{aligned} \quad (85)$$

A further simplification could be achieved by ignoring secondary effects at the anode, that is, by putting

$$\Gamma_{pa} = 0,$$

or

$$Y(\xi)/X(\xi) = 1.$$

Except in cases requiring extreme accuracy this will be a valid step.

(ii) *Second Approximation*.—For larger values of λh still, a further approximation is justified, this involving the neglect of all image terms. The associated simplification is achieved by assuming that

$$\lambda^2 \gg |\xi^2 - 2\alpha\lambda| \quad (86)$$

for all significant values of ξ (see (iv) below).

Subject to this condition,

$$\beta_1 = \alpha - \xi^2/2\lambda; \quad \beta_2 = 2\lambda; \quad \alpha_T = \alpha;$$

while

$$|\beta_2/\beta_1| \gg 1.$$

Using these values for β_1 and β_2 , equation (83) reduces to

$$\psi(\xi, h) = \left(1 - \frac{\lambda}{\lambda_T}\right) \frac{P}{2\pi D} \cdot \frac{2 \exp(\alpha - \xi^2/2\lambda)h}{1 - \alpha M_2(\xi)}, \quad (87)$$

$$\text{where} \quad M_2(\xi) = \frac{\Gamma_{ic}}{\alpha - \xi^2/2\lambda} \{\exp(\alpha - \xi^2/2\lambda)h - 1\} + \frac{\Gamma_{pc}}{\alpha - \xi^2/2\lambda} \{\exp(\alpha - \xi^2/2\lambda - \xi)h - 1\}, \quad (88)$$

with

$$\Gamma_{ic} = 2(1 - W/\bar{c})\gamma_{ic} \simeq 2\gamma_{ic}, \quad (89)$$

$$\Gamma_{pc} = (\eta/\alpha)(1 - W/\bar{c})\gamma_{pc} \simeq (\eta/\alpha)\gamma_{pc}. \quad (90)$$

The limitation (86) can be expressed in terms of λh , this being done in (iv) below.

(iii) *Low Pressure Limit*.—For sufficiently low pressures, a somewhat different type of approximation is possible. Using the scaling laws given in Section II(d), inspection of equations (83), (84), and (85) shows that for a given E/p the terms involving the Γ 's are more pressure dependent than the associated terms $\lambda - \beta_2$ and 1. The difference between the two sets of terms is not simple but involves at least one order of p . This means that for sufficiently low p , secondary emission is unimportant and (83) reduces to

$$\psi(\xi, h) = -(1 - \lambda/\lambda_T)(P/2\pi D) 2 \exp \beta_1 h. \quad (91)$$

It follows that

$$\frac{\psi(\xi, h)}{\psi(0, h)} = \exp\{\lambda' - (\xi^2 + \lambda'^2)^{1/2}\}h, \quad (92)$$

this leading to the Huxley-Crompton relationship with, however, λ' replacing λ . This result was originally derived by Huxley (1959).

An associated result of equal importance defines the Townsend first ionization coefficient α_T . From equations (12), (80), and (91)

$$\frac{i_z(\infty, h)}{i_z(\infty, 0)} = \exp(\lambda - \lambda')h \quad (93)$$

in the limit $p \rightarrow 0$. (It is to be noted that in the absence of secondary effects, ionization does not alter the cathode source distribution.) Therefore, by definition,

$$\lambda - \lambda' = \alpha_T \quad (\text{Townsend's coefficient}). \quad (94)$$

Since $\lambda'^2 = \lambda^2 - 2\alpha\lambda$, it follows that

$$\alpha = \alpha_T(1 - \alpha_T/2\lambda). \quad (95)$$

(iv) *Numerical Integration*.—In the general case, R cannot be expressed in a simple analytic form. Instead, even though an infinite integral, it must be evaluated by numerical methods. That this can be done to any required degree of accuracy is readily confirmed. Consider

$$R(x_r) = \int_0^{x_r} J_1(x) F(x) dx,$$

where x_r is the r th root of $J_1(x)$. Then provided $F(x)$ is a monotonically decreasing function of x for $x > x_0$, where $x_0 < x_r$, the damped oscillatory nature of $J_1(x)$ ensures that

$$\begin{aligned} |R(\infty) - R(x_r)| &< \left| \int_{x_r}^{x_{r+1}} J_1(x) F(x) dx \right| \\ &< |J_1(x_m) (x_{r+1} - x_r)| |F(x_r)|, \end{aligned}$$

x_{r+1} being the $(r+1)$ th root of $J_1(x)$. $J_1(x_m)$ is the maximum value of $J_1(x)$ in the interval $x_r < x < x_{r+1}$.

If the maximum tolerable error in the computation is denoted by ϵ , then

$$\frac{|R(\infty) - R(x_r)|}{R(\infty)} \leq \epsilon.$$

This is satisfied provided

$$\frac{|J_1(x_m) (x_{r+1} - x_r)| |F(x_r)|}{R(\infty)} \leq \epsilon.$$

This inequality may now be regarded as an equation for x_r , defining the greatest significant value of x . The maximum value of $|J_1(x_m) (x_{r+1} - x_r)|$ occurs between the first and second roots of $J_1(x)$, being 2.22. Hence, we may write

$$F(x_r) = \frac{\epsilon R(\infty)}{2.22}. \quad (96)$$

In the application of the results of this analysis, $R(\infty)$ is given, being an experimentally determined quantity. Again,

$$F(x_r) \equiv \frac{\psi(\xi, h)}{\psi(0, h)},$$

with

$$x_r = \xi_r b.$$

Therefore, from equation (83),

$$F(x_r) = K(x_r) \exp \beta_1 h, \quad (97)$$

where $K(x_r)$ is of the order of unity. Using this expression for $F(x_r)$ in (96), it may be shown that

$$(\xi_r^2 - 2\lambda\alpha)h^2 \simeq \left(\ln \frac{2.22}{\epsilon R(\infty)} \right)^2 + 2\lambda h \left(\ln \frac{2.22}{\epsilon R(\infty)} \right). \quad (98)$$

Upon insertion of values of practical interest in this equation, it is found that x_r should never have to be greater than the third root of $J_1(x)$, that is 7.02, while the inequality (86) can be rewritten as

$$\lambda h \gg 20. \quad (99)$$

(c) *Breakdown Phenomena*

Conditions in the diffusion chamber will approach those of electrical breakdown when

$$i_z(\infty, h) \rightarrow \infty, \quad (100)$$

or, referring to equation (12),

$$\psi(0, h) \rightarrow \infty. \quad (101)$$

From equation (81) it follows that the actual breakdown condition is

$$C(0) = D(0). \quad (102)$$

There is nothing to be gained by recording this relationship in detail, and only the conditions corresponding to the approximations (83) and (87) will be considered.

(i) *First Approximation*.—For values of λh of the order of 10 or greater, it follows from equations (83), (84), and (85) that the breakdown condition is

$$(\Gamma_{ic} + \Gamma_{pc}) \left(\{ \exp(\lambda - \lambda')h - 1 \} - \frac{Y(0)}{X(0)} \frac{\lambda - \lambda'}{\lambda + \lambda'} \exp(\lambda - \lambda')h \right) = 1, \quad (103)$$

where
$$\frac{Y(0)}{X(0)} = \left[1 - \frac{\lambda'}{\lambda_T - \lambda} \Gamma_{pa} [1 - \exp\{-(\lambda - \lambda')h\}] \right] \left[1 - \frac{\lambda'(\lambda - \lambda')}{(\lambda_T - \lambda)(\lambda + \lambda')} \Gamma_{pa} \right]^{-1} \quad (104)$$

and $\lambda - \lambda' = \alpha_T$.

(ii) *Second Approximation*.—For $\lambda h \gg 20$, the corresponding approximation immediately follows, being

$$1 = (\Gamma_{ic} + \Gamma_{pc})(e^{\alpha h} - 1), \quad (105)$$

$$\alpha = \alpha_T,$$

with $\Gamma_{ic} \simeq 2\gamma_{ic}, \quad (106)$

$$\Gamma_{pc} \simeq (\eta/\alpha)\gamma_{pc} \quad (\equiv \delta/\alpha). \quad (107)$$

The factor of 2 in (106), and which is also implicit in (107), is due to the fact that γ_{ic} was defined in terms of the axial component of flux only. If secondary electrons are emitted isotropically from the cathode surface, then the associated axial component of flux will only account for one half of the total secondaries introduced. This effect is similar to that already noted in equation (80), being clearly dependent on the true nature of the "reflection" coefficients.

(iii) *The Ratio R*.—As $\psi(0, h)$ approaches infinity, R approaches zero, this being due to the spread of the secondary source at the cathode. If in such circumstances

the effects of secondary emission are ignored, and equation (52) is used to determine λ or λ' , the corresponding values of k_1 will also tend to infinity as $\psi(0, h)$ approaches infinity. That is, unless realized, the spread of the secondary source of electrons at the cathode would be misinterpreted, being explained as a rapid radial diffusion associated with a high electron temperature.

(d) Applications

As already remarked in the Introduction, the application of the results of this analysis is given elsewhere. It is, however, worth noting here how these results may be applied.

For low values of E/p , $i_z(\infty, h) = i_z(\infty, 0)$. Denote this value of $i_z(\infty, 0)$ by $i_z^1(\infty, 0)$, then for higher values of E/p the ratio

$$\frac{i_z(\infty, h)}{i_z^1(\infty, 0)}$$

may be found from experiment. If now, for a given value of E/p , this ratio is determined as a function of p , apparent values of α_T/p may be obtained by using equation (93). These values will tend to an asymptotic limit as p approaches zero, this limit being the real value of α_T/p for the particular value of E/p chosen. In a similar manner, the real value of λ'/p may be found as a low pressure limit from the Huxley-Crompton relationship, R being experimentally determined as a function of p for given E/p . Hence λ/p , k_1 , and α/p may also be found.

Again, since R approaches zero at breakdown, the value of p at which breakdown occurs, for a given E/p , may be experimentally determined. Knowing α_T/p and λ'/p , and ignoring secondary emission at the anode, we may then find $\Gamma_{ic} + \Gamma_{pc}$, from equation (103). Therefore, at an intermediate pressure value, R may be evaluated as a function of either Γ_{ic} or Γ_{pc} by using (13) and (83). Comparison with the corresponding experimental value of R then determines Γ_{ic} and Γ_{pc} . Finally, since W is known from other experiments, while assuming a Maxwellian distribution (i.e. $\Lambda = 1$) \bar{c} may be found from k_1 , the secondary coefficients $2\gamma_{ic}$ and $(\eta/\alpha)\gamma_{pc}$ may also be determined.

(If higher accuracy is required, Γ_{pa} in equations (103) and (83) could now be put equal to the value of Γ_{pc} already found, and the cycle of calculations repeated.)

V. DISCUSSION

Although apparently well defined by equation (25), and the corresponding equation at the anode, the subsequent analysis proved that the "reflection" coefficients have no simple physical meaning. They appear as parameters that can be chosen only after experiment. This is not particularly surprising since they actually account for a variety of phenomena, the most important of which are:

- (1) The true physical reflecting properties of the electrodes.
- (2) In general, in the vicinity of a boundary it is to be expected that the electron velocity distribution will be anisotropic, while relatively large density gradients could occur. Such effects could lead to non-uniform and non-scalar electron temperatures and associated transport coefficients. This means that

within a few mean free paths of a boundary, equations (2), (3), and (21) will inadequately describe the behaviour of the electrons. Therefore, if a solution of these particular equations is to be used to determine n , j_z , R etc., it follows that the boundary conditions, that is the "reflection" coefficients, must be of an artificial nature.

- (3) The ζ 's have only been defined in terms of the axial components of flux. Since, at a boundary, particles may also be introduced as a pure radial component, the ζ 's must be capable of accounting for this.
- (4) Associated with (2) and (3), but still an independent possibility, the source of axial flux could be inadequately defined, the true nature and distribution of this source only being accounted for by the ζ 's.

As previously emphasized, it follows that the ζ 's must be experimentally determined quantities. Their value, however, as meaningful theoretical parameters, depends on their behaviour with varying E/p and p and, in particular, with changes in geometrical dimensions and the magnitude of the total particle flux. The very fact that they are dependent only on E/p , as proved in Section III(c), and do not depend on the geometrical dimensions or particle flux, justifies their definition. The fact that the experimentally determined values, or the associated boundary conditions, appear to be unrealistic is quite immaterial. This does not mean to say, however, that a physical explanation of the associated results cannot be given. For instance, consider the case $\alpha = 0$, $\eta = 0$ of Section IV(a). In the absence of an electric field, particles leaving the cathode source hole can diffuse in a given time t into a half sphere of effective radius

$$x_D = 2(Dt)^{1/2}.$$

On the other hand, in the absence of diffusion but in the presence of an axial electric field, particles will drift in the same time a distance

$$Z = Wt.$$

On elimination of t from these two expressions, it follows that

$$x_D/Z = 2/\lambda x_D;$$

that is, for distances less than $2/\lambda$ in the vicinity of the source hole, diffusion processes dominate. Thus, particles leaving the source hole may diffuse into a half sphere of an effective radius of the order of $1/\lambda$ before the electric field has a major effect. For large λh , the effective radius of the distributed source is just $1/\lambda$ (cf. equations (78) and (79)) and, as seen by the anode, the true source of primary flux is simply the projection of this half sphere on to the cathode surface. However, as λh approaches zero, the anode boundary condition $n(r, h) = 0$ inhibits the formation of this diffusion region, and the effective source appears to be a true point of axial flux. Implicit in this explanation are two physical properties. The first is that the cathode surface is completely reflecting, as borne out by the basic pole-type solution ($N = 0$; equation (78)). The second is that electrons leaving the cathode hole have thermal energies characteristic of the electrons already in the diffusion chamber.

Associated with statement (2) above, the assumption of a uniform T_e and uniform associated transport coefficients has been implicit throughout this analysis. It

is neither obvious nor certain that such an assumption is valid. If it is, then the energy equation per unit volume is of the form

$$n_e n_g f(T_e, T_g) = n_e e_e E_z v_z; \quad (108)$$

that is, the energy gained by the electrons from the electric field is exactly equal to that lost in binary collisions with the gas atoms or molecules. Since $j_z = n v_z$, it follows from equations (4) and (5) that v_z scales as $D\lambda$, that is as W which is a function of E/p and T_e . Therefore, from (108),

$$T_e \equiv T_e(E/p, T_g).$$

However, the assumption of uniform T_e depends now on the assumption of uniform v_z . This latter assumption is obviously incorrect, at least near the anode where n is rapidly changing. It is not clear, however, to what extent, if any, this invalidates or restricts the results derived in this paper. Since these non-uniformities will be most marked near the boundaries, such effects could be accounted for by the correct choice of "reflection" coefficients and a choice of sufficiently large λh . This being the case, it follows that the real test of the adequacy of any such assumption finally rests with experiment. That is, the validity of the results derived in the present paper depends solely on the consistency of the theoretical predictions with the experimental results. In particular, if the deduced values of k_1 , Γ_1 , λ/p , etc. depend only on E/p and not on either p or b/h , then it may be assumed that these are the correct values. That this has been shown to be the case (see Crompton and Jory 1962; Crompton *et al.* 1965) proves the adequacy of the theory and of the associated assumptions.

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