

# THE INFLUENCE OF DIFFUSION ON THE TOWNSEND DISCHARGE IN A UNIFORM ELECTRIC FIELD

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[Manuscript received May 31, 1967]

## Summary

An analysis is presented to elucidate the role of diffusion in a Townsend discharge between plane parallel electrodes. Spatial independence of the electron energy distribution function is postulated, and the development proceeds by solution of the appropriate continuity equations. It is shown that experiment may be expected to verify Townsend's law in all but extreme cases, but that diffusion corrections must be applied if measured values of the primary ionization coefficient  $\alpha_T$  are to be used for determination of quantities such as the ionizing collision frequency. Further, Paschen's law is demonstrated to be unaffected by effects resulting from diffusion.

## INTRODUCTION

In Townsend's classical experiments on ionization by collision, electrons are liberated at a constant rate from a plane cathode and travel through a gas in a uniform electric field  $E$  to a plane parallel anode. Townsend showed that when the electrode separation  $d$  is increased and the potential difference  $V = Ed$  is adjusted to maintain a fixed value of  $E$ , then the current density in the circuit obeys the law

$$J = J_0 \exp(\alpha_T d), \quad (1)$$

provided secondary sources of ionization are unimportant. He also established that the quantity  $\alpha_T$  is dependent on the field strength and gas pressure  $p$  (at fixed temperature) through a functional relationship,  $\alpha_T = p f(E/p)$ .

In Townsend's theoretical analysis of these experiments he defined a quantity  $\alpha$  as "the number of molecules ionized by one electron in moving one centimetre in the direction of the force" (Townsend 1947). Then, if there be  $n dz$  electrons per unit area between the planes at  $z$  and  $z+dz$ , the cathode being at  $z = 0$ , the increase in the concentration  $n$  as these electrons advance by  $dz$  will be  $dn = \alpha n dz$  (provided each ionizing collision releases but one additional electron). It follows that the electron concentration will be distributed exponentially

$$n = n_0 \exp(\alpha z), \quad (2)$$

where  $n_0$  is the concentration at the cathode, and the current will obey equation (1)

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with  $\alpha_T = \alpha$ . It is to be noted that the derivation of equation (2) includes no provision for effects resulting from diffusion.

An alternative approach (Huxley 1959) to the problem is afforded by the equation of continuity for the electron concentration,

$$\partial n / \partial t + \nabla \cdot \mathbf{\Gamma} = \sigma, \quad (3)$$

in which  $\sigma$  is the volume rate of ionization and  $\mathbf{\Gamma}$  is the electron flux density,

$$\mathbf{\Gamma} = n\mathbf{W} - D\nabla n = \mathbf{J}_e / e, \quad (4)$$

$\mathbf{W}$  being the electron drift velocity,  $D$  the electron diffusion coefficient,  $e$  the electron charge, and  $\mathbf{J}_e$  the electronic component of the total current density. If, in equation (3), we set  $\sigma = \nu_i n$ , where  $\nu_i$  is the ionizing collision frequency per electron, then, for the one-dimensional\* steady-state case in which  $W$  is uniform and  $D\nabla n$  is negligible, we find equation (2) and

$$J = J_e(d) = en_0 W \exp(\alpha_1 d), \quad (5)$$

in which  $\alpha_1 = \nu_i / W$ .

In much current experimental work it is certain that the diffusive portion of  $\mathbf{\Gamma}$  is not negligible. However, if it be retained in the derivation above, the differential equation for  $n$  is of second order and has a most general solution of the form

$$n = A \exp(a_1 z) + B \exp(a_2 z), \quad (6)$$

which is not precisely of the form of Townsend's formula (2). Some workers (Brambring 1964; Ward 1965), noting that Townsend's definition of  $\alpha$  refers to the motion of individual electrons, have argued that  $\sigma = \alpha_T \mathbf{\Gamma}$  is a more accurate representation of the source term of equation (3) than is  $\sigma = \nu_i n$ , since diffusion will influence the net number of electrons crossing a fixed plane in any given time interval. Further, adoption of  $\sigma = \alpha_T \mathbf{\Gamma}$  leads immediately to equation (1), which is known to have a wide range of applicability. However, other writers (Crompton 1967) have pointed out that gradients of  $n$  cannot affect the volume ionization rate in the absence of interactions amongst the electrons, a view to which the present authors subscribe. It might be that, in cases in which the electron energy distribution function is non-uniform, the flux density could have spatial dependence similar to that of the product  $\nu_i n$ ; however, in such an event, Townsend's defined parameter  $\alpha$  would also be expected to be position dependent, precluding simple integration of equation (3).

The question remains whether equation (6) is compatible with experimental evidence, manifested by the validity of Townsend's law. In the work we are reporting in this paper we sought an answer to this question by investigation of the consequences of diffusion and the presence of metal electrodes in the one-dimensional steady-

\* Integration of the three-dimensional equation of continuity with respect to  $x$  and  $y$  shows that lateral diffusion plays no part in the linear growth of the total current; the results derived in this paper do not, therefore, depend on this simplifying assumption.

state case with uniform electron energy distribution and ionization at a rate proportional to the electron concentration. For generality, we have included drift and diffusion of positive ions, but the effect of the latter is then shown to be negligible. Also, we have incorporated secondary emission of electrons at the cathode by ion bombardment. The resulting expression for the total current density is analysed and found to be equivalent to Townsend's law to a high order of approximation except in extreme cases, but with modified expressions for  $\alpha_T$  and  $J_0$ . In addition, we show that the validity of Paschen's law is unaffected by diffusive effects.

### DEVELOPMENT

It is convenient to define the parameters

$$\lambda = W/2D,$$

$$\theta = (\lambda^2 - \nu_1/D)^{\frac{1}{2}},$$

and

$$\lambda_+ = W_+/2D_+,$$

in which  $W_+$  and  $D_+$  are the drift velocity and diffusion coefficient respectively of the positive ions. Letting  $N$  be the concentration of positive ions and using primes to signify differentiation with respect to  $z$ , we may write the equations to be solved in the forms

$$n'' - 2\lambda n' + (\nu_1/D)n = 0, \quad (7)$$

$$N'' + 2\lambda_+ N' = -(\nu_1/D_+)n. \quad (8)$$

The solution of (7) is equation (6) with the roots of the characteristic equation providing  $a_1$  and  $a_2$ :

$$a_1 = \lambda - \theta, \quad (9)$$

$$a_2 = \lambda + \theta. \quad (10)$$

For boundary conditions, we assume that there is a concentration  $n_0$  of electrons at the cathode, which is also a perfect sink for positive ions, while the anode is taken to be a perfect sink for both electrons and ions:

$$n(0) = n_0,$$

$$n(d) = N(d) = N(0) = 0.$$

The solution of equation (7) which satisfies these boundary conditions is

$$n = n_0 \exp(\lambda z) \frac{\sinh \theta(d-z)}{\sinh \theta d}. \quad (11)$$

From (11) and (4) we find the electron flux density

$$\Gamma = \mathbf{k} \frac{n_0 \nu_1 \exp(\lambda z)}{2 \sinh \theta d} \left( \frac{\exp\{\theta(d-z)\}}{\lambda - \theta} - \frac{\exp\{-\theta(d-z)\}}{\lambda + \theta} \right), \quad (12)$$

where  $\mathbf{k}$  is the unit vector in the positive  $z$  direction. For reference, note that the electron current density at the anode is

$$J_e(d) = \frac{\epsilon n_0 v_1 \theta \exp(\lambda d)}{(\lambda^2 - \theta^2) \sinh \theta d}. \quad (13)$$

Use of equation (11) permits us to integrate equation (8), which yields

$$N' + 2\lambda_+ N = A_+ - \Gamma/D_+, \quad (14)$$

in which  $A_+$  is an integration constant and  $\Gamma$  is given by equation (12). The positive ion flux density is

$$\Gamma_+ = -\mathbf{k}(NW_+ + D_+ N'). \quad (15)$$

Combining (14) and (15), we obtain an expression for the total current density:

$$\mathbf{J} = \mathbf{k}\epsilon(\Gamma_+ - \Gamma) = -\mathbf{k}\epsilon D_+ A_+, \quad (16)$$

and we see that to find the total current density we need only evaluate the constant  $A_+$ .

The most general solution of equation (14) may be written

$$N = B_+ \exp(-2\lambda_+ z) + \frac{A_+}{2\lambda_+} + \left( \frac{v_1 n_0 \exp(\lambda z)}{2D_+ \sinh \theta d} \right) \left( \frac{\exp\{-\theta(d-z)\}}{(\lambda+\theta)(\lambda+\theta+2\lambda_+)} - \frac{\exp\{\theta(d-z)\}}{(\lambda-\theta)(\lambda-\theta+2\lambda_+)} \right). \quad (17)$$

Imposition of the boundary conditions leads to evaluation of  $A_+$ , which can then be inserted into (16) to give

$$J = \frac{\epsilon \lambda_+ v_1 n_0 \exp(\lambda d)}{1 - \exp(-2\lambda_+ d) \sinh \theta d} \left( \frac{1 - \exp\{-(\lambda - \theta + 2\lambda_+)d\}}{(\lambda - \theta)[(\lambda - \theta) + 2\lambda_+]} - \frac{1 - \exp\{-(\lambda + \theta + 2\lambda_+)d\}}{(\lambda + \theta)[(\lambda + \theta) + 2\lambda_+]} \right). \quad (18)$$

Equation (18) gives the total current density, including the positive ion component, with no approximations. It may also be obtained from (16) by evaluation of the flux densities at any plane, and, in particular, at  $z = d$

$$\begin{aligned} J &= (J_e - \epsilon D_+ N')_{z=d} \\ &= J_e(d)(1 - f), \end{aligned} \quad (19)$$

in which  $f = f(\lambda, \theta, \lambda_+, d)$  and  $fJ_e(d)$  represents the positive ion contribution to the anodic current density. The function  $f$  can be found by using (13) in (19) and equating the result with equation (18). At such low  $E/p$  that there can be no ionization,  $f$  will vanish, and we will now show that  $f$  is negligibly small if  $E/p$  is such that there may be ionization. First, we note that

$$2\lambda_+ = W_+/D_+ \simeq \epsilon E/kT \gg W/D = 2\lambda. \quad (20)$$

Further,

$$\lambda \pm \theta = \lambda \left[ 1 \pm \left( 1 - \frac{4\alpha_i/W}{W/D} \right)^{\frac{1}{2}} \right] < 2\lambda, \quad (21)$$

which, along with (20), gives

$$2\lambda_+ \gg \lambda \pm \theta. \quad (22)$$

Now, when (22) is applied to equation (18), factors involving  $\lambda_+$  cancel to leave

$$J \simeq \frac{\epsilon \nu_i n_0 \exp(\lambda d)}{2 \sinh \theta d} \left( \frac{1}{\lambda - \theta} - \frac{1}{\lambda + \theta} \right), \quad (23)$$

which is identical with (13).

### SECONDARY EMISSION

Let us suppose that photoelectrons are emitted from the cathode with a current density of magnitude  $J_0$  and that electrons are also emitted as a result of bombardment of the cathode by positive ions, each ion releasing  $\gamma$  electrons. The total flux density of electrons at the cathode will be

$$\Gamma(0) = J_0/\epsilon + \gamma \Gamma_+(0). \quad (24)$$

We may use equation (16) to eliminate  $\Gamma_+(0)$ , thus obtaining

$$(1 + \gamma)\Gamma(0) = (J_0 + \gamma J)/\epsilon. \quad (25)$$

Equations (12), (13), and (19), employed in conjunction with (25), give

$$\frac{\epsilon n_0 \nu_i}{(\lambda^2 - \theta^2) \sinh \theta d} = J_0 / [(1 + \gamma)(\lambda \sinh \theta d + \theta \cosh \theta d) - \gamma(1 - f)\theta \exp(\lambda d)]. \quad (26)$$

Equation (26) is now substituted back into (19) to give

$$J = J_0 \frac{(1 - f)\{2\theta/(\lambda + \theta)\} \exp\{(\lambda - \theta)d\}}{(1 + \gamma)\{1 - [(\lambda - \theta)/(\lambda + \theta)] \exp(-2\theta d)\} - \gamma(1 - f)\{2\theta/(\lambda + \theta)\} \exp\{(\lambda - \theta)d\}}. \quad (27)$$

### APPLICATION AND DISCUSSION

To investigate the behaviour of equation (27), let us first ignore  $f$  and inquire into the situation when secondary emission is also negligible, that is,  $\gamma = 0$ . Then,

$$J = \frac{J_0[1 - (\lambda - \theta)/(\lambda + \theta)] \exp\{(\lambda - \theta)d\}}{1 - \{(\lambda - \theta)/(\lambda + \theta)\} \exp\{-(\lambda + \theta)d\}}, \quad (28)$$

which, save for the denominator, is in the form of Townsend's law. Equation (21) may be rewritten in the form

$$\frac{\lambda \pm \theta}{p} = \frac{E/p}{2D/\mu} \left\{ 1 \pm (1 - 4y)^{\frac{1}{2}} \right\}, \quad (29)$$

where  $y = (D/\mu)(\alpha_i/p)/(E/p)$  and  $\mu$  is the electron mobility. In estimating  $\lambda \pm \theta$  we may safely set  $\nu_i/W \simeq \alpha_T$ . Values of some parameters of interest for hydrogen, calculated from data in the literature and equation (29), are shown in Table 1.

It may be seen from the table that  $y$  is always much less than unity, which justifies replacement of the radical in (29) by the first few terms of its expansion, thus giving

$$J \simeq \frac{J_0(1-y)}{1-y \exp\{-(E/p)(pd)/(D/\mu)\}} \exp\{(\alpha_i)(1+y)d\}. \quad (30)$$

If the denominator of this expression is to provide observable variation of  $J$  as  $pd$  is varied,  $E/p$  being held constant, it is required (1) that  $y$  be not inappreciable, and (2) that  $Ed/(D/\mu)$  be small. For example, if 1% represents a measurable variation in  $J$ , and  $y$  is of the order of 0.05, then  $Ed/(D/\mu) \lesssim 1$  or  $pd \lesssim (D/\mu)/(E/p)$ . Condition (1) necessitates large  $E/p$  and condition (2) then requires exceedingly small values of  $pd$ . We note from Table 1 that, in hydrogen at  $E/p = 100 \text{ V cm}^{-1} \text{ torr}^{-1}$ , the maximum difference between the value of  $J$  predicted by equation (30) and the value

TABLE 1  
PARAMETERS FOR HYDROGEN

$E/p$ (V cm <sup>-1</sup> torr <sup>-1</sup> )	$\alpha_T/p^*$ (cm <sup>-1</sup> torr <sup>-1</sup> )	$D/\mu^\dagger$ (V)	$y$	$(\lambda+\theta)/p$ (cm <sup>-1</sup> torr <sup>-1</sup> )	$(\lambda-\theta)/p$ (cm <sup>-1</sup> torr <sup>-1</sup> )
10	0.012	1.20	0.00144	8.3	0.012
50	0.35	3.34	0.023	14.6	0.34
100	1.3	4.82	0.062	19.5	1.3

\* Reference: Rose (1956).

† Reference: Crompton *et al.* (1966).

given by a simple exponential law is but 6% at  $pd = 0$ , and that the effect would be negligible at  $pd = 1 \text{ cm torr}$ . A survey of the situation in other gases for which sufficient data are available shows that, as in hydrogen, the difference between equation (30) and Townsend's law will be undetectable in most experimental circumstances.

Diffusion does play a role, however; we see from equation (30) and Table 1 that, in hydrogen at  $E/p = 100 \text{ V cm}^{-1} \text{ torr}^{-1}$ ,  $\alpha_T = \alpha_i(1+y)$  is approximately 6% larger than  $\alpha_i$ . Further, equation (30) reveals that the intercept of a plot of  $\ln J$  versus  $d$  will lie somewhat lower than at  $\ln J_0$ , because of back diffusion.

Finally, the exact expression for the current density, equation (27), provides the sparking criterion

$$(1+\gamma)\left(1-\frac{\lambda-\theta}{\lambda+\theta}\right)\exp(-2\theta d) = \gamma(1-f)\left(\frac{2\theta}{\lambda+\theta}\right)\exp\{(\lambda-\theta)d\}. \quad (31)$$

The quantities  $\gamma$ ,  $\lambda/p$ ,  $\theta/p$ ,  $\lambda_i/p$ , and  $f$  are all functions of  $E/p = V/pd$ . Thus, equation (31) is an implicit equation for the sparking potential as a function of  $pd$ , which is Paschen's law. The actual value of the sparking potential is somewhat influenced

by diffusion, however; when the approximations used above are employed, equation (31) reduces to

$$(1 + 1/\gamma)(1 + y) = \exp(\alpha_T d), \quad (32)$$

indicating that values of  $\gamma$  found by determination of the sparking potential should be corrected for diffusive effects.

### CONCLUSIONS

The results of the present investigation may be summarized as follows:

- (1) With respect to the question posed in the Introduction, we conclude that the choice of  $\sigma = v_i n$  for the source term of the continuity equation does not lead to predictions in conflict with available experimental evidence obtained from steady-state discharges.
- (2) The expressions for the "initial" current density  $J(0)$  and for the ionization coefficient  $\alpha_T$  are altered as a result of diffusive effects. No simply worded definition of  $\alpha_T$  can be given (or used) when diffusion is non-negligible.
- (3) Paschen's law holds exactly, even in the presence of strong effects resulting from diffusion.

### ACKNOWLEDGMENTS

We wish to thank our colleagues of the Ion Diffusion Unit for their helpful participation in discussions concerning the work presented in this paper. One of us (D.S.B.) is especially grateful to the Australian National University for the Visiting Fellowship grant which enabled him to work on the project.

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