

AN APPROXIMATION TO THE REACTION MATRIX IN NUCLEAR MATTER

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Abstract

It has been shown by Butler *et al.* that a good approximation to the Bethe-Goldstone wavefunction can be constructed from eigenfunctions of the free two-nucleon system. The approximation is therefore closely related to the T -matrix. In this paper, it is used to derive an approximate G -matrix in terms of the T -matrix. As an illustration of this approach, the resulting approximate G -matrix is compared with the reference spectrum approximation of Bethe, Brandow, and Petschek for the simple case of a pure hard core potential.

I. INTRODUCTION

In a recent paper by Butler *et al.* (1970), it was shown that a good approximation to the Bethe-Goldstone wavefunction could be constructed from eigenfunctions of the isolated two-nucleon system. This approximation was later used by Truelove and Nicholls (1970) as a starting point for exact shell model calculations of the reaction matrix and energy levels. Their method has been discussed by Barrett, Hewitt, and McCarthy (1970*a*), who also developed an alternative version which has calculational advantages (1970*b*). In the present paper, the properties of the original approximation to the G -matrix are further investigated for the case of nuclear matter.

The approximation of Butler *et al.* (1970) expresses the G -matrix in terms of the scattering states of two nucleons. This suggests that it has a close relation to the scattering matrix (T -matrix) for the two nucleon system, which is not surprising as it is well known that the G -matrix of Brueckner theory is closely related to T (e.g. Baranger 1969). Moreover, another approximation to G , the "reference G -matrix" with plane wave intermediate states, is simply a negative energy, fully off-shell T -matrix element (Sprung 1970).

In Section II(*a*) below the approximate G -matrix is expressed in terms of the half-shell T -matrix. The result is valid for any finite potential V . For potentials with a hard core, there is a small correction which is discussed in Section II(*b*). For nuclear matter with an interaction consisting of a hard core only, the half-shell T -matrix can be derived analytically. This is done in Section III and enables calculation of the G -matrix approximation directly in this case. Finally, in Section IV the results are compared with the corresponding calculations using the reference spectrum method of Bethe, Brandow, and Petschek (1963).

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II. APPROXIMATION TO REACTION MATRIX IN NUCLEAR MATTER

(a) *Well-behaved Potentials*

We define three two-body wavefunctions, namely Φ_μ , the unperturbed wavefunction of two non-interacting nucleons; Ψ_γ , the wavefunction of two nucleons interacting via a two-nucleon potential V ; and Ω_α the Bethe-Goldstone wavefunction of two nucleons with interaction V in the presence of occupied states described by the Pauli operator Q . The reaction matrix is then

$$G_{\mu\alpha} = \langle \Phi_\mu | V | \Omega_\alpha \rangle.$$

In order to evaluate this matrix element we need an approximation for Ω_α at particle separations r within the range of V .

For nucleon-nucleon potentials with strong repulsion for small r , Ω_α is nearly zero in the repulsive region and heals onto Φ_α , the unperturbed wavefunction, at large separations. We expand Ω_α in terms of the complete set Ψ_γ which has similar short-distance behaviour,

$$|\Omega_\alpha\rangle = \sum_\gamma |\Psi_\gamma\rangle \langle \Psi_\gamma | \Omega_\alpha \rangle. \quad (1)$$

This series would be expected to converge rapidly within the range of V and hence we may introduce a high energy cutoff in the sum over γ . Provided the wavefunction Ψ_γ does not rise abruptly outside the repulsive region, the matrix element $\langle \Psi_\gamma | \Omega_\alpha \rangle$ derives its major contributions from the region where Ω_α has healed onto Φ_α . We therefore choose the γ sum cutoff to be well above the energies of interest but low enough to ensure that Ω_α may be approximated by Φ_α in the expansion coefficient $\langle \Psi_\gamma | \Omega_\alpha \rangle$; the selection of the cutoff is discussed in more detail in Section IV. Hence we have the approximation of Butler *et al.* (1970)

$$|\Omega_\alpha^{(0)}\rangle = \sum'_\gamma |\Psi_\gamma\rangle \langle \Psi_\gamma | \Phi_\alpha \rangle, \quad (2)$$

where the prime on the summation is used to indicate the cutoff.

In nuclear matter with a spin-independent potential V , we use the outgoing scattered wavefunctions $\Psi_{k_1 k_2}^+$ for the Ψ_γ . The Φ_μ are plane waves $|\mathbf{p}_1 \mathbf{p}_2\rangle$. The reaction matrix G is given by

$$\langle \mathbf{p}'_1 \mathbf{p}'_2 | G | \mathbf{p}_1 \mathbf{p}_2 \rangle = \langle \mathbf{p}'_1 \mathbf{p}'_2 | V | \Omega_{\mathbf{p}_1 \mathbf{p}_2} \rangle. \quad (3)$$

Substituting (2) into (3) we get an approximate G -matrix $G^{(0)}$, which is

$$\langle \mathbf{p}'_1 \mathbf{p}'_2 | G^{(0)} | \mathbf{p}_1 \mathbf{p}_2 \rangle = \sum'_{k_1 k_2} \langle \mathbf{p}'_1 \mathbf{p}'_2 | V | \Psi_{k_1 k_2}^+ \rangle \langle \Psi_{k_1 k_2}^+ | \mathbf{p}_1 \mathbf{p}_2 \rangle. \quad (4)$$

We now change to relative-centre of mass coordinates $\mathbf{p} = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2)$, $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$, etc. Since V is translationally invariant so that

$$\langle \mathbf{P}' \mathbf{p}' | V | \Psi_{\mathbf{k} \mathbf{k}}^+ \rangle = \langle \mathbf{P}' | \mathbf{K} \rangle \langle \mathbf{p}' | V | \Psi_{\mathbf{k}}^+ \rangle,$$

we see that G and $G^{(0)}$ will also be translationally invariant. We therefore define a barycentric G -matrix $\langle \mathbf{p}' | G | \mathbf{p} \rangle$ by

$$\langle \mathbf{P}' | \mathbf{P} \rangle \langle \mathbf{p}' | G | \mathbf{p} \rangle = \langle \mathbf{P}' \mathbf{p}' | G | \mathbf{P} \mathbf{p} \rangle \quad (5)$$

and find that

$$\langle \mathbf{p}' | G^{(0)} | \mathbf{p} \rangle = \sum_{\mathbf{k}}' \langle \mathbf{p}' | V | \Psi_{\mathbf{k}}^+ \rangle \langle \Psi_{\mathbf{k}}^+ | \mathbf{p} \rangle. \quad (6)$$

The major result of this paper is the observation that the right-hand side of equation (6) may be expressed in terms of the half-shell T -matrix for two-particle scattering, $\langle \mathbf{p}' | T(k^2 + i\eta) | \mathbf{k} \rangle$.

Firstly we observe that

$$\langle \mathbf{p}' | V | \Psi_{\mathbf{k}}^+ \rangle = \langle \mathbf{p}' | T(k^2 + i\eta) | \mathbf{k} \rangle, \quad (7)$$

and also make use of the Lippman-Schwinger equation

$$\langle \mathbf{p} | \Psi_{\mathbf{k}}^+ \rangle = \langle \mathbf{p} | \mathbf{k} \rangle + \frac{\langle \mathbf{p} | V | \Psi_{\mathbf{k}}^+ \rangle}{\epsilon(\mathbf{k}) - \epsilon(\mathbf{p}) + i\eta}, \quad (8)$$

to obtain the result

$$\langle \mathbf{p}' | G^{(0)} | \mathbf{p} \rangle = \sum_{\mathbf{k}}' \langle \mathbf{p}' | T(k^2 + i\eta) | \mathbf{k} \rangle \left(\frac{\langle \mathbf{p} | T(k^2 + i\eta) | \mathbf{k} \rangle^*}{\epsilon(\mathbf{k}) - \epsilon(\mathbf{p}) - i\eta} + \langle \mathbf{p} | \mathbf{k} \rangle^* \right). \quad (9)$$

We now expand (9) into partial waves. It should be noted that

$$\sum_{\mathbf{k}}' = \frac{1}{(2\pi)^3} \int d\hat{\mathbf{k}} \int_0^{k_m} k^2 dk,$$

where we have taken the cutoff at k_m . The T -matrix can be expanded into partial waves as

$$\langle \mathbf{p}' | T(p^2 + i\eta) | \mathbf{p} \rangle = 4\pi \sum_l (2l+1) P_l(\hat{\mathbf{p}}' \cdot \hat{\mathbf{p}}) T_l(p', p). \quad (10)$$

Also $\epsilon(\mathbf{k}) = \epsilon(k)$ so that

$$\begin{aligned} \frac{1}{\epsilon(\mathbf{k}) - \epsilon(\mathbf{p}) - i\eta} &= \frac{\mathcal{P}}{\epsilon(\mathbf{k}) - \epsilon(\mathbf{p})} + i\pi \delta(\epsilon(\mathbf{k}) - \epsilon(\mathbf{p})) \\ &= \frac{\mathcal{P}}{\epsilon(\mathbf{k}) - \epsilon(\mathbf{p})} + \frac{i\pi}{d\epsilon/dp} \delta(k - p) \end{aligned}$$

and we have normalized the plane wave states by

$$\langle \mathbf{p} | \mathbf{k} \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{k}).$$

Then, after performing the angle integrals, (9) gives

$$\begin{aligned} \langle \mathbf{p}' | G^{(0)} | \mathbf{p} \rangle &= 4\pi \sum_l (2l+1) P_l(\hat{\mathbf{p}}' \cdot \hat{\mathbf{p}}) T_l(p', p) \left(\frac{2}{\pi} \frac{i\pi p^2}{d\epsilon/dp} T_l^*(p, p) + 1 \right) \\ &\quad + \frac{2}{\pi} \int_0^{k_m} k^2 dk \frac{\mathcal{P}}{\epsilon(k) - \epsilon(p)} 4\pi \sum_l (2l+1) P_l(\hat{\mathbf{p}}' \cdot \hat{\mathbf{p}}) T_l(p', k) T_l^*(p, k), \end{aligned}$$

since we have chosen k_m to be greater than p . Now the on-shell T -matrix is related to the phase shifts by

$$T_l(p, p) = -\frac{d\epsilon/dp}{2p^2} \sin\{\delta_l(p)\} \exp\{i\delta_l(p)\}. \quad (11)$$

Hence

$$\begin{aligned} \langle \mathbf{p}' | G^{(0)} | \mathbf{p} \rangle &= 4\pi \sum_l (2l+1) P_l(\hat{\mathbf{p}}' \cdot \hat{\mathbf{p}}) \\ &\times \left(T_l(p', p) \cos\{\delta_l(p)\} \exp\{i\delta_l(p)\} + \frac{2}{\pi} \int_0^{k_m} \mathcal{P} \frac{k^2 dk}{\epsilon(k) - \epsilon(p)} T_l(p', k) T_l^*(p, k) \right), \end{aligned} \quad (12)$$

which completes the derivation of an expression for $G^{(0)}$ in terms of the partial wave half-shell T -matrix $T_l(p', p)$. In Section III, we derive $T_l(p, k)$ analytically for a pure hard core potential and use the result to evaluate (12).

(b) *Core Volume Contribution to Reaction Matrix for Hard Core Potentials†*

For a hard core potential, the matrix element $\langle \Phi | V | \Omega \rangle$ contains contributions from inside the core which are not included using (1) as $\langle \Phi | V | \Psi \rangle$ contains no contribution from this region of configuration space. Equation (12) then gives an approximation to G which we will call G_a . The remaining core volume correction $G_c = G - G_a$ can be calculated from the matrix elements of G_a . This was done by Trulove and Nicholls (1970) in the shell model basis and the same arguments can be adapted to the present case.

To calculate this correction, we define

$$\langle \psi | \phi \rangle_c = \int_0^{c-} d\mathbf{r} \psi^* \phi, \quad (13)$$

where c is the hard core radius. Then using the Bethe–Goldstone equation it can be shown that

$$\langle \mathbf{p}' | G | \mathbf{p} \rangle_c = \sum_{k < k_F, k \neq p} \langle \mathbf{k} | G | \mathbf{p} \rangle \langle \mathbf{p}' | \mathbf{k} \rangle_c, \quad (14)$$

k_F being the Fermi momentum. One can compute G_c from this equation iteratively, substituting G_a for G on the right-hand side and obtaining a first approximation G_c^1 and then repeating the procedure with $G_a + G_c^1$ for G and so on until self consistency is obtained. In practice, as will be seen in Section III(b), G_c^1 is less than 2% of G_a and it is not necessary to continue the process beyond the first step.

We can evaluate (14) by separating out the dependence of (12) on \mathbf{p}' . Notice that (12) is a linear combination of terms like $P_l(\hat{\mathbf{p}}' \cdot \hat{\mathbf{p}}) T_l(p', q)$ summed over q and l . We define the linear operator M by

$$\langle \mathbf{p}' | G^{(0)} | \mathbf{p} \rangle_a = M P_l(\hat{\mathbf{p}}' \cdot \hat{\mathbf{p}}) T_l(p', q). \quad (15)$$

Substituting (15) into (14) and performing the angle integrations gives

$$\langle \mathbf{p}' | G^{(0)} | \mathbf{p} \rangle_c = M P_l(\hat{\mathbf{p}}' \cdot \hat{\mathbf{p}}) \frac{2}{\pi} \int_0^{k_F} k'^2 dk' \int_0^c r'^2 dr' T_l(k', q) j_l(k' r') j_l(p' r'). \quad (16)$$

† This correction was pointed out by Hewitt in a communication with Trulove and Nicholls.

Comparing (16) with (15) we see that the core contribution to $G^{(0)}$ can be obtained by replacing $T_l(p', q)$ in (12) by

$$\frac{2}{\pi} \int_0^{k_F} k'^2 dk' \int_0^c r'^2 dr' T_l(k', q) j_l(k' r') j_l(p' r'). \quad (17)$$

III. EVALUATION OF REACTION MATRIX FOR PURE HARD CORE POTENTIAL

(a) Half-shell T -matrix

We now proceed to calculate $T_l(p', p)$ for a pure hard core potential of radius c . For this potential the phase shifts δ_l are given by

$$\tan \delta_l(p) = j_l(pc)/n_l(pc). \quad (18)$$

Working in units with $\hbar = \hbar^2/M = 1$, we have $\epsilon(p) = p^2$ so that (11) gives

$$T_l(p, p) = -p^{-1} \sin\{\delta_l(p)\} \exp\{i\delta_l(p)\}. \quad (19)$$

In order to calculate $T_l(p', p)$, we note that Ψ_p^+ satisfies the Schrödinger equation

$$\left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + p^2 - V(r) \right) r U_{pl}^+(r) = 0, \quad (20)$$

where $U_{pl}^+(r)$ is the l th partial wave of Ψ_p^+ and $V(r)$ is an infinite hard core for $r < c$ and zero for $r > c$. The half-shell T -matrix is given by

$$T_l(p', p) = \langle j_l(p' r) | V | U_{pl}^+(r) \rangle. \quad (21)$$

One can see from equation (20) that

$$V(r) r U_{pl}^+(r) = A(p, l, c) \delta(r - c),$$

where A is independent of r . Now

$$\langle j_l(p' r) | V | U_{pl}^+(r) \rangle = \int_0^\infty dr r j_l(p' r) V r U_{pl}^+(r) = c j_l(p' c) A(p, l, c). \quad (22)$$

Hence*

$$\frac{T_l(p', p)}{T_l(p, p)} = \frac{\langle j_l(p' r) | V | U_{pl}^+(r) \rangle}{\langle j_l(p r) | V | U_{pl}^+(r) \rangle} = \frac{j_l(p' c)}{j_l(p c)}. \quad (23)$$

This is the half-shell extension function. Combining this with (19) gives

$$T_l(p', p) = \frac{j_l(p' c)}{j_l(p c)} \left(-\frac{\sin\{\delta_l(p)\}}{p} \right) \exp\{i\delta_l(p)\}. \quad (24)$$

(b) Evaluation of G -matrix

If we now substitute (18) and (24) into (12) we get

$$\begin{aligned} \langle \mathbf{p}' | G^{(0)} | \mathbf{p} \rangle_a &= 4\pi c \sum_l (2l+1) P_l(\hat{\mathbf{p}}' \cdot \hat{\mathbf{p}}) j_l(p' c) \\ &\times \left(-\frac{1}{pc} \frac{n_l(pc)}{j_l^2(pc) + n_l^2(pc)} + \frac{2j_l(pc)}{\pi} \int_0^{k_m c} \mathcal{P} \frac{dx}{x^2 - p^2 c^2} \frac{1}{j_l^2(x) + n_l^2(x)} \right). \end{aligned} \quad (25)$$

* This derivation was pointed out to us by G. N. Epstein.

Comparing this with (17) and (23) we see that $\langle \mathbf{p}' | G^{(0)} | \mathbf{p} \rangle_c$ is given by the right-hand side of equation (25) with $j_l(p'c)$ replaced by $\mathcal{J}_l(p', c)$ where

$$\mathcal{J}_l(p', c) \equiv \frac{2}{\pi} \int_0^{k_F} k'^2 dk' \int_0^c r'^2 dr' j_l(k'c) j_l(k'r') j_l(p'r'). \quad (26)$$

In the following we use $k_F = 1.5 \text{ fm}^{-1}$ and $c = 0.4 \text{ fm}$. For $l \gtrsim k_F c = 0.6$, the approximation

$$j_l(\rho) \simeq \frac{\rho^l}{(2l+1)(2l-1) \dots 5.3.1} = \frac{\rho^l}{(2l+1)!!}$$

becomes useful, and we can integrate (26) directly to give

$$\frac{\mathcal{J}_l(p', c)}{j_l(p'c)} \simeq \frac{2}{\pi} \frac{(k_F c)^{2l+3}}{\{(2l+3)!!\}^2}. \quad (27)$$

For $l = 0$, we have evaluated (26) numerically and find that it is approximately constant at 1.7% of $j_0(p'c)$. Equation (27) gives this ratio as 1.5% for $l = 0$ and 0.02% for $l = 1$, and it is hence negligible for $l > 0$.

The numerical results of the evaluation of G are discussed in the next section.

TABLE 1
COMPARISON OF $G^{(0)}$ AND G^R FOR PURE HARD CORE POTENTIAL

k_1/k_F	pc	$G^{(0)}$ (fm)	G^R (fm)	k_1/k_F	pc	$G^{(0)}$ (fm)	G^R (fm)
0.0	0.23	8.93	9.13	0.6	0.29	8.70	9.04
0.1	0.23	8.92	9.13	0.7	0.31	8.61	9.00
0.2	0.24	8.90	9.12	0.8	0.33	8.51	8.96
0.3	0.25	8.87	9.11	0.9	0.36	8.39	8.92
0.4	0.26	8.83	9.09	1.0	0.38	8.25	8.86
0.5	0.28	8.77	9.07				

IV. NUMERICAL RESULTS AND COMPARISON WITH REFERENCE SPECTRUM METHOD

To investigate the validity of the present approximation we compare our numerical results for the matrix elements $\langle \mathbf{k}_1 \mathbf{k}_2 | G | \mathbf{k}_1 \mathbf{k}_2 \rangle$, where both \mathbf{k}_1 and \mathbf{k}_2 are inside the Fermi Sea, with the results of the reference spectrum calculations of Bethe, Brandow, and Petschek (1963). For states inside the Fermi Sea, Dahll, Ostgaard, and Brandow (1969) have shown that corrections to the reference spectrum approximation G^R are small (at most 10%). The close agreement between the present approximation and the reference spectrum approximation exhibited in Table 1 thus shows that we have constructed a good approximation to the exact G matrix.

The calculation of G^R for a hard core potential is described by Bethe, Brandow, and Petschek (1963). Their solution depends on the effective energy of the interacting particles, $-\hbar^2 \gamma^2 / m$, which is determined by the assumed form of the single-particle energies (called the reference spectrum). This may be characterized by the gap in the spectrum at the Fermi momentum, $\Delta(\hbar^2 k_F^2 / m)$. For particles within the Fermi Sea

$$\gamma^2 = 2\Delta k_F^2 - p^2, \quad (28)$$

where p is the relative momentum of the interacting particles. In the present calculation we used the value $\Delta = 0.66$, following Day (1967). The function γ is related to the healing distance d by

$$d = c + \gamma^{-1}.$$

The cutoff parameter k_m of the present approximation is also related to d . As shown by Butler *et al.* (1970),

$$d = c + x_1/k_m.$$

We therefore select k_m to give the same healing distance as γ ,

$$k_m = x_1 \gamma. \quad (29)$$

Values of $\langle p | G | p \rangle$ are given for both approximations in Table 1. The range of values of p is determined by the fact that $\langle p | G | p \rangle$ must be the barycentric G -matrix element corresponding to $\langle \mathbf{k}_1 \mathbf{k}_2 | G | \mathbf{k}_1 \mathbf{k}_2 \rangle$ where both \mathbf{k}_1 and \mathbf{k}_2 are inside the Fermi Sea. We fix \mathbf{k}_1 and then use the value of $p = \frac{1}{2} |\mathbf{k}_1 - \mathbf{k}_2|$ obtained by averaging \mathbf{k}_2 over all occupied states. This gives

$$p^2 = \frac{1}{4}(k_1^2 + \frac{3}{5}k_F^2). \quad (30)$$

As can be seen from Table 1, the present approximation agrees quite well with the reference spectrum G -matrix, and hence with the exact G -matrix, for states inside the Fermi Sea. As discussed by Barrett, Hewitt, and McCarthy (1970*b*) and Truelove and Nicholls (1970), the approximation may be improved in a systematic way to yield exact G -matrix elements. For the case of nuclear matter, as distinct from finite nuclei, this technique does not seem to us to provide advantages over the present methods of obtaining exact G -matrices which were compared by Dahll, Ostgaard, and Brandow (1969). However, the approximation discussed here would appear to be useful where a simple approximation to G -matrix elements and the Bethe-Goldstone wavefunction suffices.

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