SOLUTIONS OF THE RELATIVISTIC TWO-BODY PROBLEM

I. CLASSICAL MECHANICS

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Abstract

The chief difficulty of lack of simultaneity of events in all Lorentz frames in relativistic mechanics is overcome using a proper time calibration method. The electromagnetic and gravitational point source interactions are derived. A Lagrangian and Hamiltonian formalism is shown to be valid. The mechanics can be quantized easily. Relativistic corrections are applied to the problem of planetary motion, a model for the relativistic Coulomb interaction is explored, and the relativistic harmonic oscillator is evaluated.

I. Introduction

It appears reasonable to state that the problem of the motion of several interacting relativistic particles in classical and quantum mechanics has not been solved (Wigner 1969). In constructing such a theory, one encounters relative time coordinates whose meaning is often obscure. Furthermore, in following rigorously the historical methods of analysis, there arises a relativistic orbital angular momentum tensor which has been exhaustively investigated within the framework of group theory (Kahan 1965) but as yet plays no significant role in physical theories of motion.

Earlier work involving a many-time formalism by Dirac, Fock, and Podolsky (1932) led to difficulties with regard to Lorentz invariance. The later equation of Bethe and Salpeter (1951) which generalized the Feynman propagator treatment to many-particle systems gave solutions for neutral exchange bosons in the two-body case, as found by Cutkosky (1954) and Wick (1954), but these solutions could not be related to physical experience. Even the most recent work (Haymaker and Blankenbecler 1969) confines itself to evaluating formal properties of the equation without any detailed application to a specific physical process. It remains unclear as to whether the sum over a special class of Feynman diagrams leads to a physically applicable equation.

Repeated frustration with attempts to apply these theories to strong interactions led the author to review the possible alternatives. It seems apparent that the connection between the Bethe–Salpeter equation and classical physics is tenuous and that by approaching the problem from a basis more closely related to the main historical developments in mechanics, one might be able to establish a viable two-body classical theory, then a quantum theory, and finally a workable field theory for strong interactions. This paper deals with the solutions for the motion of two relativistic particles in classical physics, where the formulation was deliberately chosen so as to be easily quantized.

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In the following section the means for calibrating nonlocal times are discussed and an invariant time calibration procedure is proposed. The electromagnetic interaction between point charges is next evaluated using this calibration scheme. The gravitational field representation in the proper time formalism is then explored and the gravitational potential found.

The peculiarities of the proper time calibration permit one to establish a Lagrangian and Hamiltonian mechanics that is closely analogous to the classical equivalent. The relativistic angular momentum tensor has an important role in such mechanics. Three typical models are evaluated: the simple harmonic oscillator and the inverse square and inverse cube laws of 4-force. The inverse square law is applied to motion within the solar system, and the corrections due to special relativity are found.

It is shown that this mechanics can be quantized easily in the original Bohr (1913) sense. The clock paradox can be investigated readily and conditions for the paradox to exist can be found.

II. Motion of Several Particles

The calibration of ordinary nonlocal time coordinates has been discussed systematically by Moller (1952) and Synge (1958), but the system used in the present paper is different from these authors. Suppose we relax the Newtonian assumption that the time coordinate defines simultaneous events which span the entire configuration space of a system of N particles, and instead associate one time coordinate t_i with each specific point which may be taken to be the centre of mass r_i of each discrete particle in the system. We now try to relate each of these local ordinary times to an overall time coordinate τ measured by a clock placed at a reference position, say the overall centre of mass.

There will be a kinetic energy

$$T_i = \tfrac{1}{2} M_i g_{mn} \frac{\mathrm{d}x_i^m}{\mathrm{d}t_i}\frac{\mathrm{d}x_i^n}{\mathrm{d}t_i} \tag{1}$$

for the ith particle, with metric g_{mn} and mass M_i, and a Hamiltonian associated with each

$$H_i = T_i + V_i \qquad (r_1, r_2, \dots r_N; t_1, t_2, \dots t_N).$$

A composite Hamiltonian can be defined by

$$H\,\mathrm{d}\tau^2 = \sum_i H_i\,\mathrm{d}t_i^2, \tag{2}$$

giving

$$H = \tfrac{1}{2}\sum M_i \gamma_i^2 g_{mn} v_i^m v_i^n + \sum V_i \gamma_i^2, \tag{3}$$

where v_i^m is the velocity component and $\gamma_i = \mathrm{d}t_i/\mathrm{d}\tau$ is the relative rate of variation of each particle clock to the c.m. clock. Now a relative time coordinate appears when transformations are made to c.m. coordinates.

Following Smith (1960), a sequence of mass centres may be defined by

$$\boldsymbol{R}_j = \mathbf{U}_{ii}^N \boldsymbol{r}_i \tag{4a}$$

and associated with these points are times

$$T_j = \mathbf{U}^N_{ji}\, t_i. \tag{4b}$$

For example, if the mass of each particle is shown explicitly

$$\mathbf{U}^4_{ji} = \begin{bmatrix} M_1/M & M_2/M & M_3/M & M_4/M \\ M_1/(M-M_4) & M_2/(M-M_4) & M_3/(M-M_4) & -1 \\ M_1/(M_1+M_2) & M_2/(M_1+M_2) & -1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \tag{5}$$

is one such construction.

To introduce momenta, some properties of the matrices $\mathbf{U}$ are considered. Let

$$\mathbf{M} = M_i\,\delta_{ij}, \qquad \mathbf{G} = \mu_i\,\delta_{ij}, \qquad \mu_j = M_j \sum_{i=1}^{j-1} M_i \Big/ \sum_{i=1}^{j} M_j, \tag{6}$$

where the μ_j are the reduced masses appropriate to each $\boldsymbol{R}_j$. Hence

$$\mathbf{G} = (\mathbf{U}^{-1})^{\sim} . \mathbf{M} . \mathbf{U}^{-1}, \tag{7}$$

where the tilde denotes transposition. Defining the particle momenta as

$$\boldsymbol{p}_i = M_i\, \mathrm{d}\boldsymbol{r}_i/\mathrm{d}t_i, \tag{8}$$

then the c.m. momenta are given as

$$\boldsymbol{q}_j = (\mathbf{U}^{-1}_{ji})^{\sim} \boldsymbol{p}_i, \tag{9}$$

which yields

$$H = \sum_i \{\gamma_i^2(p_i^2/2M_i + V_i)\} = \sum_j (\xi_j^2 q_j'^2/2\mu_j + W_j), \tag{10}$$

where

$$\xi_j\, \boldsymbol{q}_j' = \boldsymbol{q}_j, \qquad \boldsymbol{q}_j' = \mathrm{d}\boldsymbol{R}_j/\mathrm{d}T_j, \qquad \xi_j = \mathrm{d}T_j/\mathrm{d}\tau, \qquad \sum \gamma_i^2\, V_i = \sum W_j.$$

No progress can be made beyond this point without specifying how the array of N nonlocal clocks is to be calibrated. In Newtonian mechanics, of course, one introduces the concept of simultaneity by calibrating across a surface

$$t_1 = t_2 = \dots t_N = \tau, \qquad T_1 = \tau, \qquad T_j = 0, \quad j \neq 1, \tag{11}$$

of equal ordinary times, which ensures that the Hamiltonian

$$H(\boldsymbol{r}_i, \boldsymbol{p}_i, t) \equiv H(\boldsymbol{R}_j, \boldsymbol{q}_j, T_1)$$

is invariant under the Galilean transformations (Moller 1952)

$$t_i' = t_i, \qquad \boldsymbol{r}_i' = \boldsymbol{r}_i - \boldsymbol{V} t_i, \qquad \tau' = \tau, \qquad T_j' = T_j. \tag{12}$$

The prime denotes coordinates measured by an observer moving with velocity $\boldsymbol{V}$ relative to the centre of mass $\boldsymbol{R}_1$, with respect to a second observer who may be at rest relative to the centre of mass. Thus events that are simultaneous in one frame are simultaneous in all frames, with respect to ordinary times, and one has a universal time coordinate for reference to find the Newtonian trajectories $\boldsymbol{r}_i(\tau)$ or $\boldsymbol{R}_j(\tau)$.

It is well known that the above situation does not hold within the framework of the special theory of relativity (Einstein 1922). Nonlocal events that are simultaneous with respect to ordinary time in one frame of reference are generally not simultaneous in any other frame of reference.

The generalization of equations (1)–(7) which takes account of Lorentz invariance is obtained by adding time-like components to each vector. Let

$$2\mathscr{T}\,\mathrm{d}\tau^2 = -\sum_i M_i c\, g_{\mu\nu}\,\mathrm{d}x_i^\mu \mathrm{d}x_i^\nu, \tag{13}$$

where

$$g_{\mu\nu} = \begin{bmatrix} 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & -1 \end{bmatrix},$$

$\mathrm{d}x_i^4 = c\,\mathrm{d}t_i$, c is the velocity of light, $\mathscr{T}$ is the covariant analogue of the composite kinetic energy (1), and τ is the c.m. standard time as before. Hence, the c.m. transformations (4) yield

$$2\mathscr{T}\,\mathrm{d}\tau^2 = \mathbf{dr}\,.\,\mathbf{M}\,.\,\mathbf{dr} = \mathbf{dR}\,.\,\mathbf{G}\,.\,\mathbf{dR}, \tag{14}$$

where the $\mathbf{r}_i$ are now covariant 4-vectors $(\boldsymbol{r}_i, -\mathrm{i}ct_i)$ and the $\mathbf{R}_j$ are similarly the components $(\boldsymbol{R}_j, -\mathrm{i}cT_j)$. The corresponding 4-momenta are

$$\mathbf{P}_i = M_i\,\mathrm{d}\mathbf{r}_i/\mathrm{d}\tau = \gamma_i(\boldsymbol{p}_i, -\mathrm{i}E_i/c) = \gamma_i(M_i\beta_i\boldsymbol{v}_i, -\mathrm{i}E_i/c),$$

$$\gamma_i = \mathrm{d}\tau_i/\mathrm{d}\tau, \qquad \beta_i = \mathrm{d}t_i/\mathrm{d}\tau, \qquad E_i = M_i\beta_i c^2, \tag{15}$$

E_i being the total energy of the ith particle and τ_i the local proper time while γ_i describes the effect of translating a stationary clock from $\boldsymbol{r}_i$ to the centre of mass. The c.m. 4-momenta are

$$\mathbf{Q}_j = \mu_j\,\mathrm{d}\mathbf{R}_j/\mathrm{d}\tau = (\boldsymbol{q}_j, -\mathrm{i}\epsilon_j/c) = (\mathbf{U}^{-1})^{\sim}\mathbf{P}_i \tag{16}$$

with relative energy components

$$\epsilon_j = (\mathbf{U}_{ji}^{-1})^{\sim}E_i\,.$$

In an ordinary time calibration it would be customary to choose a particular frame of reference where the t_i are taken to be equal over a reference space-like surface. However, under local Lorentz transformations (Moller 1952)

$$\mathrm{d}\boldsymbol{r}_i' = \beta(\mathrm{d}\boldsymbol{r}_i - \boldsymbol{V}\,\mathrm{d}t_i), \qquad \mathrm{d}t_i' = \beta(\mathrm{d}t_i - \boldsymbol{V}\,.\,\mathrm{d}\boldsymbol{r}_i/c^2),$$

where the primes denote reference to a frame moving with velocity V relative to a frame where coordinates are not primed and

$$\beta = (1-V^2/c^2)^{-\frac{1}{2}};$$

the relative times transform as

$$T'_j = \beta(T_j + \boldsymbol{V}\cdot\boldsymbol{R}_j/c^2), \tag{17}$$

so events calibrated to be simultaneous in a reference frame, that is, $t.T_j = 0$, are not simultaneous in other frames, where

$$T'_j = \beta \boldsymbol{V}\cdot\boldsymbol{R}_j/c^2.$$

Therefore, ordinary time does not provide either an invariant concept of nonlocal instantaneousness or an invariant reference coordinate which permits a concept of trajectories $\mathbf{R}_j(t)$ applicable in all Lorentz frames. This leads to excessively complicated expressions for the equations of motion and represents a major obstacle that has so far prevented the determination of solutions to the relativistic two-body problem.

An alternative system of calibration is to define a space-like surface over which all local proper times are equal:

$$\tau_1 = \tau_2 = \dots \tau_N = \tau, \tag{18}$$

where

$$d\tau_i^2 = dr_i^2 - c^2\, dt_i^2.$$

In this type of calibration, ordinary times are given by

$$t_i = \int \beta_i(\tau)\, d\tau$$

and, for free particles,

$$t_i = \beta_i \tau.$$

For two particles, the relative time becomes

$$T_2 = \int (\beta_1 - \beta_2)\, d\tau \tag{19}$$

and nonlocal events are instantaneous with respect to ordinary time in the frame where the integral is zero, e.g. for free particles

$$T_2 = (\beta_1 - \beta_2)\tau \tag{20}$$

and $T_2 = 0$ if $\beta_1 = \beta_2$ and hence $v_1 = v_2$. This is the special frame where the particles have equal speed.

This calibration has two distinct advantages. Firstly, events that are instantaneous with respect to proper time in one frame are instantaneous in all other frames. We must introduce the restriction that τ refers to arc lengths of particles of finite mass. For photons, τ can refer to a variation along a null cone which for a particle implies that β_i is infinite, thus permitting a finite ordinary time increment to be nonzero, i.e.

$$\mathrm{d}\tau^2 = \mathrm{d}t_i^2/\beta_i = 0\,, \qquad \mathrm{d}t_i^2 = \beta_i\,\mathrm{d}\tau^2 \neq 0\,.$$

Secondly, the theory now behaves as if one has a Newtonian mechanics in a 4-space, with three real and one imaginary component of a 4-vector. One has a single parameter to which all trajectories $\mathbf{r}_i(\tau)$ can be referred. Its main disadvantage is that interactions over surfaces of equal proper times are not known and must be determined from field equations.

III. Electrodynamics

Let us consider a system of N moving charges for which we assume that Maxwell's field equations hold over the whole space–time. The electromagnetic field tensor (Moller 1952) therefore satisfies

$$\frac{\partial F^{(3)}_{\mu\nu}}{\partial x_\sigma} + \frac{\partial F^{(3)}_{\nu\sigma}}{\partial x_\mu} + \frac{\partial F^{(3)}_{\sigma\mu}}{\partial x_\nu} = 0 \tag{21}$$

everywhere, and the current densities in the system are given by

$$\partial F^{(3)}_{\mu\nu}/\partial x_\nu = J^{(3)}_\mu\,. \tag{22}$$

The equations are in standard tensor notation with $\mu, \nu, \sigma = 1, 2, 3, 4$. However, the term density here refers to unit volume in 3-space and hence the superfix (3) has been employed. An electromagnetic potential $A^{(3)}_\mu$ can be chosen in any system of inertia to satisfy

$$F^{(3)}_{\mu\nu} = \partial A^{(3)}_\mu/\partial x_\nu - \partial A^{(3)}_\nu/\partial x_\mu\,, \tag{23a}$$

$$\partial A^{(3)}_\mu/\partial x_\mu = 0\,, \tag{23b}$$

and hence

$$\partial^2 A^{(3)}_\mu/\partial x_\nu\,\partial x^\nu = -J^{(3)}_\mu\,. \tag{23c}$$

As a purely formal device, we introduce a scalar potential $\Phi^{(4)}$ such that when densities are referred to 4-space

$$F^{(4)}_{\mu\nu} = \partial A^{(4)}_\mu/\partial x_\nu - \partial A^{(4)}_\nu/\partial x_\mu\,, \tag{24a}$$

$$\partial A^{(4)}_\mu/\partial x_\mu + c^{-1}\,\partial\Phi^{(4)}/\partial\tau = 0\,, \tag{24b}$$

$$\partial^2 A^{(4)}_\mu/\partial x_\mu\,\partial x^\mu = -J^{(4)}_\mu\,, \tag{24c}$$

$$\partial J^{(4)}_\mu/\partial x_\mu + c^{-1}\,\partial\rho^{(4)}/\partial\tau = 0 \tag{24d}$$

such that in the proper time calibration

$$A^{(3)}_\mu = \int A^{(4)}_\mu\,\mathrm{d}\tau\,, \tag{25a}$$

$$J_{\mu}^{(3)} = \int J_{\mu}^{(4)} \, \mathrm{d}\tau \,, \tag{25b}$$

$$F_{\mu\nu}^{(3)} = \int F_{\mu\nu}^{(4)} \, \mathrm{d}\tau \,. \tag{25c}$$

$\rho^{(4)}$ is a charge density (per unit of 4-space) and refers to the local proper time at the sources. The equations (25) are implicit in the work of Wheeler and Feynman (1949).

The electromagnetic potential $A_{\mu}^{(3)}$ can be evaluated from the current densities using (23c) (Moller 1952) as

$$4\pi^2 A_{\mu}^{(3)}(x_{\nu}) = \iiiint \{J_{\mu}^{(3)}(x'_{\nu})/|\,\mathbf{S}-\mathbf{S}'\,|^2\} \, \mathrm{d}^4 x'_{\nu} \,, \tag{26}$$

where

$$|\,\mathbf{S}-\mathbf{S}'\,|^2 \equiv (x_{\nu}-x'_{\nu})(x^{\nu}-x'^{\nu}) \,.$$

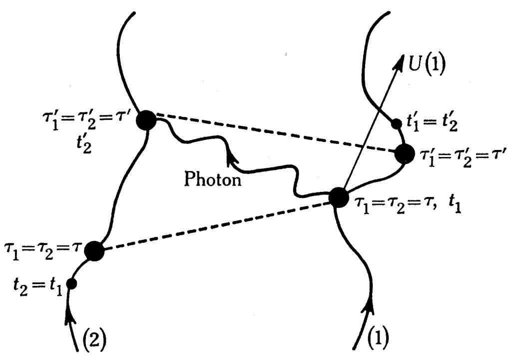

Fig. 1.—Photon exchange in an equal proper time calibration.

Consider two charges moving along time tracks denoted in Figure 1 by (1) and (2). Let a photon leave (1) at proper time τ and arrive at (2) at proper time τ', after travelling along a null cone given by

$$\{\boldsymbol{r}_1(\tau) - \boldsymbol{r}_2(\tau')\}^2 = c^2\{t_1(\tau) - t'_2(\tau')\} \,. \tag{27}$$

Following Wheeler and Feynman (1949), the current along (1) seen at (2) is

$$J_{\mu}^{(4)}(1) = e_1 b \, U_{\mu}(1) \, \delta\big(x_{\mu}(1) - x'_{\mu}(2)\big) \,, \tag{28}$$

where e_1 is the charge on (1) and $U_{\mu}(1)$ is its 4-velocity at time τ' and b is a constant to be determined. However,

$$x'_{\mu}(2) - x_{\mu}(1) = x'_{\mu}(2) - x'_{\mu}(1) + x'_{\mu}(1) - x_{\mu}(1)$$

and for low accelerations

$$x'_{\mu}(1) - x_{\mu}(1) \approx U_{\mu}(1)\,(\tau'-\tau) + \frac{\mathrm{d}U_{\mu}(1)}{\mathrm{d}\tau'}\frac{(\tau'-\tau)^2}{2} + \dots$$

so that retardations are allowed for approximately by putting

$$J^{(4)}_{\mu}(1) \approx e_1 b\, U_{\mu}(1)\, \delta(x'_{\mu}(1) - x'_{\mu}(2) - U_{\mu}(1)(\tau' - \tau)). \tag{29}$$

We now introduce the pseudospherical relative coordinates

$$\left.\begin{aligned} X_1 = R\sin\theta\cos\phi, \qquad X_2 = R\sin\theta\sin\phi, \qquad X_3 = R\cos\theta, \\ R = S\cosh\gamma, \qquad cT = S\sinh\gamma, \end{aligned}\right\} \tag{30}$$

where

$$\mathbf{X} = \mathbf{x}(1) - \mathbf{x}(2)$$

is the relative c.m. coordinate. These coordinates are all real for space-like separation of the particles. In these coordinates, the delta functions in (29) can be written

$$\delta(x_{\mu}(1) - x'_{\mu}(2)) = \frac{\delta(S' - |\mathbf{U}(1)|\,\tau)\, \delta(\gamma \pm \tfrac{1}{2}\mathrm{i}\pi - \delta)\, \delta(\theta - \alpha)\, \delta(\phi - \beta)}{S^3 \cosh^2\gamma \sin\theta}, \tag{31}$$

where the plus and minus signs refer to advanced and retarded solutions respectively and δ, α, and β are the pseudospherical components of $\mathbf{U}$. Taking (31), substituting into (29) and using (25b), we obtain after removing the radial delta function

$$J^{(3)}_{\mu}(1) = e_1 b\{U_{\mu}(1)/U(1)\}\delta(\gamma \pm \tfrac{1}{2}\mathrm{i}\pi - \delta)\, \delta(\theta - \alpha)\, \delta(\phi - \beta) \tag{32}$$

and hence, on integrating the angles over all 4-space,

$$4\pi^2 A^{(3)}_{\mu}(\mathbf{S}) = \iiiint \frac{1}{|\mathbf{S} - \mathbf{S}'|^2} \frac{e_1 b}{U(1)} U_{\mu}(1)\, \delta(\gamma' \pm \tfrac{1}{2}\mathrm{i}\pi - \delta)$$

$$\times \frac{\delta(\theta' - \alpha)\, \delta(\phi' - \beta) \cosh^2\gamma' \sin\theta' \,\mathrm{d}S'\, \mathrm{d}\gamma'\, \mathrm{d}\theta'\, \mathrm{d}\phi'}{\sinh^2\delta \sin\alpha}$$

$$= \{U(1)\}^{-1} \int \frac{e_1 b\, U_{\mu}(1)\, \mathrm{d}S'}{S^2 \pm 2\mathrm{i}SS'Z + S'^2},$$

where

$$Z = \sinh\gamma\cosh\delta - (\cosh\gamma\sinh\delta)z.$$

Therefore

$$4\pi^2 A^{(3)}_{\mu}(\mathbf{S}) \approx \frac{e_1 U_{\mu}(1)}{U(1)\, S(1+Z^2)^{\frac{1}{2}}} \left[A \arctan\left(\frac{S' - \mathrm{i}SZ}{S(1+Z^2)^{\frac{1}{2}}}\right) + B \arctan\left(\frac{S' + \mathrm{i}Z}{S(1+Z^2)^{\frac{1}{2}}}\right)\right]_0^{\infty},$$

where A and B are coefficients of the advanced and retarded solutions. Let $A = B = b$ be the required solution, thus giving

$$4\pi^2 A^{(3)}_{\mu}(\mathbf{S}) = \tfrac{1}{2}\pi\, e_1 b\, U_{\mu}(1)/S(1+Z^2)^{\frac{1}{2}}.$$

If this potential is to become the Coulomb potential at low velocities, then

$$b = 8\pi\, U(1)$$

and

$$A^{(3)}_{\mu}(\mathbf{S}) = e_1 U_{\mu}(1)/[\mathbf{S}^2+\{\mathbf{S}.\mathbf{U}(1)\}^2]^{\frac{1}{2}} \tag{33}$$

$$\approx e_1 U_{\mu}(1)/R \qquad \text{when} \qquad U_{\mu}(1) \approx (0, -\mathrm{i}).$$

We shall take (33) to be the covariant potential applicable across surfaces of equal proper time for the Coulomb field. This is the 4-potential at (2) due to the motion of the charge at (1). There is a similar potential at (1) due to the motion at (2). How do we combine these potentials to give a form applicable to the relative and c.m. equivalents? The centres of charge do not coincide generally with the centres of masses, and therefore a simple vector sum of currents will not suffice. Instead we define c.m. currents of mass

$$e_{\mathrm{c}} J^{(4)}_{\mu}(\text{c.m.}) = M^{-1}\{m_1(e_2) J^{(4)}_{\mu}(1) + m_2(e_1) J^{(4)}_{\mu}(2)\}, \tag{34a}$$

$$e_{\mathrm{r}} J^{(4)}_{\mu}(\text{rel.}) = \{(e_2) J^{(4)}_{\mu}(1) - (e_1) J^{(4)}_{\mu}(2)\}, \tag{34b}$$

with

$$e_{\mathrm{c}} = |e_1| + |e_2|, \qquad e_{\mathrm{r}} = e_1 e_2/e_{\mathrm{c}},$$

both of which still satisfy the continuity equation, and, using (26) with (33), give potentials

$$e_{\mathrm{c}} A^{(3)}_{\mu}(\text{c.m.}) \approx \left(\frac{m_1}{M}\frac{U_{\mu}(1)}{(1+Z_1^2)^{\frac{1}{2}}} + \frac{m_2}{M}\frac{U_{\mu}(2)}{(1+Z_2^2)^{\frac{1}{2}}}\right)\frac{e_1 e_2}{S}, \tag{35a}$$

$$e_{\mathrm{r}} A^{(3)}_{\mu}(\text{rel.}) \approx \mu\left(\frac{U_{\mu}(1)}{(1+Z_1^2)^{\frac{1}{2}}} - \frac{U_{\mu}(2)}{(1+Z_2^2)^{\frac{1}{2}}}\right)\frac{e_1 e_2}{S}, \tag{35b}$$

where

$$Z_1 = S_{\mu} U^{\mu}(1)/S, \qquad Z_2 = S_{\mu} U^{\mu}(2)/S.$$

The pseudoangles Z_1 and Z_2 for a momentary equal speed frame are given by

$$Z_1 = \hat{\boldsymbol{R}}.\hat{\boldsymbol{v}}_1 \beta_1/c, \qquad Z_2 = \hat{\boldsymbol{R}}.\hat{\boldsymbol{v}}_2 \beta_2/c,$$

since $t_1 = t_2$, which if the interaction is weak produces low velocities in this frame, thus giving

$$Z_1 \sim Z_2 \ll 1$$

and therefore

$$e_{\mathrm{c}} A^{(3)}_{\mu}(\text{c.m.}) \approx U_{\mu}(\text{c.m.})\, e_1 e_2/S, \tag{36a}$$

$$e_{\mathrm{r}} A^{(3)}_{\mu}(\text{rel.}) \approx U_{\mu}(\text{rel.})\, e_1 e_2/S. \tag{36b}$$

It follows from the current (29) and equation (26) that S is the same in (36a) and (36b). The approximations (36) do not satisfy Maxwell's equations or the Lorentz condition exactly. They are approximations for low charge accelerations, are covariant, and hold across the surface of equal proper time.

IV. Gravitation

The derivation of the gravitational potential for weak fields follows closely the conventional lines (Moller 1952). We introduce five components

$$(X_1,X_2,X_3)\equiv \boldsymbol{R}\,,\qquad X_0=cT\,,\qquad X_4=c\tau\,, \tag{37}$$

in which (X_0,X_1,X_2,X_3) are relative coordinates and X_4 defines the surface of equal proper time, and plays the role of ordinary time in the five-dimensional theory.

A weak field means that we can write the metric tensor

$$g_{ik}=G_{ik}+h_{ik}\,, \tag{38}$$

where

$$G_{ik}=\begin{bmatrix} -1 & . & . & . & . \\ . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & . & -1 \end{bmatrix}$$

is the metric tensor applicable to the special theory and the h_{ik} are small general relativistic corrections. The Ricci curvature tensor is

$$R_{ik}=\frac{\partial\Gamma^l_{il}}{\partial x^k}-\frac{\partial\Gamma^l_{ik}}{\partial x^l}+\Gamma^r_{il}\Gamma^l_{kr}-\Gamma^r_{ik}\Gamma^l_{lr}\,, \tag{39}$$

where the Γ^i_{kl} are the Christoffel symbols defined as

$$\Gamma_{i,kl}=\frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^l}+\frac{\partial g_{il}}{\partial x^k}-\frac{\partial g_{kl}}{\partial x^i}\right)=g_{ir}\Gamma^r_{kl}\,. \tag{40}$$

Neglecting squares of the Christoffel symbols, we get

$$\begin{aligned} R_{ik} &\approx \frac{\partial\Gamma^r_{ir}}{\partial X^k}-\frac{\partial\Gamma^r_{ik}}{\partial X^r} \\ &= \frac{G^{rs}}{2}\frac{\partial^2 h_{ik}}{\partial X^r\,\partial X^s}+\frac{1}{2}\left(\frac{\partial^2 h}{\partial X^i\,\partial X^k}-\frac{\partial^2 h^r_i}{\partial X^r\,\partial X^k}-\frac{\partial^2 h^r_k}{\partial X^i\,\partial X^r}\right), \end{aligned} \tag{41}$$

where

$$h^r_i=G^{rs}h_{is}\,,\qquad h=h^r_r=G^{rs}h_{rs}\,,$$

and hence

$$R_{44}=-c^{-2}\,\square^2\chi\,, \tag{42a}$$

$$\square^2=\sum_{i=0}^{3}G_{ik}\frac{\partial^2}{\partial X_i\,\partial X_k}\,, \tag{42b}$$

where

$$g_{44}=-(1+2\chi/c^2) \tag{42c}$$

and χ is the scalar gravitational potential. Neglecting elastic stresses, the energy–momentum tensor in this space is

$$T_{ik} = G_{il}\, G_{km}\, \mu_0\, U^l\, U^m\,, \tag{43}$$

where μ_0 is the matter 4-density and U^l the component of 5-velocity. We arrive by standard methods at the field equation

$$R_{ik}+\lambda g_{ik} = -\kappa(T_{ik}-\tfrac{1}{2}Tg_{ik})\,, \tag{44}$$

where λ is a cosmological constant, assumed to be small, κ is the gravitational constant, and $T = T^i_i$. We get from (44) using equations (42)

$$\Box^2\chi - c^2 g_{44} = \tfrac{1}{2}\kappa c^2 \mu_0\,. \tag{45}$$

Defining

$$\chi^k_i = h^k_i - \tfrac{1}{2}\delta^k_i\, h\,, \qquad \chi_{ik} = h_{ik} - \tfrac{1}{2}G_{ik}\, h\,,$$

with $h = h^i_i$, and employing $\partial\chi^k_i/\partial X^k = 0$, we get

$$\Box^2\chi_{ik} = -2\kappa T_{ik}\,, \tag{46}$$

that is,

$$\chi^{(4)}_{ik}(X^\mu,\tau) = \frac{2\kappa}{4\pi^2}\iiiint \frac{T^{(4)}_{ik}}{|\,\mathbf{S}-\mathbf{S}'\,|^2}\, \mathrm{d}X'^4_\mu\,, \qquad \mu = 0,1,2,3, \tag{47}$$

where the superscript (4) shows reference to matter 4-density. Therefore we have

$$\partial T^{k(4)}_i/\partial X^k = 0\,, \tag{48}$$

expressing conservation of energy–momentum, and

$$\chi^l_l = -h\,, \qquad h_{ik} = \chi_{ik} + \tfrac{1}{2}hG_{ik}\,.$$

We then get

$$\chi^{l(4)}_l(\mathbf{S}) = \frac{\kappa_4}{4\pi^2}\iiiint \frac{T(\mathbf{S}')\, \mathrm{d}^4X'_\mu}{|\,\mathbf{S}-\mathbf{S}'\,|^2}\,,$$

$$h_{ik} = \frac{\kappa_4}{4\pi^2}\iiiint \frac{(T_{ik}-\tfrac{1}{2}G_{ik}\,T)\, \mathrm{d}^4X'_\mu}{|\,\mathbf{S}-\mathbf{S}'\,|^2}\,. \tag{49}$$

For a static system

$$T_{ik} = \delta_{i4}\,\delta_{k4}\,\mu_0\, c^2\,, \qquad\qquad T = -\mu_0\, c^2\,,$$

$$h_{44} = \frac{\kappa_4 c^2}{8\pi^2}\iiiint \frac{\mu_0(\mathbf{S}')\, \mathrm{d}V'_4}{|\,\mathbf{S}-\mathbf{S}'\,|^2}\,, \qquad\qquad \mathrm{d}V'_4 = \mathrm{d}^4x'_\mu\,,$$

$$h_{i4} = 0\,, \qquad h_{ik} = \frac{\kappa_4 c^2}{8\pi^2}\iiiint \frac{\mu_0(\mathbf{S}')\, \mathrm{d}V'_4}{|\,\mathbf{S}-\mathbf{S}'\,|^2}\,\delta_{ik} = h_{44}\,\delta_{ik}\,,$$

$$\begin{aligned}\chi^{(3)} &= c\int \chi^{(4)}\,\mathrm{d}\tau' = -\tfrac{1}{2}c^3\int h_{44}\,\mathrm{d}\tau' \\ &= -\frac{\kappa_4 c^3}{8\pi^2}\iiiint\!\!\int \frac{\mu_0(\mathbf{S}')\,\mathrm{d}V_4'}{|\,\mathbf{S}-\mathbf{S}'\,|^2}\,\mathrm{d}\tau' \\ &= -\kappa_3 \sum_i M_{0i}/\{S_i(1+Z_i^2)^{\frac{1}{2}}\}\end{aligned} \tag{50}$$

as in the electromagnetic case, where

$$\mu_{0i} = \sum_i M_{0i}\,\delta\big(\mathbf{S}(\tau)-\mathbf{S}_i'(\tau')\big), \qquad \kappa_3 = \kappa_4 c^2 U/\pi^2 .$$

The line element is

$$\begin{aligned}\mathrm{d}\sigma^2 &= (G_{ik}+h_{ik}^{(3)})\,\mathrm{d}X^i\,\mathrm{d}X^k \\ &= (1-2\chi^{(3)}/c^2)(-\mathrm{d}X_0^2+\mathrm{d}X_1^2+\mathrm{d}X_2^2+\mathrm{d}X_3^2)-(1+2\chi^{(3)}/c^2)c^2\,\mathrm{d}\tau^2 .\end{aligned} \tag{51}$$

For low accelerations, we therefore find

$$\chi^{(3)} = -\kappa_3 \sum_i M_{0i}/S_i , \qquad Z_i \ll 1 . \tag{52}$$

V. Forces, Lagrangian, and Hamiltonian

(a) *Covariant Angular Momentum*

A concept essential to relativistic mechanics is that of the covariant angular momentum tensor

$$\Lambda^i_{\mu\nu} = X^i_\mu P^i_\nu - X^i_\nu P^i_\mu \equiv \mathbf{r}_i \times \mathbf{P}_i , \tag{53}$$

which is the moment of momentum relative to an event distance $\mathbf{r}_i$ from the ith particle in the system. It is analogous to the orbital angular momentum $\boldsymbol{L}_i = \boldsymbol{r}_i \times \boldsymbol{p}_i$ and can be written

$$\boldsymbol{\Lambda} \equiv \boldsymbol{L}-\mathrm{i}\boldsymbol{A} , \qquad \boldsymbol{A} = (E/c)\boldsymbol{r}-\boldsymbol{p}ct , \tag{54}$$

$\boldsymbol{L}$ being an axial 3-vector and $\boldsymbol{A}$ a polar 3-vector. Two quantities which are Lorentz invariant are

$$\tfrac{1}{2}\Lambda^{\mu\nu}\Lambda_{\mu\nu} = \Lambda^2 = \boldsymbol{L}^2-\boldsymbol{A}^2 \qquad \text{and} \qquad \boldsymbol{L}\,.\,\boldsymbol{A} = 0 . \tag{55}$$

(i) For fields which impart no 4-torque to a particle's orbit, $\Lambda^i_{\mu\nu}$ is conserved in the proper time calibration, that is,

$$\mathrm{d}\Lambda^i_{\mu\nu}/\mathrm{d}\tau = X^i_\mu \dot{P}^i_\nu - X^i_\nu \dot{P}^i_\mu = M^i_{\mu\nu} = \dot{\boldsymbol{L}}-\mathrm{i}\dot{\boldsymbol{A}} = 0 , \tag{56}$$

and hence each component of $\boldsymbol{L}$ and $\boldsymbol{A}$ is conserved.

(ii) For an isolated system, the total covariant angular momentum is conserved in the proper time calibration, that is,

$$\mathrm{d}\Lambda_{\mu\nu}/\mathrm{d}\tau = \sum_i \mathrm{d}\Lambda^i_{\mu\nu}/\mathrm{d}\tau = \sum_i M^i_{\mu\nu} = M_{\mu\nu} = 0 , \tag{57}$$

since no external 4-torque acts. Using the transformations (4) we find

$$\dot{\mathbf{\Lambda}} = \sum_i \mathbf{r}_i \times \dot{\mathbf{P}}_i = \sum_j \boldsymbol{R}_j \times \dot{\boldsymbol{Q}}_j = \sum_j \mathbf{M}_j = 0. \tag{58}$$

In the two-body case, consider the example where each $\mathbf{M}_j$ is zero. With a proper time calibration, the relative coordinates give

$$\dot{\boldsymbol{L}} = \boldsymbol{R} \times \dot{\boldsymbol{q}}, \qquad \dot{\boldsymbol{A}} = \dot{\epsilon}\boldsymbol{R} - \dot{\boldsymbol{q}}T,$$

where $\mathbf{Q} = (\boldsymbol{q}, -\mathrm{i}\epsilon/c)$. The 4-force is $\mathbf{F} = (\dot{\boldsymbol{q}}, -\mathrm{i}\dot{\epsilon}/c)$, so that if $\boldsymbol{L}$ and $\boldsymbol{A}$ are both zero not only is $\dot{\boldsymbol{q}}$ parallel to $\boldsymbol{R}$, giving a central force, but

$$\dot{q}/\dot{\epsilon} = R/cT = \coth\gamma$$

and the 4-vector $\mathbf{F}$ is parallel to the 4-vector $\mathbf{R}$. Such forces we shall call "hypercentral".

For the relative cooordinates

$$\Lambda^2 = \boldsymbol{L}^2 - \boldsymbol{A}^2 > 0 \qquad \text{always,} \tag{59a}$$

$$\Lambda^2 = Q^2S^2(1-Z^2) > 0, \qquad Z = \mathbf{Q}\,.\,\mathbf{R}/QS, \tag{59b}$$

$$L^2 = q^2R^2(1-z^2) > 0, \qquad z = \boldsymbol{q}\,.\,\boldsymbol{R}/qR. \tag{59c}$$

(b) *Lagrangian*

Having established the form of two interactions we wish to know how to apply them in a point-particle mechanics. The proper time calibration allows the use of conventional Lagrangian theory (Corben and Stehle 1950) in a Minkowski 4-space. The components of relative acceleration are, in curvilinear coordinates y^μ,

$$a^\rho = \ddot{y}^\rho + \Gamma^\rho_{\mu\nu}\,\dot{y}^\mu\,\dot{y}^\nu, \tag{60}$$

where $\mu, \nu, \rho = 0, 1, 2, 3$, the dots denote differentiation with respect to τ, and $\Gamma^\rho_{\mu\nu}$ is the Christoffel symbol (40) defined in the 4-space. For the relative pseudospherical coordinates (30)

$$g_{00} = 1, \quad g_{11} = -S^2, \quad g_{22} = S^2\cosh^2\gamma, \quad g_{33} = S^2\cosh^2\gamma\sin^2\theta, \tag{61}$$

from which we obtain the acceleration components

$$a^0 = a_S = \ddot{S} + S\{\dot{\gamma}^2 - \cosh^2\gamma\,(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)\}, \tag{62a}$$

$$a^1 = a_\gamma = -S^2\ddot{\gamma} - 2S\dot{S}\dot{\gamma} - 2S^2\sinh\gamma\cosh\gamma\,(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2), \tag{62b}$$

$$a^2 = a_\theta = R^2\{\ddot{\theta} + 2(\dot{S}/S)\dot{\theta} + 2\tanh\gamma\,\dot{\gamma}\dot{\theta}\} = R^2\{\ddot{\theta} + 2(\dot{R}/R)\dot{\theta}\}, \tag{62c}$$

$$a^3 = a_\phi = R^2\sin^2\theta\,\{\ddot{\phi} + 2(\dot{R}/R)\dot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi}\}. \tag{62d}$$

The Euler–Lagrange equation for the relative motion is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial \mathscr{L}}{\partial \dot{y}^{\mu}}\right)-\frac{\partial \mathscr{L}}{\partial y^{\mu}}=0, \tag{63}$$

with

$$\begin{aligned}\mathscr{L} &= \tfrac{1}{2}\mu(\dot{S}^2-S^2\dot{\gamma}^2+S^2\cosh^2\gamma\,\dot{\theta}^2+S^2\cosh^2\gamma\sin^2\theta\,\dot{\phi}^2)-\mathscr{V}\\ &=\mathscr{T}-\mathscr{V},\end{aligned} \tag{64}$$

$\mathscr{V}$ being the interaction potential and $\mathscr{T}$ the analogue of the kinetic energy. The single-particle definition of $\mathscr{T}$ is given by Corben and Stehle (1950), Moller (1952), and Synge (1958).

The components of 4-force along the four unit vectors

$$\mathbf{e}_S=(\cosh\gamma\,\hat{\boldsymbol{R}},-\mathrm{i}\sinh\gamma), \qquad \mathbf{e}_\gamma=(\sinh\gamma\,\hat{\boldsymbol{R}},-\mathrm{i}\cosh\gamma),$$

$$\mathbf{e}_\theta=(\cos\theta\cos\phi,\cos\theta\sin\phi,-\sin\theta,0), \quad \mathbf{e}_\phi=(-\sin\phi,\cos\phi,0,0),$$

such that

$$\mathbf{e}_S^2=\mathbf{e}_\theta^2=\mathbf{e}_\phi^2=1, \qquad \mathbf{e}_\gamma^2=-1,$$

are

$$F_S=\dot{Q}_S-\partial\mathscr{T}/\partial S=\mu a_S, \qquad F_\gamma=\dot{Q}_\gamma-\partial\mathscr{T}/\partial\gamma=\mu a_\gamma,$$

$$F_\theta=\dot{Q}_\theta-\partial\mathscr{T}/\partial\theta=\mu a_\theta, \qquad F_\phi=\dot{Q}_\phi-\partial\mathscr{T}/\partial\phi=\mu a_\phi,$$

where

$$\mathbf{Q}=Q_S\,\mathbf{e}_S+Q_\gamma\,\mathbf{e}_\gamma+Q_\theta\,\mathbf{e}_\theta+Q_\phi\,\mathbf{e}_\phi \tag{65a}$$

is the relative 4-momentum and

$$\mathbf{F}=\dot{\mathbf{Q}}=F_S\,\mathbf{e}_S+F_\gamma\,\mathbf{e}_\gamma+F_\theta\,\mathbf{e}_\theta+F_\phi\,\mathbf{e}_\phi \tag{65b}$$

is the relative 4-force. The canonical momenta are

$$\frac{\partial\mathscr{L}}{\partial\dot{S}}=Q_S=\mu\dot{S}-\frac{\partial\mathscr{V}}{\partial\dot{S}}, \qquad \frac{\partial\mathscr{L}}{\partial\dot{\gamma}}=Q_\gamma=-\mu S^2\dot{\gamma}-\frac{\partial\mathscr{V}}{\partial\dot{\gamma}},$$

$$\frac{\partial\mathscr{L}}{\partial\dot{\theta}}=Q_\theta=\mu R^2\dot{\theta}-\frac{\partial\mathscr{V}}{\partial\dot{\theta}}, \qquad \frac{\partial\mathscr{L}}{\partial\dot{\phi}}=Q_\phi=\mu R^2\sin^2\theta\,\dot{\phi}-\frac{\partial\mathscr{V}}{\partial\dot{\phi}}.$$

(c) *Hamiltonian*

The kinetic energy can be partitioned into a sum of contributions, each term of which is a function of the associated c.m. coordinate and its proper time derivative. We assume that this is true in special cases for interactions as well, for the purpose of investigating simple models, that is,

$$\mathscr{L}_{\text{total}}=\sum_j \mathscr{L}(\mathbf{R}_j,\dot{\mathbf{R}}_j,\tau). \tag{66}$$

The systems dealt with are isolated so the c.m. motion always obeys free particle equations and only interaction with respect to relative motion is considered. As above, the index (2) is omitted for the two-body problem.

The Hamiltonian

$$\mathscr{H} = \mathbf{Q} . \mathbf{R} - \mathscr{L}$$

obeys the canonical relations

$$\partial\mathscr{H}/\partial Q_\nu = \dot{R}_\nu, \qquad \partial\mathscr{H}/\partial R_\nu = -\dot{Q}_\nu. \tag{67}$$

For the solution of problems with hypercentral motion it is useful to define an action variable $\mathscr{S}$ such that

$$Q_\mu = \partial\mathscr{S}/\partial R^\mu, \tag{68}$$

which when substituted into the Hamiltonian for the relative motion

$$\mathscr{H} = \mathbf{Q}^2/2\mu + \mathscr{V}(\mathbf{R}, \dot{\mathbf{R}}, \tau) \tag{69}$$

leads to the Hamilton–Jacobi differential equation.

VI. Relativistic Models and Bohr Quantization

The Bohr–Sommerfeld model of the atom is now largely of historical interest only. It does, however, bear a close relationship to the Brillouin (1926), Kramers (1926), and Wentzel (1926) approximation to quantum mechanics, and it is instructive to show that the quantized covariant theory based upon the proper time calibration does indeed give meaningful results.

Bohr (1913) postulated that the action integrals

$$\begin{aligned} J_i &= \mathscr{S}_i \,(\text{one period}) \\ &= \oint Q_i \, \mathrm{d}y_i = 2\pi n(h/2\pi) = nh, \end{aligned} \tag{70}$$

where (Q_i, y_i) are a canonically conjugate pair of variables, can have only values that are integral multiples of Planck's constant. We shall refer to this postulate in the following models. The following postulates are also used in these models unless otherwise indicated.

(i) The non-relativistic interaction $V(\mathbf{R})$ can be replaced by $\mathscr{V}(\mathbf{S})$ in the covariant model.

(ii) The interaction is a function only of relative configurational coordinates.

(a) *Harmonic Oscillator*

For motion confined to the relative X axis, we write

$$\mathscr{T} = \tfrac{1}{2}\mu(\dot{X}^2 - c^2\dot{T}^2), \qquad \mathscr{V} = \tfrac{1}{2}k(X^2 - c^2T^2), \tag{71}$$

where X is the distance between the two particles, T the relative time defining the surface of constant proper time, μ the reduced mass, and k an elastic force constant.

Lagrange's equations give

$$\mu\ddot{X}+kX = 0\,, \qquad \mu\ddot{T}+kT = 0\,,$$

with solutions

$$X = X_0 \sin\{\omega_X(\tau-\tau_0)\}\,, \qquad T = T_0 \sin\{\omega_T(\tau-\tau_0)\}\,, \tag{72}$$

and, if $\omega_X = \omega_T = \omega$,

$$S = S_0 \sin\{\omega(\tau-\tau_0)\}\,,$$

where $S_0^2 = X_0^2-c^2T_0^2$ and $\omega = (k/\mu)^{\frac{1}{2}}$.

The energy can be separated such that

$$\mathscr{E} = \mathscr{E}_X+\mathscr{E}_T\,, \tag{73a}$$

where

$$\mathscr{E}_X = q^2/2\mu+\tfrac{1}{2}kX^2 = \tfrac{1}{2}kX_0^2\,, \tag{73b}$$

$$\mathscr{E}_T = -\epsilon^2/2\mu-\tfrac{1}{2}kT^2 = -\tfrac{1}{2}kT_0^2\,, \tag{73c}$$

giving

$$\mathscr{E} = \tfrac{1}{2}kS_0^2\,.$$

The oscillating relative time $T(\tau)$ and relative energy

$$\epsilon = \mu\left(\frac{E_1(\tau)}{m_1}-\frac{E_2(\tau)}{m_2}\right) = \mu T_0\,\omega\cos\{\omega(\tau-\tau_0)\} \tag{74}$$

are observable only at relativistic speeds when $\mathscr{V}_1 \sim c$ or $\mathscr{V}_2 \sim c$.

The energy levels in the Bohr theory are derived from the relations

$$n_X h = \oint Q_X\,\mathrm{d}X\,, \qquad n_T h = c\oint Q_T\,\mathrm{d}T\,,$$

where

$$Q_X^2 = 2\mu(\mathscr{E}_X-\tfrac{1}{2}kX^2) = q^2\,, \tag{75a}$$

$$Q_T^2 = 2\mu(\mathscr{E}_T-\tfrac{1}{2}kc^2T^2) = \epsilon^2/c^2\,. \tag{75b}$$

Integrating over the periods from $-X_0 \leqslant X \leqslant X_0$, $-T_0 \leqslant T \leqslant T_0$, we obtain

$$\mathscr{E}_X = n_X h\omega/2\pi\,, \qquad \mathscr{E}_T = n_T h\omega/2\pi \tag{76}$$

and therefore, by using (4a), (4b), and (16) in the total Hamiltonian, we get an energy

$$\mathscr{E} = (Q_1^2-M^2)/2M = (W^2-M^2)/2M = (n_X+n_T)h\omega/2\pi\,,$$

where $W = E_1+E_2 = Q_1$ is the conventional notation for c.m. energy, or mass, and the mass levels are

$$W = Q_1 = M^{\frac{1}{2}}(M+nh\omega/\pi c^2)^{\frac{1}{2}}\,, \qquad n = n_X+n_T\,. \tag{77}$$

This simple result for the mass level of a two-body dynamical system is characteristic of the proper time calibration. Equation (77) shows that the relative time oscillations contribute directly to the mass levels of a system. Very high energies would be required to excite these states since we require $\mathscr{E}_T \sim \mathscr{E}_X$, and hence $\epsilon \sim qc$, which implies an extremely powerful elastic force. Such forces may not occur in nature.

In the non-relativistic limit, $T_0 \to 0$ and $\epsilon \to 0$ as $c \to \infty$ and the equations become the ordinary equations for the non-relativistic harmonic oscillator.

(b) *Inverse Cube Law of Force*

Let us suppose that the S^{-2} behaviour obtained from solving Maxwell's equations or the gravitational field equations with matter 4-densities is interpreted as a potential energy. Then we have

$$\mathscr{V} = k/S^2. \tag{78}$$

The Hamilton–Jacobi equation is easily solved to give

$$d_1+\tau = \mu \int \{-2\mu\,|\,\mathscr{E}\,|+(2\mu k-\Lambda^2)/S^2\}^{-\frac{1}{2}}\,\mathrm{d}S$$

$$= -\{(2\mu k-\Lambda^2)-2\mu\,|\,\mathscr{E}\,|\,S^2\}^{\frac{1}{2}}/4\,|\,\mathscr{E}\,|, \tag{79a}$$

$$d_2 = -\arcsin\{(\Lambda/A)\sin\gamma\}+(\Lambda/\bar{\Lambda})\operatorname{arcosh}(\bar{\Lambda}/2\mu\mathscr{E}S), \tag{79b}$$

with

$$\bar{\Lambda}^2 = 2\mu k-\Lambda^2, \qquad \tanh\gamma = (A/L)\cos\theta,$$

and

$$d_4 = \phi, \tag{79c}$$

where the d_i are constants of the motion. That is, the orbital equation is

$$\frac{|\bar{\Lambda}|}{2\mu\,|\,\mathscr{E}\,|\,S} = \cosh\left[\frac{\bar{\Lambda}}{\Lambda}\left\{\arccos\left(\frac{\Lambda}{A}\sinh\gamma\right)\right\}\right] \approx \cosh\left(\frac{\bar{\Lambda}}{\Lambda}\theta\right), \tag{80}$$

$$\approx \cos(|\bar{\Lambda}|\,\theta/\Lambda) \qquad \text{for} \qquad \bar{\Lambda}^2 < 0,$$

at low velocities where $\Lambda \approx L$. Equation (80) is that for a spiral. The solution has no stable orbits, the Bohr integrals do not exist, and bound states do not occur.

(c) *Coulomb Law of Force*

The Hamiltonian is taken to be

$$\mathscr{H} = (\mathbf{Q}_1-e_\mathrm{c}\,\mathbf{A}_1)^2/2M+(\mathbf{Q}_2-e_\mathrm{r}\,\mathbf{A}_2)^2/2\mu, \tag{81}$$

where

$$e_\mathrm{c}\,\mathbf{A}_1 = (e^2/c)\mathbf{U}_{\mathrm{cm}}\,g_1/S, \qquad e_\mathrm{r}\,\mathbf{A}_2 = (e^2/c)\mathbf{U}_{\mathrm{cm}}\,g_2/S$$

for convenience, which are approximate potentials corresponding to equations (35). We choose g_1 and g_2 to satisfy

$$(Q_1/M)g_1-\mathbf{V}\,.\,\mathbf{U}_{\mathrm{cm}}\,g_2 = f_1, \qquad (\mu/M)g_1^2+g_2^2 = f_2. \tag{82}$$

Equation (81) becomes

$$\mathscr{H} = \frac{1}{2M}\left(Q_1^2 - \frac{2e^2 f_1 Q_1}{cS} + \frac{e^4 f_2}{c^2 S^2}\right) + \frac{1}{2\mu}\left(Q_S^2 - \frac{Q_\gamma^2}{S^2} + \frac{Q_\theta^2}{R^2} + \frac{Q_\phi^2}{(R\sin\theta)^2}\right). \tag{83}$$

Using the Hamilton–Jacobi equation we obtain

$$d_2 = -\arcsin\left(\frac{\Lambda}{\bar{\Lambda}}\sinh\gamma\right) + \frac{\Lambda}{\bar{\Lambda}}\arcsin\left(\frac{S^{-1} + Q_1 e^2 f_1 \mu/\bar{\Lambda}^2 cM}{(-E^2/\bar{\Lambda}^2 + Q_1 e^2 \mu/\Lambda^2)^{\frac{1}{2}}}\right) \tag{84a}$$

and

$$\tanh\gamma = (\Lambda/L)\cos\theta, \tag{84b}$$

where

$$\bar{\Lambda}^2 = \Lambda^2 - (e^4/c^2)f_2, \qquad E = -\,|\,(\mu/M)Q_1^2 + 2\mu\mathscr{E}\,|, \quad \mathscr{E} = \mathscr{H}.$$

This leads to the complicated orbital equation

$$\frac{1}{S} = B\left(1 + e_1\cos\left[\frac{\bar{\Lambda}}{\Lambda}\left\{\arccos\left(\frac{\Lambda}{L}\frac{\cos\theta}{\{1-(\Lambda^2/L^2)\cos^2\theta\}^{\frac{1}{2}}}\right)\right\}\right]\right) \tag{85}$$

with

$$B = \frac{Q_1^2 e^2 f_1 \mu}{\bar{\Lambda}^2 cM}, \qquad Be_1 = \left(-\frac{E}{\bar{\Lambda}^2} + \frac{Q_1 e^2 f_1 \mu}{\Lambda^2 cM}\right)^{\frac{1}{2}}$$

which tends to the form of Sommerfeld's equations (McCrea 1954) as $\Lambda \to L$.

For the bound state problem

$$Q^2 = |\,Q_2\,|^2 = \mu(M^2 - Q_1^2)/M.$$

Bohr quantization of these equations gives

$$\frac{\mu}{M}\frac{Q_1(e^2/c)f_1}{Q_2(\infty)} = n_S\hbar + (\lambda^2 - f_2\alpha^2)^{\frac{1}{2}}\hbar, \tag{86}$$

where $\lambda = \Lambda/\hbar$ and $Q_2(\infty)$ is the limiting value of Q_2 for large S. This yields atomic two-body mass levels

$$W = Q_1 = \frac{M}{\{1 + (\mu/M)\alpha^2 f_1^2/\bar{n}^2\}^{\frac{1}{2}}} \tag{87}$$

with

$$\bar{n} = n_S + (\lambda^2 - \alpha^2 f_2)^{\frac{1}{2}}, \qquad n = n_S + l, \qquad l = n_\theta + n_\phi,$$

$$\oint Q_S\,\mathrm{d}S = n_S h, \qquad \oint Q_\gamma\,\mathrm{d}\gamma = n_\gamma h,$$

$$\oint Q_\theta\,\mathrm{d}\theta = n_\theta h, \qquad \oint Q_\phi\,\mathrm{d}\phi = n_\phi h,$$

and energy levels in the limit $m_1 \gg m_2$ of

$$E_1 = \frac{m_1 c^2}{[1+\alpha^2/\{n_r+(l^2-\alpha^2)^{\frac{1}{2}}\}]^{\frac{1}{2}}}, \qquad n = n_r + l,$$

which is the conventional result. In detail however

$$Q_1 = \frac{M}{\{1+(\mu/M)\alpha^2 f_1^2/\bar{n}^2\}^{\frac{1}{2}}} \approx M\left(1-\frac{\mu}{2M}\frac{\alpha^2 f_1^2}{\bar{n}^2}+\frac{3\mu^2}{8M^2}\frac{\alpha^4 f_1^4}{\bar{n}^4}\dots\right),$$

that is, for $m_2 \gg m_1$, $M \gg \mu$,

$$E_1 - m_1 c^2 \approx -\tfrac{1}{2} m_1 c^2 \alpha^2 f_1/n^2 - \tfrac{1}{2} m_1 c^2 \alpha^4 f_1^2 f_2^2/n^3\lambda + O(\alpha^6), \tag{88}$$

within which the well-established term $\frac{3}{8}\, m_1 c^2 \alpha^4/n^4$ is missing. This can be allowed for by correcting the parameters f_1 or f_2 but as yet the author has found no other adequate reason for doing so.

(d) Planetary Motion under Gravity

With an attractive potential

$$\mathscr{V}(\mathbf{R}) = -K/S, \tag{89}$$

one can integrate the Hamilton–Jacobi equation to obtain

$$d_1+\tau = \frac{\mu}{(2\mu\,|\mathscr{H}|)^{\frac{1}{2}}}\left\{\frac{K^2}{4\,|\mathscr{H}|^2} - \Lambda^2 - \left(S - \frac{K}{2\mu\,|\mathscr{H}|}\right)^2\right\}^{\frac{1}{2}}$$
$$-\frac{K}{2\mu\,|\mathscr{H}|}\arcsin\left(\frac{S-K/2\,|\mathscr{H}|}{(K^2/4\mathscr{H}^2 - \Lambda^2/2\mu\,|\mathscr{H}|)^{\frac{1}{2}}}\right), \tag{90a}$$

$$d_2 = -\arcsin\left(\frac{\Lambda}{A}\sinh\gamma\right) + \arcsin\left(\frac{\Lambda^2/S - \mu K}{(\mu^2K^2 - 2\mu\Lambda^2\,|\mathscr{H}|)^{\frac{1}{2}}}\right), \tag{90b}$$

$$\cos\theta = -(L/A)\tanh\gamma, \tag{90c}$$

where we choose

$$\mathscr{H} = -|\mathscr{H}| = \mathscr{E} = -|\mathscr{E}|$$

and $\mathscr{E}$ is the analogue of the total energy of the system. In practice it is one-half the sum of the rest masses at infinite separation. The orbital equation is therefore

$$\frac{1}{S} = \frac{\mu K}{\Lambda^2}\left\{1+\left(1-\frac{2\,|\mathscr{H}|\,\Lambda^2}{\mu K^2}\right)^{\frac{1}{2}}\frac{\Lambda}{A}\sinh\gamma\right\}$$
$$= \frac{\mu K}{\Lambda^2}\left\{1-\left(1-\frac{2\,|\mathscr{H}|\,\Lambda^2}{\mu K^2}\right)^{\frac{1}{2}}\cos\theta\,\frac{A}{L}\left(1-\frac{A^2}{L^2}\cos^2\theta\right)^{-\frac{1}{2}}\right\}$$

or

$$\frac{1}{R} = \frac{\mu K}{\Lambda^2}\left\{\left(1-\frac{A^2}{L^2}\cos^2\theta\right)^{\frac{1}{2}} - \frac{\Lambda}{L}\left(1-\frac{2\,|\mathscr{H}\,|\Lambda^2}{\mu K^2}\right)^{\frac{1}{2}}\cos\theta\right\}, \tag{91}$$

which becomes the Newtonian solution as $A \to 0$ and $\Lambda \to L$ in the non-relativistic limit. Also in this limit $\tau \to t$, the ordinary time variable, and (91) becomes the equation for a conic section, in this instance an ellipse. The small corrections due to the special theory of relativity take account of the lack of simultaneity of events defined over the surface of equal proper time. S is the interbody distance in a frame where T is momentarily zero, while R is the actual distance when both are at proper time τ.

To evaluate the corrections to the Newtonian formula, we note the definition of $\boldsymbol{A}$ from (54) and calibrate the relative time to be zero when $\cos\theta$ is zero, that is,

$$T(\gamma = 0) = 0\,, \qquad \cos\theta = 0\,; \qquad \theta = \tfrac{1}{2}\pi, \tfrac{3}{2}\pi, \tfrac{5}{2}\pi, \ldots$$

and

$$A = |\boldsymbol{A}| = |(\epsilon/c)\boldsymbol{R} - \boldsymbol{q}cT| = (\epsilon_0/c)\,|\boldsymbol{R}_0|\,, \tag{92}$$

where ϵ_0 is ϵ when $\cos\theta = 0$, and $\boldsymbol{R}_0$ is evaluated at the same point.

If one of the bodies has a large mass m_1, then

$$\epsilon \approx \frac{m_1 m_2}{m_1+m_2}\left(1-\frac{1}{(1-v_0^2/c^2)^{\frac{1}{2}}}\right)c^2,$$

where $v_0 = |\boldsymbol{v}_2|$ at $\cos\theta = 0$. At low velocities

$$\epsilon \approx \mu\{1-(1-\tfrac{1}{2}v_0^2/c^2+\ldots)\}c^2 \approx \tfrac{1}{2}\mu v_0^2$$

and $A = \frac{1}{2}\mu(v_0^2/c)R_0$. However, at this point in the orbit

$$L \approx \mu v_0 R_0$$

and therefore

$$A/L \approx \tfrac{1}{2}v_0/c\,, \tag{93}$$

which shows clearly that $A \ll L$ and $\Lambda \approx L$. Using this approximation, we find

$$\frac{1}{R} = \frac{\mu K}{L^2}\left\{\left(1-\frac{v_0^2}{4c^2}\cos^2\theta\right)^{\frac{1}{2}} - e_1\cos\theta\right\}, \tag{94}$$

where $e_1 = (1-2\,|\mathscr{H}\,|\,\Lambda^2/\mu K^2)^{\frac{1}{2}}$ is the eccentricity. The corrections to (94) for motion in the solar system are therefore of order v_0^2/c^2 and are very small.

The gravitational interaction $K = Gm_1 m_2$ was used in (94) and gave the approximations

$$\Lambda/L = 4Gm_1 c^2/\{4Gm_1 c^2 + d_{\max}(1-e_1)v_0^4\}\,, \tag{95}$$

where $d_{\max}$, the maximum radius of orbit, is given by

$$d_{\max} = (1-e_1)^{-1}e_1 s\,, \tag{96}$$

TABLE 1

DEVIATIONS IN ORBITS AND TIME STANDARDS

θ (rad)	Radius (10^7 miles)	Deviation (miles)	Time variation (s)	θ (rad)	Radius (10^7 miles)	Deviation (miles)	Time variation (s)
		Mercury				Venus	
0·0	4·3355	−0·156	0·0197	0·0	6·7653	−0·115	0·021
0·314	4·2813	−0·136	0·0185	0·314	6·7653	−0·104	0·020
0·628	4·1313	−0·087	0·0152	0·628	6·7565	−0·075	0·017
0·943	3·9175	−0·036	0·0105	0·943	6·7463	−0·039	0·012
1·257	3·6777	−0·0045	0·0052	1·257	6·7335	−0·011	0·0065
1·571	3·4440	0·0	0·0	1·571	6·7194	0·0	0·0
1·885	3·2382	−0·0174	−0·0046	1·885	6·7053	−0·011	−0·0065
2·199	3·0726	−0·046	−0·0082	2·199	6·6927	−0·040	−0·012
2·513	2·9528	−0·075	−0·0109	2·513	6·6827	−0·075	−0·017
2·827	2·8807	−0·095	−0·0125	2·827	6·6763	−0·103	−0·020
3·142	2·8566	−0·103	−0·013	3·142	6·6740	−0·113	−0·022
		Earth				Mars	
0·0	9·4452	−0·116	0·0251	0·0	15·4476	−0·128	0·0338
0·314	9·4373	−0·105	0·0239	0·314	15·3983	−0·114	0·0320
0·628	9·4146	−0·075	0·0203	0·628	15·1775	−0·079	0·0268
0·943	9·3794	−0·039	0·0147	0·943	14·8457	−0·038	0·0191
1·257	9·3354	−0·010	0·0077	1·257	14·4478	−0·0082	0·0097
1·571	9·2872	0·0	0·0	1·571	14·0310	0·0	0·0
1·885	9·2394	−0·011	−0·0076	1·885	13·6375	−0·014	−0·0092
2·199	9·1967	−0·040	−0·014	2·199	13·3010	−0·042	−0·017
2·513	9·1632	−0·074	−0·020	2·513	13·0455	−0·073	−0·023
2·827	9·1417	−0·102	−0·023	2·827	12·8866	−0·097	−0·027
3·142	9·1343	−0·112	−0·024	3·142	12·8328	−0·106	−0·028
		Jupiter				Saturn	
0·0	50·6671	−0·121	0·059	0·0	93·5570	−0·121	0·081
0·628	50·1851	−0·076	0·048	0·628	92·5527	−0·077	0·065
1·257	48·9559	−0·0095	0·018	1·257	90·0229	−0·0094	0·024
1·885	47·5173	−0·012	−0·017	1·885	87·0807	−0·012	−0·023
2·513	46·4139	−0·074	−0·044	2·513	84·8376	−0·074	−0·059
3·142	46·0059	−0·109	−0·054	3·142	84·0109	−0·109	−0·073
		Uranus					
0·0	186·68	−0·12	0·114				
1·257	180·86	−0·010	0·033				
2·513	172·16	−0·074	−0·084				
3·142	170·76	−0·110	−0·10				
						Halley's Comet	
		Neptune		0·0	328·48	−13·0	1·57
0·0	281·74	−0·116	0·14	0·314	134·6	+1·02	0·61
1·257	279·40	−0·011	0·042	0·628	49·61	+1·15	0·19
2·513	275·70	−0·074	−0·108	0·943	25·01	+0·51	0·070
3·142	275·08	−0·113	−0·13	1·257	15·39	+0·17	0·023
				1·571	10·79	0·0	0·0
		Pluto		1·885	8·310	−0·10	0·012
0·0	460·0	−0·170	0·212	2·199	6·882	−0·16	−0·019
1·127	375·6	−0·003	0·05	2·513	6·056	−0·19	−0·023
2·513	289·5	−0·076	−0·108	2·827	5·622	−0·21	−0·026
3·142	278·6	−0·103	−0·129	3·142	5·487	−0·22	−0·026

s being the distance between the focus and the directrix. The major semi-axis is

$$a = \tfrac{1}{2}(d_{\max}+d_{\min}) = e_1 s/(1-e_1^2) \tag{97}$$

and for

$$\theta = \tfrac{1}{2}\pi, \tfrac{3}{2}\pi, \tfrac{5}{2}\pi, \ldots, \qquad a = e_1 s,$$

and hence

$$d_{\max} = (1-e_1)^{-1}a\,, \qquad d_{\min} = (1+e_1)^{-1}a\,.$$

However, the covariant Hamiltonian satisfies

$$a = K/2\,|\mathscr{H}| \tag{98}$$

and therefore from

$$\mathscr{H} = \mathscr{T}+\mathscr{V}\,,$$

$$-\frac{K}{2a}+\frac{K}{(1-e_1^2)a} = \mu c^2\left(\frac{1}{(1-v_0^2/c^2)^{\frac{1}{2}}}-1\right),$$

from which we obtain

$$v_0 = \left(1-\frac{1}{(1+P/\mu c^2)^2}\right)^{\frac{1}{2}}, \qquad P = \frac{K}{2a}\frac{1+e_1^2}{1-e_1^2}\,. \tag{99}$$

Data for $(e_1, d_{\max})$ were taken from the World Almanac (1964) for the nine major planets and Halley's comet and calculations were performed in double precision on the AAEC IBM 360/50H computer, giving an accuracy to 14 decimal places. A test hyperbola was also chosen for a comet which travels very close to the Sun and escapes, a motion which accentuates relativistic corrections. The results for the deviations in orbit and relative time variations are shown in Table 1.

It is quite clear from these results that there is no body within the solar system which moves sufficiently fast for a detectable deviation to occur from the Newtonian orbit, on the basis of published measurements. Also the length of the sidereal year remains the same, as the equation for $S(\tau)$ is the same as the non-relativistic form for $R(t)$.

Another point is that it follows from equations (90) that after one full period in θ, the quantities (γ, S) return to their initial values, and so therefore does $T = (S/c)\sinh\gamma$. Hence a space ship which is launched on a highly elongated elliptical orbit and allowed to coast will return to its starting point in such a way that the relative time returns to its initial value. In this case, there is no clock paradox. More generally, such a paradox exists in the proper time theory only if the system is non-conservative, and T is not a true periodic function of τ.

An extension of the models discussed in this paper to a relativistic quantum theory is fairly straightforward and details of work in this direction are presented in Part II (Cook 1972, present issue pp. 141–65).

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