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Mode Coupling in the Solar Corona. II* Oblique Incidence

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Abstract

Previous discussions of mode coupling at a QT region have assumed vertical incidence and have thus invoked magnetic structures which violate $\mathrm{div}\,\boldsymbol{B} = 0$. A new method is developed here for calculating the coupling coefficients for oblique incidence so that coupling at a QT region can be treated without invoking nonphysical magnetic structures. The method involves solving the Booker quartic equation implicitly in terms of the familiar formulae of magnetoionic theory. A coupling approximation is introduced which involves one step in an iterative procedure to find explicit solutions from the implicit ones. The approximation is necessarily valid in a finite range about the critical coupling points. The present method is used to generalize the results of Cohen to allow oblique incidence. The results of the existing discussions of mode coupling for vertical incidence and nonphysical magnetic structures can be justified both qualitatively and semiquantitatively (although with a slightly different physical interpretation).

1. Introduction

In the preceding Part I (Melrose 1974, present issue pp. 31–42) mode coupling in the solar corona was discussed for waves vertically incident on a slowly-varying stratified medium. In this paper the assumption of vertical incidence is relaxed, i.e. the waves are allowed to be obliquely incident on the stratified medium.

Existing treatments of coupling caused by spatial variations in the magnetic field are nonphysical in that they violate $\mathrm{div}\,\boldsymbol{B} = 0$. For stratification along the z axis $\mathrm{div}\,\boldsymbol{B} = 0$ implies $B_z' = 0$ and so $B_z = \text{const.}$, although B_x and B_y can be functions of z. (The notation is that used in Part I.) To treat coupling at a QT region assuming vertical incidence, Cohen (1960) and Zhelezniakov and Zlotnik (1963) had to assume impossible magnetic structures; in particular they assumed that B_z changes sign as a function of z. Zhelezniakov (1970, p. 354) suggested that one could satisfy $\mathrm{div}\,\boldsymbol{B} = 0$ by allowing $\boldsymbol{B}$ to be a function of two of the coordinates. An alternative way is to allow oblique incidence, since the sign of the component of $\boldsymbol{B}$ along the wave vector can then change as a function of z. This is the simplest way of treating coupling at a QT region without violating $\mathrm{div}\,\boldsymbol{B} = 0$.

The assumed magnetic structures also imply $\mathrm{curl}\,\boldsymbol{B} \neq 0$, and the associated current affects the coupling between hydromagnetic waves (Frisch 1964). However, its implicit neglect in existing treatments of the coupling between magnetoionic waves can be justified provided that the required average velocity of the electrons is much less than the phase velocity of the magnetoionic waves.

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The coupling matrix in the general case of oblique incidence was calculated explicitly by Budden and Clemmow (1957) (see also Budden 1961, Ch. 18). A new approximate method for calculating the coupling matrix is developed in Section 2 below. This method leads to a direct generalization of the formulae derived for vertical incidence in Part I.

2. New Approximate Method for Oblique Incidence

Method

The general approach described in Section 2 of Part I involves calculating a coupling matrix $\mathbf{\Gamma}$ or $\boldsymbol{\gamma}$ by solving the Booker quartic equation for the four roots, q_i with $i = 1,...,4$ say; one then finds the four characteristic column matrices $\mathbf{v}_i$ and constructs the matrix $\mathbf{R}$ and so $\mathbf{\Gamma}$ and $\boldsymbol{\gamma}$ using equations (6) and (8) of Part I. This was done by Budden and Clemmow (1957), whose results are expressed in terms of the q_i, the elements of the matrix $\mathbf{T}$ and their derivatives. The expressions are so cumbersome that they do not seem to have been applied to other than the special cases considered by Budden and Clemmow and by Swift (1962). Inoue and Horowitz (1966*a*, 1966*b*) carried out the calculation in a slightly different way and presented some numerical results appropriate for mode coupling in the ionosphere.

We shall use the well-known formulae of magnetoionic theory to write down implicit expressions for the q_i and $\mathbf{v}_i$. It will then be argued that whenever coupling is important one is justified in using an iterative method to find approximate explicit solutions. These approximate solutions lead to formulae which are manifest generalizations of those for vertical incidence. The basic idea behind this alternative procedure is that the solutions for q_i and $\mathbf{v}_i$ are physically equivalent to the solutions for the refractive indices and polarization vectors of magnetoionic theory. The two sets of solutions, which involve different choices of variables, can be obtained from each other by appropriate change of variables, i.e. they are related by a conformal transformation. In standard magnetoionic theory the independent variables are ω and θ (θ being the angle between $\boldsymbol{k}$ and $\boldsymbol{B}$) while the dependent variable is the refractive index μ. Let us define our labelling of the modes by writing the four solutions

$$\mu = \mu_i(\omega, \theta), \qquad i = 1,...,4,$$

as

$$\mu_1(\omega,\theta) = -\mu_3(\omega,\theta) = \mu_o(\omega,\theta), \qquad \mu_2(\omega,\theta) = -\mu_4(\omega,\theta) = \mu_x(\omega,\theta), \tag{1}$$

where o and x refer to the ordinary and extraordinary modes.

On the other hand, the independent variables in the Booker quartic equation are ω, in the relation

$$r = (k_x^2 + k_y^2)^{\frac{1}{2}}\, c/\omega, \tag{2}$$

and two angles, while the dependent variable is

$$q = k_z\, c/\omega. \tag{3}$$

If the direction of the magnetic field is written as

$$\boldsymbol{b} = (\sin\psi\cos\phi,\ \sin\psi\sin\phi,\ \cos\psi) \tag{4}$$

and the coordinate axes are chosen such that k_y vanishes, the two angular variables may be chosen as ψ and ϕ. The angle θ is related to the other variables by

$$\cos\theta = (q\cos\psi + r\sin\psi\cos\phi)/(q^2+r^2)^{\frac{1}{2}}. \tag{5}$$

The solutions of the Booker quartic may be written as

$$q = q_i(\omega, r, \psi, \phi), \tag{6}$$

the implicit expressions for the q_i being

$$q_1 = \{\mu_o^2(\omega,\theta_1)-r^2\}^{\frac{1}{2}}, \qquad q_2 = \{\mu_x^2(\omega,\theta_2)-r^2\}^{\frac{1}{2}}, \tag{7a, b}$$

$$q_3 = -\{\mu_o^2(\omega,\theta_3)-r^2\}^{\frac{1}{2}}, \qquad q_4 = -\{\mu_x^2(\omega,\theta_4)-r^2\}^{\frac{1}{2}}, \tag{7c, d}$$

where

$$\theta_i = \theta(q_i, r, \psi, \phi) \tag{8}$$

is the function obtained by substituting $q = q_i$ in equation (5).

Characteristic Matrices

To construct the characteristic matrices $\mathbf{v}_i$ the ratios $E_x : -E_y : B_x : B_y$ are required for each of the four modes. Maxwell's equations imply

$$B_x : B_y = -qE_y : (qE_x - rE_z). \tag{9}$$

The three components of the electric vector are just the three components of the polarization vector; e.g. in the notation used by Melrose and Sy (1972, Appendix I) one has

$$\boldsymbol{E} \propto (K\boldsymbol{\kappa} + T\boldsymbol{\tau} + \mathrm{i}\boldsymbol{a}). \tag{10}$$

With the present choice of coordinate axes the vectors $\boldsymbol{\kappa}$, $\boldsymbol{\tau}$ and $\boldsymbol{a}$, as defined by Melrose and Sy, take on the following explicit forms:

$$\boldsymbol{\kappa} = (r, 0, q)/(r^2+q^2)^{\frac{1}{2}}, \tag{11a}$$

$$\boldsymbol{\tau} = (-\alpha q,\ -(r^2+q^2)^{\frac{1}{2}}\beta,\ \alpha r)/(\alpha^2+\beta^2)^{\frac{1}{2}}, \tag{11b}$$

$$\boldsymbol{a} = (\beta q,\ -(r^2+q^2)^{\frac{1}{2}}\alpha,\ -\beta r)/(\alpha^2+\beta^2)^{\frac{1}{2}}, \tag{11c}$$

where the quantities

$$\alpha = (q\sin\psi\cos\phi - r\cos\psi)/(q^2+r^2)^{\frac{1}{2}}, \qquad \beta = \sin\psi\sin\phi \tag{12}$$

have been introduced for simplicity in writing. From equations (5) and (12),

$$\alpha^2+\beta^2 = \sin^2\theta. \tag{13}$$

Thus we find

$$E_x : E_y : E_z$$
$$= \{-q(\alpha T - \mathrm{i}\beta) + rK\sin\theta\} : -(r^2+q^2)^{\frac{1}{2}}(\beta T + \mathrm{i}\alpha) : \{r(\alpha T - \mathrm{i}\beta) + qK\sin\theta\}. \tag{14}$$

Let us write

$$R = \frac{\alpha T - \mathrm{i}\beta}{\beta T + \mathrm{i}\alpha}, \qquad P = -\frac{K\sin\theta}{\beta + T\mathrm{i}\alpha}. \tag{15}$$

The quantity R is an appropriate generalization of R_σ introduced in equation (13) of Part I; for vertical incidence the two quantities are the same for $q > 0$ (see comments following equation (14) of Part I in connection with the relation between upgoing and downgoing waves). The quantity P is an additional term which involves the longitudinal electric field, i.e. the component of $\boldsymbol{E}$ along $\boldsymbol{k}$; P does not appear in the ratio $E_x : E_y$ for vertical incidence because one then has $r = 0$ in equation (14).

The characteristic matrices are found by substituting $q = q_i$ in

$$\mathbf{v} = \begin{bmatrix} (qR+rP)/(q^2+r^2)^{\frac{1}{2}} \\ -1 \\ -q \\ (q^2+r^2)^{\frac{1}{2}}R \end{bmatrix}. \tag{16}$$

In the matrix (16) R and P remain implicit functions of q through the dependence of T and K on θ and therefore on q through equation (5).

Coupling Approximation

Coupling tends to be important near the critical coupling points where the two appropriate roots of the Booker quartic equation are equal (see e.g. Budden 1961, p. 413). Consequently, in treating coupling between modes 1 and 2, say, it is reasonable to set $q_1 = q_2$ as a first approximation and then to iterate using equations (5) and (7) to find corrections to any desired order. (Because r, which is constant according to Snell's law, is one of the independent variables, $r_1 = r_2$ is implicit.) To lowest order it follows from equation (5) that the two functions θ_1 and θ_2 obtained by setting $q = q_1$ and $q = q_2$ are the same, and thus to this approximation θ could be chosen as one of the independent variables. This leads to great simplification from the cumbersome implicit independences in the general case.

Let the *coupling approximation* be defined as that in which one sets $q_1 = q_2$ to a first approximation and thereby ensures that the functions θ_1 and θ_2 are the same to first approximation. The only quantity required in the next approximation is $q_1 - q_2$; this follows from equations (7a, b) with $\theta_1 = \theta_2$. The coupling approximation must be valid for some range of the parameters centred on the values at which critical coupling obtains. To a first approximation one may write

$$\bar{\mu}(\omega,\theta) = \tfrac{1}{2}\{\mu_o(\omega,\theta)+\mu_x(\omega,\theta)\} \tag{17}$$

and

$$q = \bar{\mu}\cos\rho\,, \qquad r = \bar{\mu}\sin\rho\,, \tag{18}$$

where ρ is the angle between $\boldsymbol{k}$ and the z axis. The angle θ follows from equation (5):

$$\cos\theta = \cos\rho\cos\psi + \sin\rho\sin\psi\cos\phi\,. \tag{19}$$

By implication θ is a function of ρ, ψ and ϕ. The difference $q_1 - q_2$ follows from equations (7):

$$q_1 - q_2 = \frac{q_1^2 - q_2^2}{q_1 + q_2} = \frac{\mu_o^2(\omega,\theta) - \mu_x^2(\omega,\theta)}{2\bar{\mu}(\omega,\theta)\cos\rho} = \frac{\mu_o(\omega,\theta) - \mu_x(\omega,\theta)}{\cos\rho}. \tag{20}$$

The quantity α (from equations (12)) reduces to

$$\alpha = \cos\rho\sin\psi\cos\phi - \sin\rho\cos\psi = -(\partial\theta/\partial\rho)\sin\theta\,. \tag{21}$$

Because the difference between q_1 and q_2 no longer appears to first approximation in α or in θ ($T(\omega,\theta)$ is a function of θ) the identities

$$R_1 R_2^* = -1 = R_1^* R_2, \qquad R_3 R_4^* = -1 = R_3^* R_4 \tag{22}$$

hold. However, unlike the case of vertical incidence where $R_1 = -R_3 = R_o$ and $R_2 = -R_4 = R_x$, there is no simple relation between R_1 and R_3 or between R_2 and R_4. To see this, set $\cos\rho = \pm|\cos\rho|$ in equation (19) to define angles θ_+ (θ_-) between upgoing (downgoing) waves and the z axis:

$$\cos\theta_\pm = \pm|\cos\rho|\cos\psi + \sin\rho\sin\psi\cos\phi\,. \tag{23}$$

For vertical incidence one has $\theta_+ + \theta_- = \pi$ and so $R_1 = -R_3$ and $R_2 = -R_4$. Except for

$$\cos\rho = 0,\quad \sin\rho = 0,\quad \cos\psi = 0,\quad \sin\psi = 0,\quad \cos\phi = 0 \tag{24}$$

there is no simple relation between θ_+ and θ_-. In general one has $|T_\sigma(\omega,\theta_+)| \neq |T_\sigma(\omega,\theta_-)|$ and $|\alpha_+| \neq |\alpha_-|$, where

$$\alpha_\pm = -(\partial\theta_\pm/\partial\rho)\sin\theta_\pm\,, \tag{25}$$

i.e. there is no simple relation between R_1 and R_3 or R_2 and R_4.

3. Coupling Matrices and Coupling Ratio

In this section the method developed in Section 2 is applied to generalize the results of Part I to the case of oblique incidence.

Coupling Matrix

When the characteristic matrices (16) are used to write out the matrix $\mathbf{R}$ (cf. equation (11) of Part I) the resulting expression for $\mathbf{R}^{-1}\mathbf{R}'$ remains cumbersome even when the coupling approximation discussed in Section 2 is used. Two further approximations can often be justified.

One approximation is to set $\mu_o = \mu_x = \bar{\mu}$ in the matrix $\mathbf{R}$. From the results of Part I the only significant loss of generality which would follow from setting $\mu_o = \mu_x$ in $\mathbf{R}$ is that the elements Γ_{14}, Γ_{41}, Γ_{23} and Γ_{32} of the coupling matrix would vanish. These elements describe coupling between o$\uparrow$ and x$\downarrow$ etc., i.e. between unlike oppositely-travelling modes. Provided one is not concerned specifically with such coupling it is reasonable to set $\mu_o = \mu_x$ from the outset.

The second additional approximation is to neglect the terms involving the P_i. The finiteness of the P_i depends on the finiteness of the longitudinal part of the polarization (described by K_σ). The electric field has a significant longitudinal component at frequencies near the resonant frequencies which satisfy

$$1 - X - Y^2 + XY^2\cos^2\theta = 0\,, \tag{26}$$

but the polarization approaches transverse polarization at frequencies far removed from these resonant frequencies. In particular for $f \gg f_p, f_H$ both K_o and K_x are of the same order as $\mu_o - \mu_x$ (see equations (46a) below).

If both of the above approximations are made the matrix **R** becomes

$$\mathbf{R} = \begin{bmatrix} R_1\cos\rho & R_2\cos\rho & -R_3\cos\rho & -R_4\cos\rho \\ -1 & -1 & -1 & -1 \\ -\bar{\mu}\cos\rho & -\bar{\mu}\cos\rho & \bar{\mu}\cos\rho & \bar{\mu}\cos\rho \\ \bar{\mu}R_1 & \bar{\mu}R_2 & \bar{\mu}R_3 & \bar{\mu}R_4 \end{bmatrix}. \tag{27}$$

The elements of the coupling matrix Γ can then be found. The results are:

$$\Gamma_{12} = \frac{R_2'}{R_2-R_1}, \qquad \Gamma_{21} = \frac{R_1'}{R_1-R_2}, \tag{28a$_{1,2}$}$$

$$\Gamma_{34} = \frac{R_4'}{R_4-R_3}, \qquad \Gamma_{43} = \frac{R_3'}{R_3-R_4}; \tag{28a$_{3,4}$}$$

$$\Gamma_{14} = \frac{(\cos\rho)'}{2\cos\rho}\frac{R_4-R_2}{R_1-R_2} + \frac{\bar{\mu}'}{2\bar{\mu}}\frac{R_2+R_4}{R_1-R_2}, \tag{28b$_1$}$$

$$\Gamma_{41} = \frac{(\cos\rho)'}{2\cos\rho}\frac{R_3-R_1}{R_3-R_4} + \frac{\bar{\mu}'}{2\bar{\mu}}\frac{R_3+R_1}{R_3-R_4}, \tag{28b$_2$}$$

$$\Gamma_{23} = \frac{(\cos\rho)'}{2\cos\rho}\frac{R_1-R_3}{R_1-R_2} + \frac{\bar{\mu}'}{2\bar{\mu}}\frac{R_1+R_3}{R_1-R_2}, \tag{28b$_3$}$$

$$\Gamma_{32} = \frac{(\cos\rho)'}{2\cos\rho}\frac{R_2-R_4}{R_3-R_4} - \frac{\bar{\mu}'}{2\bar{\mu}}\frac{R_3+R_2}{R_3-R_4}; \tag{28b$_4$}$$

$$\Gamma_{13} = \frac{(\cos\rho)'}{2\cos\rho}\frac{R_3-R_2}{R_1-R_2} - \frac{\bar{\mu}'}{2\bar{\mu}}\frac{R_2+R_3}{R_1-R_2}, \tag{28c$_1$}$$

$$\Gamma_{31} = \frac{(\cos\rho)'}{2\cos\rho}\frac{R_1-R_4}{R_3-R_4} - \frac{\bar{\mu}'}{2\bar{\mu}}\frac{R_1+R_4}{R_3-R_4}, \tag{28c$_2$}$$

$$\Gamma_{24} = \frac{(\cos\rho)'}{2\cos\rho}\frac{R_4-R_2}{R_1-R_2} + \frac{\bar{\mu}'}{2\bar{\mu}}\frac{R_1+R_4}{R_1-R_2}, \tag{28c$_3$}$$

$$\Gamma_{42} = \frac{(\cos\rho)'}{2\cos\rho}\frac{R_3-R_2}{R_3-R_4} + \frac{\bar{\mu}'}{2\bar{\mu}}\frac{R_3+R_2}{R_3-R_4}. \tag{28c$_4$}$$

For vertical incidence one has

$$|\cos\rho| = 1, \qquad R_1 = -R_3 = R_o, \qquad R_2 = -R_4 = R_x. \tag{29}$$

Equations (28a), (28b) and (28c) then reduce respectively to equations (15a), (15b) and (15c) of Part I when one sets $\mu_o = \mu_x = \bar{\mu}$ in accord with the approximations made in deriving equations (28).

The mode coupling of interest is that between o↑ and x↑ or o↓ and x↓. The elements (28a) are the relevant ones in this connection. On comparing equations (28a) with (15a) of Part I it can be seen that the generalization to oblique incidence turns out to be relatively simple for this mode coupling.

When T_σ, K_σ and μ_σ are written as functions of X, Y and θ, and θ is written as a function of ρ, ψ and ϕ using equation (23), derivatives with respect to z of the following variables remain: X, Y, ρ, ψ and ϕ. Two of these derivatives can be eliminated using the results

$$\bar{\mu}\sin\rho = \text{const.} \qquad \text{and} \qquad Y\cos\psi = \text{const.} \tag{30a, b}$$

Equation (30a) follows from r = const.; it is an expression of Snell's law. Equation (30b) follows from $\operatorname{div}\boldsymbol{B} = 0$.

With the result (30a) the elements (28b) and (28c) of $\boldsymbol{\Gamma}$ involve only one derivative, ρ' say. The coupling so described can therefore be attributed to refraction, i.e. to changes in the angle ρ between $\boldsymbol{k}$ and the z axis. For example, refraction of an o↑ wave causes generation of some x↑, some o↓ and some x↓ waves. For vertical incidence no such effect occurs because there is no refraction; for vertical incidence coupling between o↑ and x↓ or x↑ and o↓ depends on the finiteness of $\mu_{\mathrm{o}}-\mu_{\mathrm{x}}$. An analogous coupling depending on $\mu_{\mathrm{o}}-\mu_{\mathrm{x}}$ exists for oblique incidence but it is not included in equations (28c) because the difference between μ_{o} and μ_{x} has been ignored.

It might be commented that the coupling approximation, when invoked to treat coupling between modes 1 and 2, may not be a valid approximation in treating coupling between modes 1 and 3, 1 and 4, etc. However, the coupling described by equations (28b) and (28c) does exist and is probably described adequately by these equations. It can be shown that the coupling approximation remains valid over a wide range of parameters whenever the refractive indices vary only slowly with θ. For $f \gg f_{\mathrm{p}}, f_H$ the coupling approximation should be valid for nearly all purposes.

The existence of coupling between say o↑ and x↓ is implied by the more familiar example of reflection and transmission at a sharp boundary: waves in a single mode incident on a sharp boundary between two different anisotropic media lead to two refracted and two transmitted waves in general (see e.g. Eckersley 1950). In ionospheric sounding this type of coupling is said to produce a 'coupling echo' (see e.g. Budden 1961, Section 19.9).

Coupling Ratio

For oblique incidence the coupling between upgoing waves, i.e. between o↑ and x↑, is different from the coupling between downgoing modes, o↓ and x↓. However, the difference arises only from the fact that upgoing and downgoing waves with the same value of $\sin\rho$ (ρ is the angle between $\boldsymbol{k}$ and the z axis) are propagating at different angles to the magnetic field, i.e. the $\theta_{\pm}$, as given by equation (23), are different. Only coupling between upgoing waves, i.e. the modes labelled 1 and 2, is considered here.

The generalizations of equations (17), (20), (21), (22) and (23) of Part I are as follows (the difference between μ_{o} and μ_{x} is ignored except where it appears explicitly):

$$d_1 = \bar{\mu}^{\frac{1}{2}}\exp\left(\int\frac{R_1'\,dz}{R_1-R_2}\right), \qquad d_2 = \bar{\mu}^{\frac{1}{2}}\exp\left(\int\frac{R_2'\,dz}{R_2-R_1}\right); \tag{31}$$

$$\chi_1 \equiv \psi_1 = \frac{R_1'}{R_1-R_2}\exp\left(\int dz\,\frac{R_2'+R_1'}{R_2-R_1}\right), \tag{32a}$$

$$\chi_2 \equiv \psi_2 = \frac{R_2'}{R_2-R_1}\exp\left(\int dz\,\frac{R_1'+R_2'}{R_1-R_2}\right); \tag{32b}$$

$$\chi_1^* = -\chi_2; \tag{33}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \exp\left((\mathrm{i}\omega/c)\int \mathrm{d}z\,\bar{\mu}\cos\rho\right); \tag{34}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} -(\mathrm{i}\omega/2c)\Delta q & \chi_2 \\ \chi_1 & (\mathrm{i}\omega/2c)\Delta q \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{35}$$

with

$$\Delta q = q_1 - q_2 = (\mu_o - \mu_x)/\cos\rho = \Delta\mu/\cos\rho \tag{36}$$

from equation (20). Thus the generalization of the coupling ratio Q becomes

$$Q = |\chi_1\chi_2/(\omega\,\Delta q/2c)^2|^{\frac{1}{2}}. \tag{37}$$

Explicit evaluation of Q proceeds as follows. In the equation

$$Q = \frac{2c}{\omega}\frac{|\cos\rho|}{|\Delta\mu|}\left(\frac{(\alpha'\beta-\beta'\alpha)(T^2-1)+\mathrm{i}(\alpha^2+\beta^2)T'}{(\alpha^2+\beta^2)(1+T^2)}\right), \tag{38}$$

one has

$$\frac{\alpha'\beta-\beta'\alpha}{\alpha^2+\beta^2} = \frac{1}{\sin\theta}\left\{\left(\psi'-\rho'\frac{\cos\theta}{\sin\rho}\right)\frac{\partial\theta}{\partial\phi} - \phi'\frac{\partial\theta}{\partial\psi}\sin\psi\right\}. \tag{39}$$

From Appendix 2 of Part I, equation (A9) shows

$$T' = \frac{T(T^2-1)}{T^2+1}\left(\frac{Y'}{Y}+\frac{X'}{1-X}+(2\cot\theta+\tan\theta)\theta'\right), \tag{40}$$

with

$$\theta' = \rho'\frac{\partial\theta}{\partial\rho}+\psi'\frac{\partial\theta}{\partial\psi}+\phi'\frac{\partial\theta}{\partial\phi}. \tag{41}$$

Also from Appendix 2 of Part I, equation (A5) gives

$$|\Delta\mu| = \frac{T^2+1}{2|T^2-1|}\frac{XY^2\sin^2\theta}{(1-X)^{3/2}}, \tag{42}$$

where $\bar{\mu}^2 = 1-X$ and $Y^2 \ll 1-X$ are assumed. Writing the result in the form of Part I, equation (29) with the approximation (30), gives

$$Q = \frac{4c}{\omega}\frac{|\cos\rho|(1-X)^{3/2}}{XY^2\sin^2\theta}\left\{(\tilde{\phi}')^2+\frac{T^2}{(T^2+1)^2}\left(\frac{Y'}{Y}+\frac{X'}{1-X}+(2\cot\theta+\tan\theta)\theta'\right)^2\right\}^{\frac{1}{2}}, \tag{43}$$

with

$$\tilde{\phi}' = \frac{1}{\sin\theta}\left\{\left(\psi'-\rho'\frac{\cos\theta}{\sin\rho}\right)\frac{\partial\theta}{\partial\phi}-\phi'\frac{\partial\theta}{\partial\psi}\sin\psi\right\}. \tag{44}$$

For vertical incidence one has $\rho = 0$ and $\theta = \psi$, in which case equation (44) reduces to $\tilde{\phi}' = \phi'$. In equation (43), θ' is to be re-expressed in terms of ρ', ψ' and ϕ' using (41). Furthermore, from equation (30a) with $\bar{\mu}^2 = 1-X$ and (30b) one finds

$$X'/(1-X) = 2\rho'\cot\rho, \qquad Y'/Y = \psi'\tan\psi, \tag{45}$$

that is, only the derivatives ρ', ψ' and ϕ' need appear in equation (43).

4. Discussion

In this section the application of the above results to mode coupling in the solar corona is discussed and then other possible uses of the method developed in Section 2 are considered.

Mode Coupling in the Corona

The detailed treatment of mode coupling for oblique incidence leads to the important but rather unproductive result that previous discussions based on the theory for vertical incidence were qualitatively and semiquantitatively correct despite the fact that nonphysical magnetic field structures had been invoked.

For vertical incidence only two directions appear in the problem: the direction of wave propagation (which is assumed to be parallel to the direction $\boldsymbol{n}$ say of the gradients of all field quantities) and the magnetic field $\boldsymbol{B}$. The important coupling results from the rate of change of the angle θ between $\boldsymbol{k}$ and $\boldsymbol{B}$ when $\theta \approx \frac{1}{2}\pi$; the assumption that θ passes through $\frac{1}{2}\pi$ is the nonphysical assumption. Twists of the plane containing $\boldsymbol{k}$ and $\boldsymbol{B}$ also contribute to mode coupling.

In the case of oblique incidence the three directions of $\boldsymbol{k}$, $\boldsymbol{B}$ and $\boldsymbol{n}$ are all different. Furthermore, unlike the case of vertical incidence, the direction of $\boldsymbol{k}$ changes as a result of refraction. Again the important coupling occurs for $\theta \approx \frac{1}{2}\pi$ but θ may pass through $\frac{1}{2}\pi$ either because of a change in the direction of $\boldsymbol{B}$ (without violating $\operatorname{div}\boldsymbol{B} = 0$) or because of a change in the direction of $\boldsymbol{k}$ caused by refraction.

Another effect which appears in the general case of oblique incidence but not for vertical incidence is the dependence of the coupling on the longitudinal part of the electric field in the waves. This has been ignored by omitting the term P in the matrix (16). The neglect of P can be justified as follows. In a QT region one has

$$K_o \approx \frac{XY\sin\theta}{1-X}, \qquad K_x \approx \frac{XY\sin\theta}{1-X-Y^2\sin^2\theta}. \tag{46a}$$

For the x-mode not to be evanescent requires $1-X > Y$. For $Y \ll 1-X$ equations (46a) imply

$$K_o \approx K_x \approx \{XY/(1-X)\}\sin\theta. \tag{46b}$$

Thus for $1-X \gg Y$, K_σ is of the same order as $\mu_o-\mu_x$. Because $\mu_o-\mu_x$ is neglected, the neglect of the term involving K_σ is justified for $1-X \gg Y$. However, very close to the cutoff frequency for the x-mode the additional term involving P in equation (16) may contribute significantly.

Other Applications

Oblique incidence is thought to be important in the generation of the Z-trace in the ionosphere (Ellis 1953, 1956). In the earlier theories of this process the critical coupling points were found and the phase integral method assuming vertical incidence was applied (see Budden 1961, Sections 19.6, 19.7; and references quoted by Ginzburg 1964, Section 29; Rawer and Suchy 1967, Sections 11, θ) and ι)). More recently this problem has been discussed by Budden and Terry (1971) and by Smith (1973) using numerical methods. In principle the method developed in Section 2 could be applied to treat this coupling. However, the neglect of the longitudinal part of the electric field would not be justified in this case, because the coupling occurs at $X \approx 1$, nor would the neglect of collisions be valid.

The same coupling as that invoked in the generation of the Z-trace has been suggested in connection with solar radio emission by Ginzburg and Zhelezniakov (1959) and in a modified form by Mollwo (1969). This involves coupling, at the level where $f \approx f_p$, of waves in the extraordinary mode into o-mode waves which escape: the extraordinary mode waves could be generated as electron plasma waves which propagate into a region of higher f_p so that the refractive index decreases. Again one needs to assume oblique incidence to treat this coupling in the general case.

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