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Series Analysis of a Three-state Potts Model with Three-site Interactions

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Abstract

Series expansions are used to investigate a three-state Potts model which has two-site interactions on the bonds of a triangular lattice and three-site interactions on alternate triangles. Although some of the exponent estimates show variations with interaction strengths, the assumption of scaling constrains the permitted exponent values to such an extent that the possible variations lie within the ranges of uncertainty of individual exponent estimates.

1. Introduction

In a recent paper Baxter *et al.* (1978) have investigated a triangular lattice Potts model in which both two-site and three-site interactions occur. Their model has three-site interactions on only half the triangles (those of '+' parity) and this form of interaction is such that it distinguishes only between those triangles for which all sites are in the same state and those with at least one pair of unlike states. This generalized model is of interest because it appears to have the same symmetry as the conventional Potts model and so should provide an important test of universality. It lies outside the class of n-vector models for which the most extensive tests of universality have been made (Stanley 1974). While renormalization group techniques have provided a framework for understanding n-vector models (see e.g. Fisher 1974; Wilson and Kogut 1974), attempts to apply such techniques to Potts models have encountered many difficulties. Typically, real-space renormalizations fail to predict first-order transitions in cases where they are expected and reciprocal-space renormalizations fail to predict expected continuous transitions (Amit and Shcherbakov 1974; Burkhardt *et al.* 1976; Dasgupta 1977). The present paper uses series expansion analysis to test the three-state generalized Potts model to see whether universality is valid.

In Section 2 below a graphical derivation of a duality relation is presented. This same relation (in the thermodynamic limit) was obtained by Baxter *et al.* (1978) using operator transformations but the graphical derivation is included here because:

(1) It provides a starting point for describing the graphs used in Section 3.

(2) It is exact for finite lattices and so would be important for describing series on finite lattices. Such series form the basis of the expansion technique of de Neef and Enting (1977).

(3) Recent work by Hilhorst *et al.* (1978) has shown how such a connection between finite lattices of slightly different size can be used to generate *exact* real-space differential renormalization group transformations. In the generalized Potts model this technique, at best, would be only valid at the critical point but this may be sufficient to determine the exponent η.

Once one has a duality relation between the high temperature and low temperature regimes the critical temperature can be located so long as one assumes that it is unique. A knowledge of the critical point is of considerable assistance in obtaining accurate exponent estimates from series expansions. Section 3 shows how series expansion for the three-state generalized Potts model can be obtained from the data given by Enting (1974*a*, 1974*b*).

The series analysis is given in Section 4. Initially the series are analysed without assuming a knowledge of the critical point so as to check that there is in fact only one critical point; lattice models with two critical points are not unknown (Wu and Lin 1974; Wu 1977). The series analysis for the generalized Potts model indicates that the critical points coincide with the self-dual points and so in Section 4 the exponent estimates are based on the assumption that the critical points are known exactly. The raw exponent estimates show a small variation as the strength of the three-site interaction is changed. The amount of variation in exponents is less than the 'spread' in the published estimates of exponents for Potts models with purely two-site interactions.

A stronger reason for regarding the apparent variation in the exponent estimates as spurious is that if three-exponent scaling is assumed then the variations in any two of the exponents α', β, γ' or δ will determine the variations in the other two. In practice the variations observed are not consistent with three-exponent scaling. The analysis in terms of scaling is given in detail in Section 5.

2. Mixed Model and Duality Relation

The model is defined on a triangular lattice and has a q-state variable $t_i = 0, 1, ..., q-1$ at each site i. The energy E of any configuration of the variables t_i is given by

$$E = -H\sum_i \delta(t_i, 0) - J\sum_{<i,j>} \delta(t_i, t_j) - K\sum_{<i,j,k>} \delta(t_i, t_j, t_k). \tag{1}$$

The first sum is over all sites, the second over all bonds and the third over all '+' triangles. The functions $\delta(a,b)$ and $\delta(a,b,c)$ are unity if all arguments are equal, and are zero otherwise.

The natural series expansion variables are u, w and μ where

$$u = \exp(-\beta J) = (1+f)^{-1}, \quad w = \exp(-\beta K) = (1+g)^{-1}, \quad \mu = \exp(-\beta H). \tag{2}$$

The variables f and g are those used by Baxter *et al.* (1978); they considered only the $H = 0$ case. For a system with N lattice sites one writes

$$Z_N = \sum_{\mathrm{C}} \exp(-\beta E), \tag{3}$$

where the C-summation is over all configurations, and for $H = 0$ one has, in the limit of large N, the duality relation

$$Z_N(f^*, g^*) = (q/y)^N Z_N(f, g) \tag{4a}$$

or, if $F = \lim_{N\to\infty} N^{-1} Z_N$,

$$F(f, y) = F(f^*, y^*) + \ln y - \ln q, \tag{4b}$$

where

$$y = g(1+f)^3 + f^3 + 3f^2 \tag{5}$$

and the duality transformation is given by

$$y^* = q^2/y, \qquad f^* = qf/y. \tag{6a, b}$$

The model is thus self-dual when

$$y_c = q. \tag{7}$$

Differentiating equation (4b) with respect to y and then setting $y = q$ one obtains

$$2F_y(f,q) + fq^{-1}F_f(f,q) = q^{-1}. \tag{8}$$

In this model we cannot solve for F_y and F_f because the relation obtained by differentiating equation (4b) with respect to f reduces to an identity at $y = q$. This means that we cannot obtain $\partial F/\partial(\beta K)$ and $\partial F/\partial(\beta J)$ independently and so cannot determine the critical value of the energy. The linear combination in equation (8) can be used, however, to define an energy-like quantity R where, at T_c,

$$R_c = f\frac{\partial F}{\partial(\beta J)} - \frac{fq+6f^2-2q}{q+3f+1}\frac{\partial F}{\partial(\beta K)} = 1+f \tag{9}$$

and for general temperatures

$$R = f_c\frac{\partial F}{\partial(\beta J)} - \frac{f_c q+6f_c^2-2q}{q+3f_c+1}\frac{\partial F}{\partial(\beta K)}. \tag{10}$$

The graphical derivation of the duality relations (4) and (6) uses three lattices, I, II and III. Lattice I is a triangular region of a triangular lattice and has $\frac{1}{2}n(n+1)$ vertices. Lattice II is obtained by taking the vertices of lattice I (denoted as the Y sites) and adding $\frac{1}{2}n(n-1)$ additional sites (X sites) at the centres of the '+' triangles of lattice I. Connection of X sites to neighbouring Y sites gives a honeycomb lattice. For pictorial convenience, dummy unfinished edges are added around the boundary so that all vertices are of degree three. Lattice III is the geometric dual of lattice II. The '+' triangles are taken as those which point the *opposite* way to the triangles of the original lattice I. Fig. 1 illustrates the case for $n = 4$.

It will now be shown that the high temperature expansion of the model (1) on lattice I (with free boundaries) maps, term by term, onto the low temperature expansion of (1) on lattice III with all boundary sites fixed at state 0.

The common expression (for the partition function) is in terms of bond states in which each bond in lattice II is labelled with an integer from 0 to $q-1$ in all possible ways such that at every vertex the sum of the labels on incident bonds is 0 modulo q. The dummy external edges all have label 0. From the set of such labellings we construct the generating function

$$Q(r,s) = \sum_{\mathrm{AL}} r^F s^G, \tag{11}$$

where F is the total number of bonds with nonzero labels, G is the number of X vertices for which at least one incident edge has a nonzero label, and the AL-summation is over all allowed labellings.

The partition function for model (1) on lattice III is by definition a sum over all possible states of the internal sites. Each term in this sum is mapped onto a bond labelling of lattice II. If C_{ij} is the label on the edge (of II) dual to the edge (of III) connecting sites i and j (on III) then the mapping is defined by

$$C_{ij} = (t_i - t_j) \bmod q. \tag{12}$$

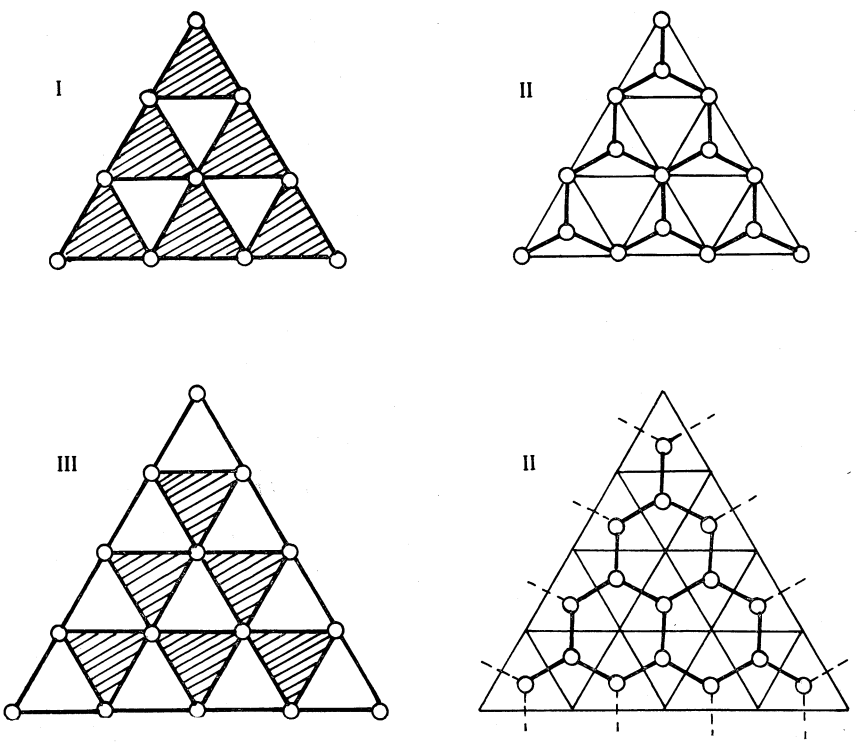

Fig. 1. Three lattices used in the graphical proof of the duality relation. The high temperature expansion on lattice I (with free boundaries) is equivalent to the low temperature expansion on lattice III (with fixed boundaries). The proof consists of showing that both expansions are equivalent to a vertex expansion on lattice II.

We have to ensure that we take these differences in a consistent manner around each vertex of II (e.g. clockwise around each X vertex and anticlockwise around each Y vertex). At each vertex we will then have incident edges such as C_{ij}, C_{ik}, C_{jk} such that

$$(C_{ij} + C_{jk} + C_{ki}) \bmod q = (t_i - t_j + t_j - t_k + t_k - t_i) \bmod q = 0. \tag{13}$$

As well as showing that each state of the system on lattice III maps onto an allowed labelling of II, the relation (13) ensures that the transformation can be reversed so that, starting from the fixed boundary sites, the bond labellings can be used to reconstruct the configuration on III unambiguously.

The state of lattice III with all t_i zero gives a leading contribution to the partition function of $(u^3 w)^{-n(n-1)/2}$. For each other state this factor is multiplied by u raised to the power of the number of neighbours for which $t_i \neq t_j$ (equivalent to the number of nonzero labels on II) and by w raised to the power of the number of '+' faces on III

for which $t_i \neq t_j$ or $t_j \neq t_k$ (equivalent to the number of X sites with nonzero labels on some incident bonds). Thus the fixed boundary partition function Z_{III} is given by

$$Z_{\mathrm{III}}(u,w) = (u^3 w)^{-n(n-1)/2} Q(u,w). \tag{14}$$

To map the high temperature form of the partition function onto $Q(r,s)$ we separate the energy into a sum of contributions associated with each '+' triangle of I, the set being indexed by i,j,k:

$$Z_{\mathrm{I}} = \sum_{\{t_i\}} \prod_{<i,j,k>} \exp(-\beta E_{ijk}) = \sum_{\{a,b,c,\ldots\}} \prod_{<i,j,k>} V_{abc}(i,j,k). \tag{15}$$

The indices $a,b,c,\ldots$ correspond to $t_i, t_j, t_k, \ldots$. We can write V_{abc} as

$$V_{abc} = \sum_{mnp} A_{am} A_{bn} A_{cp} \lambda_{mnp}, \tag{16}$$

where

$$A_{am} = q^{-\frac{1}{2}} \exp(2\pi i a m/q), \tag{17}$$

$$\lambda_{mnp} = 0, \qquad m+n+p \neq 0 \bmod q. \tag{18}$$

Thus Z_{I} becomes the sum

$$Z_{\mathrm{I}} = \sum_{\{a,b,c,\ldots\}} \prod_{<i,j,k>} \left(\sum_{mnp} A_{am} A_{bn} A_{cp} \lambda_{mnp} \right). \tag{19}$$

Each additive term in equation (19) can be mapped onto a labelling of bonds on lattice II simply by using the indices of λ_{mnp} (which are associated with a set of vertices around a '+' face on I equivalently with bonds around an X site in II).

Equation (18) ensures that the constraint on allowed bond labels is satisfied at X sites. The constraint is also satisfied at Y sites since

$$\sum_a A_{am} A_{an} A_{ap} = 1, \qquad m+n+p = 0 \bmod q, \tag{20a}$$

$$= 0 \qquad \text{otherwise.} \tag{20b}$$

If we write the nonzero values of λ as

$$\begin{aligned} \lambda_{mnp} = {} & \delta(m,0)\,\delta(n,0)\,\delta(p,0)\,(1-3r^2s+2r^3s) \\ & + (\delta(m,0)+\delta(n,0)+\delta(p,0))(r^2s-r^3s) + r^3s, \end{aligned} \tag{21}$$

then

$$Z_{\mathrm{I}} = Q(r,s) \tag{22}$$

and V_{abc} is given by

$$\begin{aligned} V_{abc} = {} & qr^3s\,\delta(a,b)\,\delta(b,c) \\ & + (r^2s-r^3s)(\delta(a,b)+\delta(b,c)+\delta(a,c)) + q^{-1}(1-3r^2s+2r^3s). \end{aligned} \tag{23}$$

The final term in V_{abc} corresponds to an additive constant in the energy but the other terms correspond to an energy of the form (1). For the model (1) we have

$$\begin{aligned} Z_{\mathrm{III}}(u,w).(u^3w)^{n(n-1)/2} &= Z_{\mathrm{I}}(u',w').\{q^{-1}(1-3u^2w+2u^3w)\}^{n(n-1)/2} \\ &= Q(u,w), \end{aligned} \tag{24}$$

where from equation (23)

$$f' = (u')^{-1} - 1 = (qu^2w - qu^3w)/(1 - 3u^2w + 2u^3w) = qf/y, \tag{25a}$$

which is equivalent to the relation (6b), and

$$y' = q^2u^3w/(1 - 3u^2w + 2u^3w) = q^2/y, \tag{25b}$$

equivalent to the relation (6a).

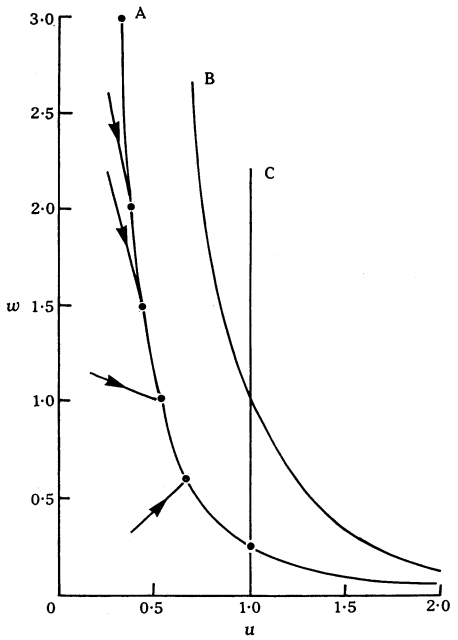

Fig. 2. Plots in the u–w plane. Curve A is the line of self-dual points which are conjectured to be critical points. Curve B represents $u^3w = 1$ and is the line along which correlations can be obtained by using the techniques described in the Appendix. Curve C corresponds to pure three-site interactions; the duality transformation maps this line into itself. The points on curve A are those investigated by series analysis and the arrows at these points show the directions along which R corresponds to a derivative of the free energy.

The duality transformation has one other interesting property. The line $u = 0$ is the dual of the curve $wu^3 = 1$. It is shown in the Appendix that the free energy and various correlations can be obtained very easily for points on the line $wu^3 = 1$ by using a generalization of the techniques described by Enting (1977*a*).

Fig. 2 shows some of the results of this section in terms of the u–w plane. The line of self-dual points (the conjectured critical points) is shown by curve A. The points on this curve are those at which the series are investigated and the arrows show the directions along which the derivative of the free energy is proportional to R. Curve B represents $u^3w = 1$. As pointed out by Baxter *et al.* (1978), the line $u = 1$ (curve C, corresponding to pure three-site interactions) is mapped into itself by the duality transformation.

The fact that for $q = 2$ the three-site interaction is equivalent to an additional contribution to the two-site interaction indicates that the duality relation described above is different from the duality relations on triplet Ising models (e.g. Wood and Griffiths 1972; Wood and Pegg 1977) and thus different from the generalizations to triplet models with q-state variables (Enting 1975). There is, however, a direct correspondence between the graphical arguments and the general formulation of duality given by Wegner (1972). The constraints (13), (18) and (20) correspond to

Wegner's 'quantum-number' conservation properties which generate the duality transformation. Alternatively the constraints can be used (implicitly) to define subsets of graphs and, following Merlini and Gruber (1972), definition of group operations on graphs can lead to a description of duality in terms of group homomorphisms.

3. High Field Expansions

The high field expansions are expansions in powers of the variable μ. The set of graphs corresponding to μ^n will be the strong graphs of n sites. For $n \leqslant 5$, Sykes *et al.* (1965) have tabulated these graphs together with the lattice counts, the number of ways of embedding these graphs on the triangular lattice. For Potts models, it is necessary to consider decorations of these graphs so that one colours sites in all possible combinations using one of $q-1$ colours at each site. The important properties of each decorated graph are:

- n, the number of sites;
- b, the number of edges for which both ends are identically coloured;
- d, the number of edges with ends of different colours;
- t, the number of '+' triangles with identically coloured vertices;
- r, the number of triangles with two vertices identical but one different;
- s, the number of triangles with all three vertices different (this must be zero for $q = 3$).

For $q = 3$, Enting (1974*b*) has enumerated all the decorations for $n \leqslant 5$. The data are published as a set of F_n which are given by sums of terms of the form

$$k(3n-b-d-2t-2r,\ 3n-2b-2d+3t+3r,\ b-3t-r,\ d-2r,\ t,\ r),$$

where k is the lattice constant. To convert these data into series in powers $\mathscr{P}$ of μ, u and w we have

$$\mathscr{P}(\mu) = n, \qquad \mathscr{P}(u) = 6n-2b-d, \qquad \mathscr{P}(w) = 3n-b-d+r. \tag{26}$$

The series in powers of μ can be regrouped to give series in powers of u; the expressions given by Enting (1974*b*) give series in u correct to u^{17}. Terms to u^{20} can be obtained by including the partial codes

$$\begin{aligned} F_6 &= 24(11,6,3,0,2)+24(11,6,3,0,1,1)+24(11,6,1,2,2) \\ &\quad +2(12,9,0,0,3)+6(12,9,0,0,2,1)+6(10,3,4,2,1) \\ &\quad +2(10,3,6,0,1)+42(12,8,2,0,2)+42(11,5,5,0,1)+\ldots, \\ F_7 &= 2(12,6,3,0,3)+30(13,8,2,0,3)+30(12,5,5,0,2)+\ldots, \\ F_8 &= 6(14,8,2,0,4)+6(13,5,5,0,3)+\ldots. \end{aligned}$$

When the series are interpreted as specified by the powers (26) one has expressions of the form

$$\ln Z = \sum_{pqr} A_{pqr}\,\mu^p u^q w^r. \tag{27}$$

These can be used as series in μ along the critical isotherm by fixing w and u so that

$$(q-2)\, wu^3+3wu^2 = 1\,, \tag{28}$$

which corresponds to $y = q$. Alternatively, if w is fixed and μ is constant at $1\cdot0$, one has series in u. For values of w other than $1\cdot0$, fixed w implies that K/J is a varying function of temperature.

The present series were obtained by direct counting of graphs and so are comparatively short. A number of more sophisticated techniques have been developed. de Neef and Enting (1977) have commented that there is a tendency to substitute algebraic complexity for combinatorial complexity in series expansion techniques apparently because of the relative ease of implementing algebraic techniques on digital computers. There are a number of possible approaches which appear to be relevant to the present model:

(1) In Ising models, Sykes *et al.* (1965, 1975) have shown how the inclusion of three-site interactions actually helps in the derivation of series expansions. This simplification uses the transformation to a honeycomb lattice and is based on the equivalence of the two triangular sublattices of the honeycomb lattice. A similar simplification will not occur in the Potts models because, as seen in Section 2, these models transform to honeycomb lattices with inequivalent sublattices.

(2) Enting (1977*b*) has shown how solutions of the type described in the Appendix impose constraints on the series expansions. It may be possible to use these constraints in conjunction with the general approach of Sykes *et al.* (1965, 1975), with the additional constraints making up for the loss of information arising from inequivalent sublattices.

(3) There are a number of cluster expansions which eliminate disconnected graphs. On square lattices an extremely compact resummation exists (de Neef and Enting 1977; Enting 1978*a*) so that it is sufficient to sum over rectangular graphs. Resummations on triangular lattices are more complicated but it should be possible to extend the series using either direct cluster expansions or some suitable resummation.

4. Analysis of Series for $q = 3$

As mentioned in the preceding section, the series were analysed by fixing the value of w and then applying conventional techniques of analysis to the series in u. The values of w chosen were $0\cdot6$, $1\cdot0$, $1\cdot5$, $2\cdot0$ and $3\cdot0$. The critical values of u are shown in Fig. 2.

The first type of analysis needed is a test to check that the critical points do in fact correspond to those which arise from combining the duality relation with the assumption of a unique transition. The magnetization was defined as

$$M = \tfrac{2}{3} - \mu\, \partial(\ln Z)/\partial\mu\,. \tag{29}$$

Pade approximations were constructed for $\mathrm{d}(\ln M)/\mathrm{d}u$ (w fixed). The positive roots of the denominator polynomials were estimates of the critical values u_c of u, and these estimates are given in Table 1. The differences between the estimates and the values predicted by duality are typically about 1%. This is probably as good an agreement as could be expected from comparatively short low temperature series. The remainder of this section is based on the assumption that there is a unique transition for each w so that the critical points are in fact the self-dual points.

To estimate the exponent β, Pade approximants to

$$(u-u_c)\frac{d}{du}\left(\ln M\right) \tag{30a}$$

were constructed and evaluated at u_c. The results are given in Table 2*a*. While the estimates become less regular as w departs from unity, there is no indication of any significant variation in β.

Table 1. Estimates of critical values of u

The estimates of the critical values u_c were derived from $[N, D]$ approximants to $d(\ln M)/du$. The results are compared with the values obtained from the duality transformation, assuming the transition to be unique

Approximant	Estimates of u_c				
$[N, D]$	$w = 0{\cdot}6$	$1{\cdot}0$	$1{\cdot}5$	$2{\cdot}0$	$3{\cdot}0$
8, 11	0·6672	0·5312	0·4418[A]	0·3851	0·3193
9, 10		0·5283[A]	0·4413	0·3861	0·3193
10, 9	0·66±0·03i	0·5313	0·4420	0·3876	0·3214
11, 8	0·6636	0·5304	0·4419	0·3873	0·3203
8, 10	0·6624[A]	0·5306	0·4417	0·3866	0·3195
9, 9	Pair[B]	0·5308	0·4423	0·3883	0·3224
10, 8	0·6659	0·5303	0·4421	0·3881	0·3221
8, 9	0·6634	0·5304	0·4429	0·3894	0·3249
9, 8	0·6603	0·5275	0·4376	0·3821	0·3182
Duality:	0·6736	0·5312	0·4402	0·3844	0·3170

[A] Not the smallest positive real zero of the approximant.
[B] Pair of roots at 0·695 and 0·699.

Estimates of the exponent γ' were obtained from Pade approximants to

$$(u_c-u)\frac{d}{du}\left\{\ln\left(u^{-6}\frac{dM}{d\mu}\right)\right\}, \tag{30b}$$

with w fixed, $\mu = 1$ and $u = u_c$. The results are given in Table 2*b*. Again the spread of estimates for each value of w is similar to the range of any possible systematic variation in γ'. Table 2*b* (and also Tables 2*c* and 2*d*) contains entries which are presumably spurious, being due to interference from singularities on or near the real axis.

Table 2*c* shows direct estimates of α', the exponent for the specific heat C. These were obtained by evaluating the Pade approximants to

$$(u_c-u)\frac{d}{du}\left(\ln(u^{-6}C)\right), \tag{30c}$$

at $u = u_c$. The quantity R defined by equation (10) was also analysed. Approximants to

$$(u-u_c)\frac{d}{du}\left(\ln(R-R_c)\right), \tag{30d}$$

evaluated at u_c, are expected to give estimates of $1-\alpha'$. The results are listed in Table 2*d*. The direct estimates of α' show a marked variation with w, with the average

Table 2. Estimates of exponents

The estimates of the exponents β, γ', α' and $1-\alpha'$ were obtained from Pade approximants to the expressions (30a)–(30d), each evaluated at $u = u_c$. For the exponent δ, the estimates are $-(M_n)^{-1}$ where M_n satisfies equation (31)

Approximant		Estimates of exponent			
$[N, D]$	$w = 0{\cdot}6$	1·0	1·5	2·0	3·0
		(a) Estimates of β			
9, 10	0·107	0·107	0·107	0·107	0·100
10, 9	0·112	0·107	0·107	0·108	0·110
9, 9	0·110	0·107	0·108	0·109	0·111
8, 9	0·134	0·105	0·109	0·110	0·112
9, 8	0·111	0·106	0·110	0·114	0·117
8, 8	0·106	0·106	0·112	0·136	0·091
7, 8	0·100	0·105	0·111	0·116	0·124
8, 7	0·164	0·107	0·111	0·118	0·139
7, 7	0·091	0·104	0·111	0·114	0·116
		(b) Estimates of γ'			
6, 7	1·60	1·53	1·54	1·59	1·77
7, 6	1·58	1·52	1·52	1·54	1·59
6, 6	1·62	1·57	1·70	2·94	0·60
5, 6	1·59	1·49	1·45	1·44	1·44
6, 5	1·50	1·44	1·39	1·37	1·35
5, 5	1·50	1·41	1·35	1·32	1·28
		(c) Estimates of α'			
6, 7	0·516	0·470	0·421	0·388	0·345
7, 6	0·223	0·463	0·421	0·387	0·339
6, 6	0·587	0·489	0·422	0·382	0·335
5, 6	0·623	0·523	0·453	0·409	0·357
6, 5	0·612	0·512	0·440	0·395	0·340
5, 5	0·551	0·542	0·390	0·348	0·295
		(d) Estimates of $1-\alpha'$			
9, 10	0·569	0·590	0·635	0·613	0·630
10, 9	0·567	0·579	0·598	0·613	0·630
9, 9	0·565	0·584	0·601	0·613	0·630
8, 9	7·13	0·544	0·596	0·613	0·631
9, 8	0·568	0·581	0·599	0·613	0·635
8, 8	0·551	0·575	0·597	0·614	0·636
7, 8	0·535	0·570	0·597	0·615	0·638
8, 7	0·448	0·489	0·515	0·520	1·19
7, 7	0·478	0·533	0·573	0·599	0·631
6, 7	0·485	0·537	0·574	0·598	0·629
7, 6	0·483	0·538	0·575	0·598	0·629
		(e) Estimates of δ			
$n = 1$	16·80	16·21	16·04	16·04	16·19
2	16·14	15·95	15·79	15·64	15·36
3	15·79	15·71	15·67	15·65	15·64
4	15·52	15·57	15·56	15·53	15·46

estimates varying from 0·58 to 0·34 as w changes from 0·6 to 3·0. The estimates obtained from R show a much smaller variation with w. In addition, for each value of w the α' estimate shows a systematic dependence on the order of the approximants used. Watts (1974) has pointed out that this type of systematic variation can be extrapolated so that the α' estimates from $[M, N]$ approximants are plotted against $(M+N)^{-1}$ and extrapolated to $(M+N)^{-1} = 0$. If this procedure is applied to the results in Table 2*d*, the data are consistent with $\alpha' \approx 0{\cdot}37$ for all values of w.

For completeness, estimates of the exponent δ have also been obtained. These were derived using the technique described by Gaunt and Sykes (1972). If w is fixed and u is set at u_c, the coefficients of $\mathrm{d}(\ln M)/\mathrm{d}\mu$ should tend to $-\delta^{-1}$. Thus the estimates of δ given in Table 2*e* are $-(M_n)^{-1}$ where

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\Big(\ln M(u_c, \mu)\Big) = \sum_n \mu^n M_n . \tag{31}$$

The estimates are consistent with $\delta = 15$ but, since the values are based on only five series terms, no great significance can be attached to this result.

5. Conclusions

To put the results of the previous section into perspective it is worth recalling the wide range of exponent estimates obtained for three-state Potts models with purely two-site interactions. Series expansion estimates include $\alpha' = 0{\cdot}05 \pm 0{\cdot}10$ (Straley and Fisher 1973), $0{\cdot}276 \pm 0{\cdot}002$ (Zwansig and Ramshaw 1977) and $0{\cdot}42 \pm 0{\cdot}05$ (de Neef and Enting 1977). Estimates from renormalization group techniques include $\alpha' = 0{\cdot}3365$ (Burkhardt *et al.* 1976), 0·192 (Berker and Wortis 1976) and 0·326 (Dasgupta 1977).

Enting (1978*b*) has pointed out that the assumption of scaling restricts the possible sets of exponents and he has described a diagrammatic technique for representing the possible sets: A set of exponents $(\alpha', \beta, \gamma', \delta)$ which satisfies scaling corresponds to a point in the α'–β plane, while each estimate of an exponent corresponds to a line. An estimate $\hat{\alpha}'$ is represented by the line $\hat{\alpha}' = \alpha'$, an estimate $\hat{\beta}$ by the line $\hat{\beta} = \beta$, an estimate $\hat{\delta}$ by the line $\alpha' + \beta(1+\hat{\delta}) = 2$ and an estimate $\hat{\gamma}$ by the line $\alpha' + 2\beta + \hat{\gamma}' = 2$. Thus a set of estimates $(\hat{\alpha}', \hat{\beta}, \hat{\gamma}', \hat{\delta})$ which satisfies scaling will correspond to four lines intersecting at one point.

For $w = 1{\cdot}0$, a possible set of exponent estimates is ($\hat{\alpha}' = 0{\cdot}45$, $\hat{\beta} = 0{\cdot}107$, $\hat{\gamma}' = 1{\cdot}50$) while for $w = 2{\cdot}0$ a possible set is ($\hat{\alpha}' = 0{\cdot}38$, $\hat{\beta} = 0{\cdot}110$, $\hat{\gamma}' = 1{\cdot}55$). In Fig. 3 the lines corresponding to these exponent estimates are plotted. For each value of w the lines fail to meet at a point but instead enclose triangular areas. If one attempts to define a set of 'best-fit' scaling exponents $(\alpha', \beta, \gamma')$ by minimizing some weighted sum

$$w_\alpha |\alpha' - \hat{\alpha}'|^a + w_\beta |\beta - \hat{\beta}|^b + w_\gamma |\gamma' - \hat{\gamma}'|^c$$

then, as long as all the weights are positive, the best-fit exponents will lie within the triangular areas. It will be seen that there is considerable overlap between the triangles for $w = 1{\cdot}0$ and 2·0 and that the $\hat{\delta} = 15$ line passes through the common region. In other words, if scaling is assumed then the uncertainties in the exponent estimates are greater than the apparent variation in the range $w = 1{\cdot}0$–$2{\cdot}0$. If the range of w

is extended from 0·6 to 3·0 then even greater variations in the exponent estimates become apparent because the two techniques for estimating α' cease to give similar results.

On the basis of the present work the most plausible conclusion is that the exponents do not depend on the strength of the three-site interaction. It would seem that any variation is either comparatively small or, less probably, that there is a larger variation accompanied by a violation of three-exponent scaling.

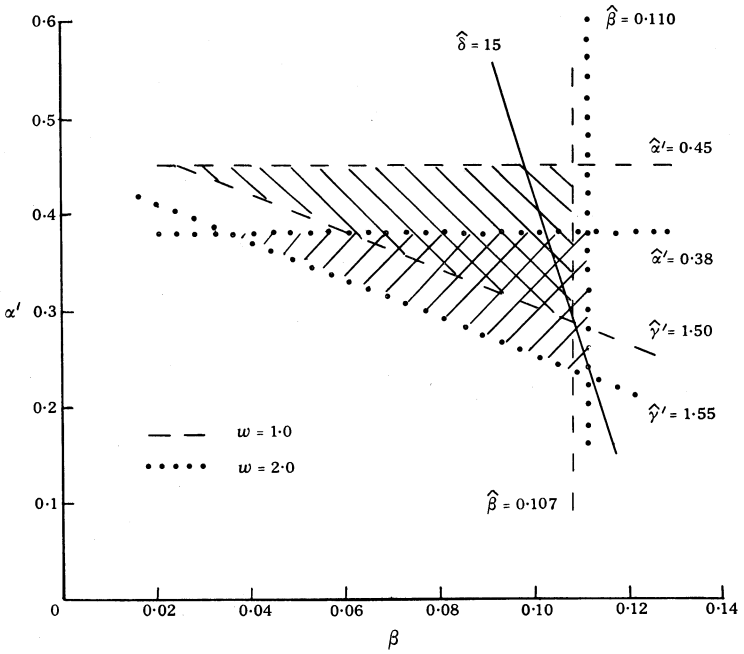

Fig. 3. Scaling plot showing the relations between exponent estimates $(\hat{\alpha}', \hat{\beta}, \hat{\gamma}', \hat{\delta})$ for values of w of 1·0 and 2·0. The shaded triangles correspond to possible best-fit sets of exponents which satisfy scaling.

The absence of any observable w dependence in the mixed Potts model is an indication that the universality class of such models depends only on the explicit symmetries of the system, as opposed to the situation occurring in the eight-vertex model. When modelling krypton layers on graphite, Berker *et al.* (1978) transformed their original model while preserving the explicit symmetries and produced generalized Potts models. The present work goes some way towards indicating that such transformations should not change the universality class of the models.

Obviously a more stringent test of universality would be possible if longer series could be obtained. A vital part of such an investigation would be establishing more precise values for critical exponents for Potts models with purely two-site interactions. The discussion at the end of Section 3 above indicates that some techniques for obtaining more extensive Potts model series might be able to be used to give series for the mixed model with little extra difficulty.

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Appendix

In lattice models with competing interactions there are sometimes special points at which it is possible to obtain simple solutions for correlations. One of the earliest such solutions was due to Gibberd (1969). A more systematic approach to Ising model solutions of this type is given by Enting (1977*a*). The lattice models are related to the conditional probability models investigated by Welberry and Galbraith (1973). (Subsequent work in this field is reviewed by Enting and Welberry 1978.)

In the present application we build up a configuration of t_i variables on a triangular lattice one row at a time, with the probability of a particular state on site i depending only on the states of m and n, its two neighbours in the previous row. The model is expressed in terms of conditional probabilities of the form $P(t_i \mid t_m, t_n)$. We choose

$$P(0\mid 0,0) = P(1\mid 1,1) = P(2\mid 2,2) = P(0\mid 1,2) = P(0\mid 2,1)$$
$$= P(1\mid 0,2) = P(1\mid 2,0) = P(2\mid 1,0) = P(2\mid 0,1) = 1-2Z, \quad \text{(A1a)}$$

and, for all other cases,

$$P(a \mid b, c) = Z, \tag{A1b}$$

with $0 \leqslant Z \leqslant \frac{1}{2}$. The connection between this conditional probability model and the mixed Potts models is established by relating the joint probability distributions for configurations. In one case the joint probability is given by a product over the conditional probabilities above. In the other case it is given by a product over Boltzmann weights. The symmetries of equations (A1) indicate that only Potts-type two-site and three-site interactions are involved. Inspection of appropriate configurations shows that the parameterization chosen corresponds to the isotropic model. Following Enting (1977*a*), inspection of particular configurations leads to the results

$$u = (1-2Z)/Z, \qquad w = Z^3/(1-2Z)^3, \tag{A2}$$

so that the conditional probability models correspond to the mixed Potts models with $u^3w = 1$.

To show how the conditional probability characterization leads to solutions for correlations, we consider site i and sites m and n, its two neighbours in the preceding row. We write

$$\begin{aligned} P(t_i = t_m) &= P(t_m \mid t_m, t_n)\, P(t_m, t_n) \\ &= (1-2Z)P(t_m = t_n) + Z\, P(t_m \neq t_n). \end{aligned} \tag{A3}$$

Then using the isotropy of the model we have

$$P(t_i = t_m) = P(t_m = t_n) = Q \tag{A4}$$

or

$$Q = (1-2Z)Q + Z(1-Q), \qquad \text{whence} \qquad Q = \tfrac{1}{3}. \tag{A5}$$

Thus the probability of two neighbours being alike is $\frac{1}{3}$, the value expected in a purely random noninteracting system. The three-site expectations, however, do show the effects of interactions:

$$P(t_i = t_m = t_n) = (1-2Z)P(t_m = t_n) = \tfrac{1}{3}(1-2Z). \tag{A6}$$

This probability only has the value 1/9 corresponding to a random noninteracting system, if in fact the interactions do vanish, that is, $Z = \frac{1}{3}$, $u = w = 1$.

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