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Equations for Pulsar Magnetospheres with Particle Inertia

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Abstract

This note deals with steadily rotating nonaxisymmetric pulsar magnetospheres, with the effects of particle inertia fully incorporated. It is pointed out that the equations of motion for the component species of a relativistically streaming nondissipative plasma can be considerably simplified by using the steady-rotation constraint together with a fluxoid conservation law and Endean's Bernoulli-type integral.

The canonical pulsar model consists of a rotating magnetized neutron star with its magnetic axis inclined to the rotation axis, but a self-consistent model of the pulsar magnetosphere is still lacking. Both the vacuum model, in which the particles are regarded as test charges in the vacuum field, and the zero-inertia model are unacceptable: the plasma is not only a source of the electromagnetic field, but also carries energy and angular momentum.

For the *axisymmetric vacuum* model, Goldreich and Julian (1969) pointed out that the component $E_{\|}$ of the electric field $\boldsymbol{E}$ parallel to the magnetic field $\boldsymbol{B}$ is sufficiently powerful near the star to pull charges out of it, so creating a charged magnetosphere. The solution for the nonaxisymmetric vacuum model with a dipolar magnetic field on the stellar surface was obtained by Deutsch (1955), in the context of the theory of normal magnetic stars. This enabled Mestel (1971) and Cohen and Toton (1971) to extend the argument of Goldreich and Julian to the *oblique rotator* model, in which the magnetic and rotation axes are not aligned or antiparallel. Subsequent investigations of the physics of neutron star surfaces have suggested that, for most pulsars, the Goldreich–Julian mechanism might not be sufficiently powerful to extract positive ions (Ruderman 1971).

The argument of Goldreich and Julian (1969) poses the problem of studying magnetospheres that are sufficiently dense near the star to make $E_{\|} \approx 0$ there. Various authors have investigated *charge-separated* plasmas with inertial and other non-electromagnetic terms neglected, so that the equation of motion is just

$$\boldsymbol{E} + c^{-1}\boldsymbol{v} \times \boldsymbol{B} \approx \boldsymbol{0},$$

where c and $\boldsymbol{v}$ are the vacuum speed of light and the plasma's fluid velocity. But in the last few years it has become clear that inertial effects are crucial in pulsar magnetospheres: the need for the fields to be nonsingular at the light cylinder leads to difficulties when inertial effects are neglected.

Mestel (1973) derived an equation for the magnetic field in a steadily rotating magnetosphere when the magnetohydrodynamic and force-free approximations are satisfied. Endean (1974) showed that the equation follows from just the steady-rotation and force-free conditions, together with the boundary condition of infinite conductivity on the stellar surface; Endean commented that the magnetohydrodynamic condition is not required, which is true for non-charge-separated plasmas, but for charge-separated plasmas the magnetohydrodynamic and force-free conditions are equivalent. Mestel (1973) applied his equation for the magnetic field to the '*cylindrical pulsar*' model, in which quantities do not vary in the direction of the rotation axis. Mestel showed that, if the equation is valid everywhere outside the star, then there is no flow of energy across the light cylinder; furthermore, the solutions between the light cylinder and infinity are standing waves and so require a reflector at infinity (Mestel *et al.* 1976). These results suggest that, with appropriate boundary conditions, the system does not reach a steady state until other forces besides the Lorentz force become significant, so that the particles are no longer tied to the field lines.

The above considerations point to the importance of developing models that fully incorporate the relativistic inertial terms in the equations of motion of the plasma species. To focus attention on the problem of treating inertia, it is reasonable provisionally to ignore all dissipative forces, and to have inertia as the only non-electromagnetic force. For the steadily rotating cylindrical model, the equations of motion of the species have been presented by Mestel *et al.* (1976). The purpose of the present note is to show how these equations can be considerably simplified, *without* restriction to either the cylindrical or axisymmetric models.

Let ϖ, ϕ and z be cylindrical polar coordinates with the z axis as the rotation axis of the pulsar. The system under consideration is steady in the rotating frame: the changes in time at points fixed in the inertial frame result only from the steady rotation of a nonaxisymmetric structure at angular frequency Ω. Hence, it follows from Faraday's law and $\nabla . \boldsymbol{B} = 0$ that $\boldsymbol{E}$ and $\boldsymbol{B}$ are connected by (Mestel 1971)

$$\boldsymbol{E} + c^{-1}\Omega\varpi\boldsymbol{t} \times \boldsymbol{B} = -\nabla\Phi, \tag{1}$$

where $\boldsymbol{t}$ is the unit toroidal vector and Φ is related to the familiar scalar and vector potentials ϕ and $\boldsymbol{A}$ by the gauge-invariant relationship (Endean 1972*a*)

$$\Phi = \phi - (\Omega\varpi/c)A_\phi .$$

Within the star, the approximation of perfect conductivity is adequate for the present purposes, so that Φ can be put equal to zero there.

In the vacuum model, the star emits an electromagnetic wave of frequency Ω, except for the axisymmetric case in which the magnetic and rotation axes are either parallel or antiparallel; with or without axisymmetry we have $\nabla\Phi \neq 0$ and $E_{\parallel} \neq 0$. In the zero-inertia model we have $\nabla\Phi = 0$, so that $E_{\parallel} = 0$.

Endean (1972*a*, 1972*b*) pointed out that, under the steady-rotation constraint (1), there exists a constant of the motion Ψ_k for particles of species k:

$$\Psi_k \equiv \Phi + \frac{\gamma_k m_k c^2}{e_k}\left(1 - \frac{\Omega\varpi}{c}\frac{v_{k\phi}}{c}\right), \tag{2}$$

where the subscript k represents either electrons or ions, while e_k, m_k, γ_k and $v_{k\phi}$

represent the charge, rest mass, Lorentz factor and ϕ-component of velocity for the particles of the species concerned. Apart from Endean's (1972*b*) own analysis of particle motion and his later treatment (Endean 1976) of certain equilibrium solutions, corresponding to a rigidly corotating magnetosphere with no outwardly or inwardly streaming particles, not much use appears to have been made of Endean's integral. The potential usefulness of this integral becomes clear when one thinks in terms of fluid streamlines rather than magnetic field lines; Endean's (1972*a*, 1972*b*, 1976) work was expressed in the language of particle dynamics rather than that of fluid dynamics.

The equation of motion for each species will now be simplified by using the steady-rotation constraint together with Endean's integral and a fluxoid conservation theorem. Since

$$\boldsymbol{\alpha} \times (\nabla \times \boldsymbol{\alpha}) \equiv \tfrac{1}{2}\nabla(\boldsymbol{\alpha}^2) - \boldsymbol{\alpha}.\nabla\boldsymbol{\alpha},$$

where $\boldsymbol{\alpha}$ is any vector, we have

$$\boldsymbol{v}_k.\nabla\boldsymbol{p}_k = -\boldsymbol{v}_k \times (\nabla \times \boldsymbol{p}_k) + m_k c^2 \nabla\gamma_k \tag{3}$$

for each species, where $\boldsymbol{p}_k \equiv \gamma_k m_k \boldsymbol{v}_k$, with $\boldsymbol{v}_k$ denoting the velocity of the species; the relation $v_k^2/c^2 = 1-\gamma_k^{-2}$ has been used to eliminate v_k^2 from the last term in equation (3). For multispecies relativistic plasmas, the flux conservation theorem of magnetohydrodynamics can be generalized to a fluxoid conservation theorem, in order to incorporate the effects of particle inertia (Buckingham *et al.* 1972, 1973). A differential form of the fluxoid theorem is obtained (Buckingham *et al.* 1973) by using equation (3) in the equation of motion for each species, and then taking the curl and eliminating $\boldsymbol{E}$ by use of Faraday's law: the quantity $\nabla \times (\boldsymbol{p}_k + e_k \boldsymbol{A}/c)$ is 'frozen-in' to species k. Using the steady-rotation condition $\partial/\partial t = -\Omega\,\partial/\partial\phi$ (Mestel 1971; Endean 1972*a*) which is valid for, in particular, cylindrical polar components of vectors, equation (3) becomes

$$(\partial/\partial t + \boldsymbol{v}_k.\nabla)\boldsymbol{p}_k = -\boldsymbol{u}_k \times (\nabla \times \boldsymbol{p}_k) + \nabla(\gamma_k m_k c^2 - \Omega\varpi p_{k\phi}), \tag{4}$$

where $\boldsymbol{u}_k \equiv \boldsymbol{v}_k - \Omega\varpi\boldsymbol{t}$. Equation (1) and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ show that

$$\boldsymbol{E} + c^{-1}\boldsymbol{v}_k \times \boldsymbol{B} = c^{-1}\boldsymbol{u}_k \times (\nabla \times \boldsymbol{A}) - \nabla\Phi. \tag{5}$$

From equations (4) and (5), the equation of motion for species k can be written in the simple form

$$\boldsymbol{u}_k \times \{\nabla \times (\boldsymbol{p}_k + e_k \boldsymbol{A}/c)\} = e_k \nabla\Psi_k. \tag{6}$$

This shows immediately that Ψ_k is constant on lines of $\boldsymbol{u}_k$ and also on lines of $\nabla \times (\boldsymbol{p}_k + e_k\boldsymbol{A}/c)$. In the language of fluid dynamics, Ψ_k is a Bernoulli-type integral, constant on streamlines of species k; the term 'streamlines' here refers to lines of $\boldsymbol{u}_k$, not of $\boldsymbol{v}_k$.

If the particles are all nonrelativistic near the star, then Ψ_k reduces to $\Phi + m_k c^2/e_k$ near the star, and the constant value $m_k c^2/e_k$ taken by Ψ_k on the surface of the star is propagated indefinitely along streamlines of species k throughout whatever portion of the magnetosphere contains particles of that species. In this case, the definition (2) shows that

$$1 - \frac{e_k\Phi}{m_k c^2} = \gamma_k\left(1 - \frac{\Omega\,\varpi}{c}\frac{v_{k\phi}}{c}\right), \tag{7}$$

and the equation of motion (6) for species k reduces to the very simple form

$$\boldsymbol{u}_k \times \{\nabla \times (\boldsymbol{p}_k + e_k \boldsymbol{A}/c)\} = \boldsymbol{0}. \tag{8}$$

Thus it is seen that use of the steady-rotation constraint, together with the fluxoid conservation law and Endean's Bernoulli-type integral, enables the equation of motion for each species, with particle inertia fully incorporated, to be expressed in simple form. Application of the resulting equations to particular models is under investigation.

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