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High-field Polynomial Expansions for the Six-state Planar Potts Model

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Abstract

High-field polynomial expansions (through order 9) are derived for the six-state planar Potts model and analysed. Suggestive evidence is found for an intermediate phase in which the exponent δ varies continuously as a function of temperature. This behaviour is consistent with recent results predicting that the model exhibits two transitions separating a 'topologically ordered' phase analogous to that found in the planar rotor model.

1. Introduction

There has been considerable interest recently in the behaviour of two-dimensional models with nearest-neighbour interactions which are invariant under a global Z_p symmetry (José *et al.* 1977; Elitzur *et al.* 1979; Wu 1979; Cardy 1980; Einhorn *et al.* 1980; Hamer and Kogut 1980; Alcaraz and Köberle 1980; Domany *et al.* 1980). A particular example of this class of models is the p-state planar Potts (1952) model (also known as the 'clock' model). The Hamiltonian of this model is $(J > 0)$

$$H = -J \sum_{<ij>} \cos(\theta_i - \theta_j), \tag{1}$$

where the sum runs over all nearest-neighbour pairs on the lattice and the variables θ_i take the values

$$\theta_i = 2\pi n_i/p; \qquad n_i = 0, 1, 2, ..., p-1. \tag{2}$$

For $p = 2$ and 4, this model reduces rather trivially to the Ising model, while for $p = 3$ it is equivalent to the conventional three-state Potts model. However, for $p > 4$, recent analyses (Cardy 1980; Alcaraz and Köberle 1980; Einhorn *et al.* 1980) have suggested that the model should exhibit *two* transitions.*

At low temperatures, the system exhibits conventional long-range order, i.e. the correlation function behaves as

$$\Gamma(\boldsymbol{r}-\boldsymbol{r}') = \langle\cos(\theta_r - \theta_{r'})\rangle \rightarrow \Gamma_0^2 \tag{3}$$

as $|\boldsymbol{r}-\boldsymbol{r}'| \rightarrow \infty$, where Γ_0 (>0) is the spontaneous order. At the first transition T_L, Γ_0 vanishes and the system enters a phase in which $\Gamma(\boldsymbol{r}-\boldsymbol{r}')$ decays algebraically:

$$\Gamma(\boldsymbol{r}-\boldsymbol{r}') \sim |\boldsymbol{r}-\boldsymbol{r}'|^{-\eta(T)}, \tag{4}$$

* This conclusion was first reached by Elitzur *et al.* (1979) for the Z_p symmetric Villian model (Z_p periodic gaussian model); see also José *et al.* (1977). For the case $p = 5$, see also Wu (1979) and Domany *et al.* (1980).

with $\eta(T)$ temperature dependent. This phase is thus similar to the low-temperature phase of the XY model. This phase then terminates at a second temperature T_{U} above which $\Gamma(r)$ decays exponentially.

In this paper, we derive high-field/low-temperature polynomial expansions for the six-state model ($p = 6$). In principle, this series allows one to analyse the behaviour of the spontaneous order as the temperature approaches the lower transition. However, Cardy (1980) using a Kosterlitz (1974) style renormalization group analysis has shown that

$$\Gamma_0(T) \approx \exp\{-a/(T_{\mathrm{L}}-T)^{\frac{1}{2}}\}, \qquad T \to T_{\mathrm{L}}, \tag{5}$$

which would be very difficult to detect in a series analysis. On the other hand, the high-field expansion can also be used to investigate the behaviour of $\Gamma_0(T,h)$ as $h \to 0$ at fixed T, where h is the appropriate symmetry breaking field. This we do in this paper, and find that over a range of temperature

$$\Gamma_0(T,h) \approx h^{1/\delta(T)}, \qquad h \to 0. \tag{6}$$

This behaviour is consistent with the expression (4) and the existence of a massless XY-like phase.

Our arguments are arranged as follows. The derivation of the basic series is described in Section 2. Section 3 is devoted to an analysis of the low-temperature expansion of the spontaneous order $\Gamma_0(T)$. As suggested above this analysis is rather inconclusive. A somewhat more successful estimation of the exponent $\delta(T)$ is carried out in Section 4. Section 5 closes the paper with an overall summary.

2. Series Expansions for Planar Potts Model

To derive high-field or low-temperature expansions for a Z_p model, we introduce a field h which singles out one of p symmetric directions $\theta_i = 2\pi n_i/p$, $n_i = 0, 1, ..., p-1$. For convenience we choose h to be in the '0' direction and consider the Hamiltonian

$$H = -h\sum_i \cos\theta_i - J\sum_{<i,j>} \cos(\theta_i - \theta_j). \tag{7}$$

The reduced partition function Λ is defined by

$$\Lambda^N = \sum_{n_1=0}^{p-1}\sum_{n_2=0}^{p-1} \cdots \sum_{n_N=0}^{p-1} \exp\{-\beta(H-NE_0)\}, \tag{8}$$

where N is the number of lattice sites and $\beta = 1/kT$ with k Boltzmann's constant, and we have subtracted off the ground state energy per spin,

$$E_0 = -2J-h. \tag{9}$$

The reduced free energy per spin has an expansion of the form

$$-\beta(F-E_0) = \ln\Lambda = \sum_{l=1}^{\infty} \mu^l L_l(x), \tag{10}$$

where

$$\mu = \exp(-h/2kT), \qquad x = \exp(-J/2kT). \tag{11a, b}$$

The basic principles of high-field expansions for lattice models have been reviewed by Domb (1960). The expansion takes the form of a summation over all perturbations from a fully aligned state. For the spin $\frac{1}{2}$ Ising model this expansion can be represented directly in terms of graphs. For other models one has to sum over all possible 'decorations' of these graphs (Sykes and Gaunt 1973).

While we are primarily interested in the case in which the field h is the same at all sites, the most efficient way of computing the series is to follow Sykes *et al.* (1965) and apply different fields h_A and h_B on each of the two sublattices of the square lattice. (Both fields still have the same direction.) Defining

$$\eta = \exp(-h_A/2kT), \qquad \nu = \exp(-h_B/2kT), \tag{12a, b}$$

we obtain

$$\ln \Lambda = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} G_{m,n}(x)\, \eta^m \nu^n . \tag{13}$$

The L_n are recovered by noting that

$$L_n(x) = \sum_{m=0}^{n} G_{m,n-m}(x) . \tag{14}$$

The computational efficiency comes from two observations (Sykes *et al.* 1965):

(1) The $G_{m,n}$ are symmetric, i.e. $G_{m,n}(x) = G_{n,m}(x)$, so that to calculate L_1 to L_9 we only need $G_{m,n}$ for $m \leqslant 4$ (and $n \leqslant 9$).

(2) If we sum over all configurations of all the 'B' sites, we create a new lattice model which can be expanded in terms of decorated graphs and which has the same reduced free energy as the original system, so that

$$\ln \Lambda = \sum_{m=0}^{\infty} H_m(x, \nu)\, \eta^m . \tag{15}$$

This means that L_1 to L_9 can be obtained from H_0 to H_4. This technique is known as the method of partial generating functions. The 'decorations' of the graphs contributing to H_m are all those that lead to a factor of η^m and for the six-state model they are:

$m = 0$: no perturbed sites on the 'A' sublattice,
$m = 1$: 1 'B' site in either state 1 or state 5,
$m = 2$: 2 'B' sites in states 1 or 5,
$m = 3$: 3 'B' sites in states 1 or 5
OR 1 'B' site in states 2 or 4,
$m = 4$: 4 'B' sites in states 1 or 5
OR 1 'B' site in state 2 or 4 and 1 in state 1 or 5
OR 1 'B' site in state 3.

For each of these values of m the first alternative listed consists of perturbing each of the m perturbed 'B' sites into one of two equivalent states. This is precisely the combinatorial problem involved in applying the method of partial generating functions (p.g.f.s) to the three-state Potts models. The appropriate combinatorial information has been given by Enting (1974*a*, 1974*b*) and can be taken over directly for the six-state planar Potts model. For $m \leqslant 4$, the additional contributions involve at most two perturbed B sites. General expressions for the combinatorial factors for any lattice model of this type and for up to three perturbed B sites have been given b Enting (1975).

The p.g.f.s for the three-state Potts model are linear combinations of products of factors denoted f_i. These functions can be expressed (Enting 1975) in a coded form so that, for instance, writing the second p.g.f. F_2 as

$$4(6,4,2)+4(6,4,0,2)+4(7,6,1)+4(7,6,0,1)-18(8,8)$$

denotes

$$F_2(\{f_i\}) = 4f_2^4 f_3^2/f_1^6 + 4f_2^4 f_4^2/f_1^6 + 4f_2^6 f_3^1/f_1^7 + 4f_2^6 f_4^1/f_1^7 - 18f_2^8/f_1^8, \tag{16}$$

where the f_i are given by Enting (1974*b*). The coefficients in parentheses are the powers of f_1^{-1}, and those of f_i for $i = 2, 3, 4, \ldots$.

The p.g.f.s $H_m(x, v)$ for the six-state planar model are obtained from the p.g.f.s F_m of the three-state model by substituting functions g_i (as listed below) for the f_i and then adding correction terms. Thus we have

$$H_0 = F_0(\{g_i\}) = \ln g_1, \tag{17a}$$

$$H_1 = F_1(\{g_i\}) = 2(g_2/g_1)^z, \tag{17b}$$

$$H_2 = F_2(\{g_i\}), \tag{17c}$$

$$H_3 = F_3(\{g_i\})+2(k_1/g_1)^z, \tag{17d}$$

$$\begin{aligned} H_4 = {} & F_4(\{g_i\}) + k_2^z/g_1^z + a g_2^{z-1} k_1^{z-1}(k_3+k_1)/g_1^{2z-1} \\ & + b g_2^{z-2} k_1^{z-2}(k_3^2+k_4^2)/g_1^{2z-2} - (2a+2b+4) g_2^z k_1^z/g_1^{2z}, \end{aligned} \tag{17e}$$

where for square and honeycomb lattices $a = 8, 12$; $b = 8, 0$; $z = 4, 3$ (the coordination number).

Enting (1974*b*) describes f_1 as the sum over all configurations of an A site with no B neighbours perturbed and f_4 as the sum over all configurations of an A site with two perturbed B neighbours which are in different states. These descriptions also apply to g_1 and g_4 respectively, and in general each of the g_i can be described in the same way as f_i for the three-state Potts model. The differences only arise in the Boltzmann weights that appear within each configurational sum. The correction terms which are not analogues of standard Potts model summations are denoted k_i and are listed below. For a lattice of coordination number z, the g_i are

$$g_1 = 1+2vx^z+2v^3x^{3z}+v^4x^{4z}, \tag{18a}$$

$$g_2 = x+vx^{z-1}+vx^{z+2}+v^3x^{3z-2}+v^3x^{3z+1}+v^4x^{4z-1}, \tag{18b}$$

$$g_3 = x^2+vx^{z-2}+vx^{z+4}+v^3x^{3z-4}+v^3x^{3z+2}+v^4x^{4z-2}, \tag{18c}$$

$$g_4 = x^2+2vx^{z+1}+2v^3x^{3z-1}+v^4x^{4z-2}, \tag{18d}$$

$$g_5 = x^3+vx^{z-3}+vx^{z+6}+v^3x^{3z-6}+v^3x^{3z+3}+v^4x^{4z-3}, \tag{18e}$$

$$g_6 = x^3+vx^z+vx^{z+3}+v^3x^{3z}+v^3x^{3z-3}+v^4x^{4z-3}, \tag{18f}$$

$$g_8 = x^4 + vx^{z-4} + vx^{z+8} + v^3x^{3z-8} + v^3x^{3z+4} + v^4x^{4z-4}, \tag{18g}$$

$$g_9 = x^4 + vx^{z-1} + vx^{z+5} + v^3x^{3z+1} + v^3x^{3z-5} + v^4x^{4z-4}, \tag{18h}$$

$$g_{10} = x^4 + 2vx^{z+2} + 2v^3x^{3z-2} + v^4x^{4z-4}. \tag{18i}$$

The f_i are defined for a general q-state Potts model. The functions f_7, f_{11} and f_{12} are associated with configurations which cannot occur for $q = 3$. Positions 7, 11 and 12 in the coded p.g.f.s given by Enting (1974*a*, 1974*b*) are therefore always zero. This means that to preserve the correspondence between the f_i of the standard model and the g_i of the six-state planar model, g_7, g_{11} and g_{12} are absent from the list above.

We need finally the correction terms in equations (17d) and (17e) which involve the functions k_i, $i = 1,2,3,4$. The function k_1 is a sum over all states of an A site with a B neighbour in state 2 or state 4, while k_2 is a similar sum for a site with its only perturbed neighbour in state 3. Finally, k_3 is the sum of a site whose two perturbed neighbours are in states 4 and 5 or 1 and 2 and k_4 is the sum for a site with neighbours in states 2 and 5 or 1 and 4. Evaluating the k_i we find

$$k_1 = g_6, \qquad k_2 = k_4 = g_{10}, \qquad k_3 = g_9. \tag{19}$$

These specifications enabled us to calculate L_1 to L_9 for the six-state planar model on the square lattice. These are given in the Appendix. The specifications for the g_i could also be used with the F_i for the three-state Potts model on the honeycomb lattice to obtain L_1 to L_9. The honeycomb p.g.f. F_5 could be used to extend the series to L_{11} once correction terms involving three perturbed sites (two in states 1 or 5 and one in 2 or 4) had been evaluated.

The high-field series can also be regrouped to give an expansion in powers of x. Since the series is dominated by configurations that have the same type of structure as those of the three-state Potts model, it is fairly easy to determine that L_1 to L_9 give the low-temperature series correctly through x^{13}. Furthermore it is easy to obtain (by hand) the graphs that are needed to extend the series to x^{14}. The additional term in $\ln \Lambda$ is $60\mu^{10}x^{14}$.

One final comment is appropriate. Unfortunately it does not seem useful to derive similar expansions for other values of $p \geqslant 5$, because in these cases a natural expansion variable does not exist as the energy gaps are not rational multiples of each other. For values of $p \leqslant 4$, the equivalence of the model to other models (Ising or three-state Potts) means that the high-field expansions do exist and have been extensively studied.

3. Spontaneous Order

From equation (10) the 'magnetization' or spontaneous order is

$$\Gamma_0(x) \equiv -(\partial F/\partial h)|_{h=0} = 1 - \tfrac{1}{2}\sum_{j=1}^{\infty} jL_j(x). \tag{20}$$

Thus from the Appendix we obtain the low-temperature expansion

$$\Gamma_0(x) = 1 - x^4 - 4x^6 - 12x^8 - 4x^9 - 28x^{10} - 36x^{11}$$
$$- 53x^{12} - 176x^{13} - 48x^{14} + O(x^{15}), \tag{21}$$

where the term of order 14 also involves the additional contribution noted towards the end of the last section.

As mentioned in the Introduction, Cardy (1980) has predicted that $\Gamma_0(x)$ should vanish at the lower critical temperature T_L with an exponential singularity of the form

$$\Gamma_0(x) \approx A \exp\{-a/(x_L - x)^{\frac{1}{2}}\}, \qquad x_L = \exp(-J/2kT_L). \tag{22}$$

Such behaviour is unfortunately rather difficult to detect in a series analysis. Guttmann (1978) has suggested that exponential singularities of this form can be usefully analysed by constructing Padé approximants to $(\mathrm{d}/\mathrm{d}x)[\ln\{(\mathrm{d}/\mathrm{d}x)\ln \Gamma_0\}]$. This quantity should have a pole at x_c with residue unity if Γ_0 vanishes algebraically, but $1+\sigma$ if Γ_0 vanishes as $\exp\{-a/(x_c - x)^\sigma\}$. We have tried this test with poor results; the Padé approximants are very defective and inconsistent.

Table 1. Estimates of x_L (and β)

The estimates were obtained from $[N, M]$ Padé approximants to $(\mathrm{d}/\mathrm{d}x)\{\ln \Gamma_0(x)\}$

			M		
N	2	3	4	5	6
2	0·490 (0·166)	0·449 (0·089)	0·515 (0·273)		
3	0·463 (0·113)	0·473 (0·132)	0·483 (0·157)	0·485 (0·162)	
4	0·486 (0·169)	0·508 (0·292)	0·485 (0·164)	0·481 (0·153)	0·483 (0·158)
5		0·492 (0·189)	0·490 (0·181)	0·483 (0·158)	
6			0·491 (0·188)		

Somewhat surprisingly, however, if we apply the simplest Padé analysis and consider the poles and zeros of Padé approximants to $(\mathrm{d}/\mathrm{d}x)\{\ln \Gamma_0\}$ in the standard way we obtain fair results as illustrated in Table 1. Taken at face value, these results imply that

$$\Gamma_0(x) \approx A'(x_L - x)^\beta, \tag{23}$$

with

$$x_L \approx 0{\cdot}49 \pm 0{\cdot}01, \qquad \beta \approx 0{\cdot}17 \pm 0{\cdot}02. \tag{24a, b}$$

Since our series is rather short and the Padé results are poorly convergent we do not wish to place too much significance on these results. Three points are however worth noting. Firstly, Padé approximants to $(x_L - x)(\mathrm{d}/\mathrm{d}x)\{\ln \Gamma_0(x)\}$ evaluated at $x = 0{\cdot}49$ rather consistently yield values near $0{\cdot}18$, which is consistent with the estimate (24b). Secondly, the estimate (24b) for β is close to what one gets from a similar analysis of the weak coupling quantum Hamiltonian series for the Z_6 model magnetization derived by Hamer and Kogut (1980). In this case, the corresponding estimate of the critical coupling is very close to the dual of the critical temperature found by Elitzur *et al.* (1979) from strong coupling series (C. J. Hamer, unpublished results). Finally, in the next section we shall present further evidence that suggests that the low-temperature critical point of the six-state vector Potts model is in the vicinity of $x \approx 0{\cdot}49$.

These points suggest that the Padé analysis is picking out the initial drop of the magnetization from unity and the behaviour is described by an effective exponent $\beta \approx 0{\cdot}18$. Near x_L the actual behaviour is presumably that predicted by Cardy (1980) although we have no evidence for it. The apparent agreement of the Padé estimate of x_L with those of other methods suggests that this cross-over occurs very close in.

4. Exponent δ

On fixing the temperature variable x, equation (10) yields an expansion of the magnetization $\Gamma(\mu, x)$ in the field variable μ about $\mu = 0$ (i.e. $H = \infty$):

$$\Gamma(\mu, x) = 1 - \tfrac{1}{2} \sum_{l=1}^{\infty} l\, L_l(x)\, \mu^l, \tag{25}$$

where the first nine coefficients are known. If the six-state planar Potts model possesses a topologically ordered phase similar to that of the $O(2)$ planar rotor model, we expect that

$$\Gamma(\mu, x) \sim D_0(x)\,(1-\mu)^{1/\delta(x)}, \qquad \mu \to 1, \quad x \text{ fixed}, \tag{26}$$

where the exponent δ varies continuously with temperature in the interval $x_L < x < x_U$. For the Villian Z_p model, the renormalization group analysis of Elitzur *et al.* (1979) gave $\eta = \frac{1}{4}$ at $x = x_U^V$ and $\eta = 4/p^2$ at x_L^V. The scaling relation $\delta = (2d-\eta)/\eta$ then gives the values with $d = 2$

$$\delta = 15 \quad \text{and} \quad \delta = p^2 - 1 \tag{27}$$

at the upper and lower transitions of this model.

To estimate δ for the six-state planar Potts model from the series (25) we have used the method of Gaunt and Sykes (1972). This method is based on the asymptotic behaviour of the coefficients in the expansion of

$$-\mu(\mathrm{d}/\mathrm{d}\mu)\{\ln \Gamma(\mu, x)\} = \sum_{n=1}^{\infty} d_n\, \mu^n. \tag{28}$$

If the expression (26) is valid, then

$$d_n = 1/\delta + O(n^{-2}) \qquad \text{as } n \to \infty. \tag{29}$$

Since there is marked even/odd oscillation in the d_n, we show in Fig. 1*a* plots of d_n^{-1} versus x for $n = 5, 7, 9$.

To facilitate interpretation of this figure, we show in Fig. 1*b* a similar plot for the two-dimensional Ising model using the same number (nine) of high-field polynomials. This plot is quite striking, the role of the critical temperature at $x_c \approx 0{\cdot}17$ being clearly seen. For $x < x_c$ there is a definite upward drift in the estimates $\delta_n = d_n^{-1}$, which are apparently growing without bound. This is consistent with the existence of a spontaneous order as $\mu \to 1$ ($x < x_c$), the series approximating this by a very large (strictly infinite) value of δ. On the other hand, for $x > x_c$, there is a marked downward drift in δ_n towards unity. This behaviour is again what one expects since in this regime

$$\Gamma(\mu, x) \approx \chi(x)\mu + O(\mu^2), \tag{30}$$

with the susceptibility finite. Of course, as $x \to x_c+$, $\chi(x)$ diverges and this affects the rate at which δ_n approaches unity. For $x = x_c$, all estimates pass very close to $\delta = 15$. This collapse of the estimates reflects the fact (recall equation 29) that δ_n

approaches δ ($=15$) with zero slope as n increases. Including additional high-field polynomials confirms these trends.

Returning to the six-state planar Potts results (Fig. 1*a*), we observe that the same type of behaviour is exhibited at high and low temperatures, although the spread in the estimates δ_n is greater than for the Ising model. A definite upward trend in the δ_n is apparent for $x < x_L \approx 0\cdot49$, while for $x > x_U \approx 0\cdot6$, δ_9 is less than δ_7. Rather

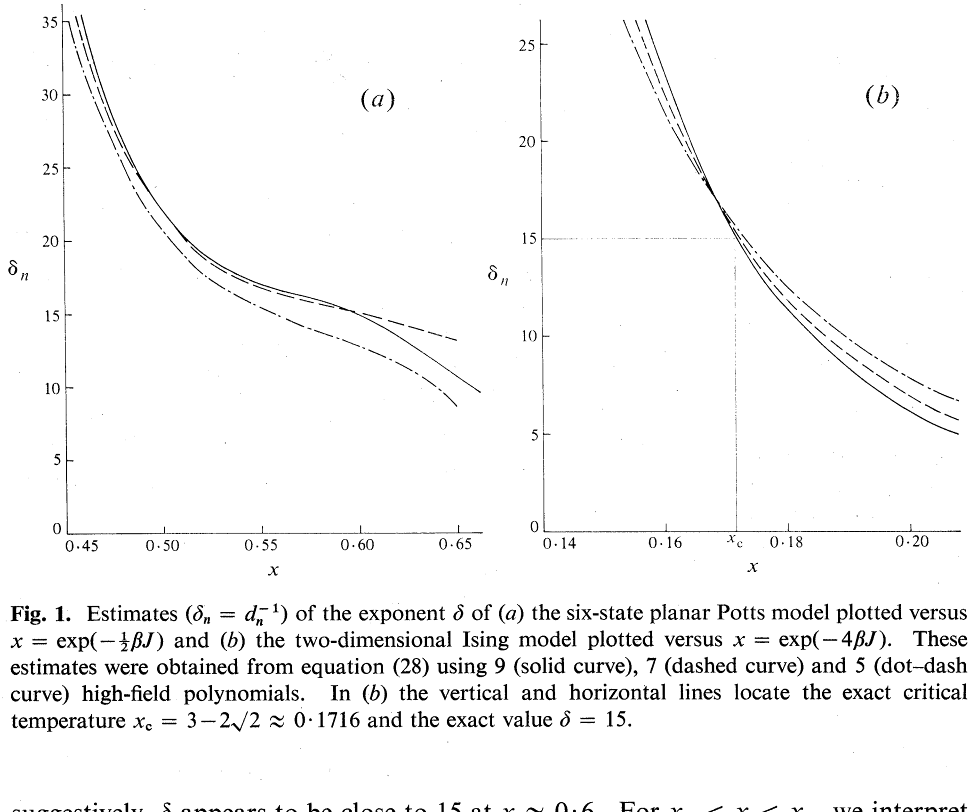

Fig. 1. Estimates ($\delta_n = d_n^{-1}$) of the exponent δ of (*a*) the six-state planar Potts model plotted versus $x = \exp(-\frac{1}{2}\beta J)$ and (*b*) the two-dimensional Ising model plotted versus $x = \exp(-4\beta J)$. These estimates were obtained from equation (28) using 9 (solid curve), 7 (dashed curve) and 5 (dot–dash curve) high-field polynomials. In (*b*) the vertical and horizontal lines locate the exact critical temperature $x_c = 3-2\sqrt{2} \approx 0\cdot1716$ and the exact value $\delta = 15$.

suggestively, δ appears to be close to 15 at $x \approx 0\cdot6$. For $x_L < x < x_U$, we interpret the close agreement between δ_9 and δ_7 as a reflection of a 'critical' limit as predicted by (29). Obviously, further polynomials beyond L_9 are necessary to confirm this interpretation. Nevertheless, Fig. 1*a* remains highly suggestive evidence for an intermediate phase in which there is no spontaneous magnetization but a divergent zero-field susceptibility.

The exponent δ can also be estimated by evaluating Padé approximants to $(1-\mu)(\mathrm{d}/\mathrm{d}x)\{\ln\Gamma(\mu,x)\}$ at $\mu = 1$. Given (26) we get

$$(1-\mu)(\mathrm{d}/\mathrm{d}x)\{\ln\Gamma(\mu,x)\,|_{\mu=1}\} = 1/\delta(x)\,. \tag{31}$$

The resulting estimates of δ at select values of x are listed in Table 2 for some central Padé approximants. These values are more or less consistent with Fig. 1*a*, but fall off rather more rapidly for $x \gtrsim 0\cdot55$. Whether this implies that x_U is considerably less than $0\cdot6$ is unclear.

Table 2. Estimates of the exponent $\delta(x)$
The estimates were obtained from Padé approximants to the expression (31)

x	Estimate of δ Padé [2, 2]	[2, 3]	[3, 3]	[3, 4]	[4, 4]
0·48	23·9	24·6	23·6	30·4	26·6
0·50	19·5	19·9	19·4	25·7	21·6
0·52	16·5	16·9	16·4	24·6	18·7
0·54	14·6	14·9	14·5	23·9	16·8
0·56	13·1	13·3	13·1	20·1	15·1
0·58	12·0	11·9	12·0	15·8	13·2
0·60	10·9	10·6	10·8	12·5	11·3

5. Summary

In the preceding two sections we have analysed the high-field polynomial expansion derived in Section 2 for the six-state planar Potts model on the square lattice. Unfortunately, the number (nine) of polynomials is insufficient to allow a precise determination of the critical parameters. Nevertheless, the following conclusions appear valid:

(1) The spontaneous magnetization vanishes at $x = x_L \approx 0 \cdot 49$: No evidence of an exponential singularity was however detected, the series analysis suggesting an effective exponent $\beta = 0 \cdot 18$.

(2) For $x < x_L$, the behaviour of high-field series is similar to that found in the Ising model, and consistent with the existence of spontaneous order.

(3) In the regime $x_L < x < x_U \approx 0 \cdot 6$, the high-field series as a function of $\mu = \exp(-h/2kT)$ can be analysed to yield δ. The results (Fig. 1*a*) are consistent (but admittedly far from conclusively so) with δ varying continuously with x.

(4) At $x = x_U \approx 0 \cdot 6$, δ appears to be 15. This agrees with the results of the renormalization group analysis of Elitzur *et al.* (1979) which predicts that the upper transition is equivalent to the Kosterlitz–Thouless transition in the planar rotor model (see also below).

(5) At $x = x_L \approx 0 \cdot 49$, δ appears however to be less than the value of $p^2 - 1$ $(=35)$ predicted by Elitzur *et al.* (1979): A value closer to 25 would be more consistent with our results.

(6) Our estimate of $x_U \approx 0 \cdot 6$ for the upper critical temperature is very close to the value $x = 0 \cdot 64$ corresponding to the critical temperature of the planar rotor model (Tobochnik and Chester 1979). It seems quite conceivable that this temperature is identically that of the Kosterlitz–Thouless transition in the planar model. It would be interesting to independently estimate x_U for the p-state planar Potts models from an analysis of *high-temperature series.*

One possible formalism for obtaining such a series would be the application of a low-temperature to high-temperature transformation. The well-known low to high transformation of the Ising model is efficient because it exploits the symmetry under field reversal which helps determine the form of the re-summed series. In models such as the three-state Potts model where there is no symmetry under reversal of the single-direction field, we find that the series given by Enting (1974*a*) would give only half as many high-temperature terms as could be obtained for the Ising model with the same number of high-field polynomials. Similarly for the six-state planar model, we can only expect a fully efficient low to high transformation if we use more general generating functions that correspond to the full symmetry of the system. The complications inherent in such an approach may render it inferior to more direct algebraic techniques for obtaining high-temperature series.

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Appendix

Six-state Planar Potts Model High-field Polynomials on Square Lattice

$$L_1 = +2x^4,$$

$$L_2 = +4x^6 - 10x^8 + 4x^9,$$

$$L_3 = +12x^8 - 64x^{10} + 24x^{11} + 84\tfrac{2}{3}x^{12} - 64x^{13} + 12x^{14},$$

$$\begin{aligned} L_4 = {} & +2x^8 + 36x^{10} - 340x^{12} + 116x^{13} + 956x^{14} - 680x^{15} \\ & - 739x^{16} + 944x^{17} - 340x^{18} + 36x^{19} + 2x^{20}, \end{aligned}$$

$$\begin{aligned} L_5 = {} & +16x^{10} + 62x^{12} + 16x^{13} - 1588x^{14} + 464x^{15} \\ & + 7504x^{16} - 4904x^{17} - 13588x^{18} + 14912x^{19} + 4768\tfrac{2}{5}x^{20} \\ & - 13552x^{21} + 7424x^{22} - 1600x^{23} + 62x^{24} + 16x^{25}, \end{aligned}$$

$$
\begin{aligned}
L_6 = {} & +4x^{10}+80x^{12}-220x^{14}+168x^{15}-6084x^{16} \\
& +1244x^{17}+48281\tfrac{1}{3}x^{18}-28836x^{19}-146192x^{20} \\
& +146204x^{21}+171860x^{22}-292816x^{23}+28252\tfrac{2}{3}x^{24} \\
& +181536x^{25}-145300x^{26}+48101\tfrac{1}{3}x^{27}-6164x^{28} \\
& -228x^{29}+80x^{30}+4x^{31},
\end{aligned}
$$

$$
\begin{aligned}
L_7 = {} & +44x^{12}+208x^{14}+56x^{15}-3034x^{16}+1064x^{17} \\
& -15036x^{18}-1456x^{19}+261712x^{20}-138168x^{21} \\
& -1218420x^{22}+1125632x^{23}+2535588x^{24}-3679432x^{25} \\
& -1586008x^{26}+5298400x^{27}-2164125\tfrac{5}{7}x^{28} \\
& -2071288x^{29}+2622520x^{30}-1212448x^{31}+262424x^{32} \\
& -15352x^{33}-3084x^{34}+208x^{35}+44x^{36},
\end{aligned}
$$

$$
\begin{aligned}
L_8 = {} & +12x^{12}+268x^{14}+8x^{15}-552x^{16}+608x^{17} \\
& -18012x^{18}+3536x^{19}+19348x^{20}-49016x^{21} \\
& +1153752x^{22}-473216x^{23}-8468430x^{24}+7190264x^{25} \\
& +26804764x^{26}-35161644x^{27}-37252512x^{28} \\
& +82850296x^{29}-2377124x^{30}-87855552x^{31} \\
& +62947709\tfrac{1}{2}x^{32}+13995804x^{33}-43373056x^{34} \\
& +27342804x^{35}-8443184x^{36}+1163272x^{37}+18396x^{38} \\
& -18344x^{39}-576x^{40}+268x^{41}+12x^{42},
\end{aligned}
$$

$$
\begin{aligned}
L_9 = {} & +2x^{12}+144x^{14}+860x^{16}+232x^{17}-10888x^{18} \\
& +4056x^{19}-62566x^{20}-6360x^{21}+525808x^{22}-378504x^{23} \\
& +3503054x^{24}-318816x^{25}-49782220x^{26}+37693480x^{27} \\
& +230051114x^{28}-27722261x^{29}-512221317\tfrac{1}{3}x^{30} \\
& +954072768x^{31}+376370604x^{32}-1701379160x^{33} \\
& +653698292x^{34}+1286988176x^{35}-1451043221\tfrac{7}{9}x^{36} \\
& +193664936x^{37}+632945708x^{38}-563904341\tfrac{1}{3}x^{39} \\
& +232473894x^{40}-49688632x^{41}+3572088x^{42} \\
& +527776x^{43}-64050x^{44}-11096x^{45}+848x^{46}+144x^{47} \\
& +2x^{48}.
\end{aligned}
$$

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