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Variational Approach to the Calculation of the Binding Energy of $^{31}_{\Lambda\Lambda}$Si

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Abstract

The double hypernucleus $^{31}_{\Lambda\Lambda}$Si has been considered as a three-body system Λ–Λ–^{29}Si and its binding energy calculated by a variational method using a trial wavefunction of the form $F(r_{C1})\,F(r_{C2})\,G(r_{12})$, where the r are interparticle triangular coordinates and F and G are of the form $z\exp(-\alpha r^2)+\exp(-\beta r^2)$. These wavefunctions allow the description of strong Λ–Λ spatial correlations which are found to be quite significant. The parameters z, α, β for the two-body wavefunctions F and G are obtained by a variational procedure in order to find the binding energies of the two-body systems Λ–^{29}Si and Λ–Λ. The parameters of the Λ–Λ wavefunction are adjusted so as to produce just a zero-energy Λ–Λ system. For the Λ–^{29}Si system the interaction potential between Λ and ^{29}Si is generated by folding a gaussian Λ–N potential into the density distribution of ^{29}Si. Parameters for the Λ–^{29}Si system are used in the three-body calculation, but those for the Λ–Λ system are kept free in the three-body variational calculation. In the first stage, our calculated value of the binding energy is 41·54 MeV, where we have used a gaussian Λ–Λ interaction having a volume integral of 610·8 MeV fm^3. This volume integral is calculated from the two-body Λ–Λ system. In the second stage we have taken the volume integral as a free parameter also, and calculated the binding energy of $^{31}_{\Lambda\Lambda}$Si to be 39·7 MeV, for a volume integral of 356·5 MeV fm^3 for the Λ–Λ potential. This value is compared with the experimental value of 38·2±6·3 MeV found by Mondal *et al.* (1975). The dependence of the binding energy on the depth of the Λ–Λ interaction has also been investigated.

1. Introduction

The discovery (Danysz *et al.* 1963) of the double hypernucleus $^{10}_{\Lambda\Lambda}$Be (or $^{11}_{\Lambda\Lambda}$Be) aroused a considerable amount of interest in $\Lambda\Lambda$ hypernuclei. This is because the study of $\Lambda\Lambda$ hypernuclei is expected to lead to significant information about the Λ–Λ interaction in the $^1S_0$ state.

The double hypernucleus $^{10}_{\Lambda\Lambda}$Be has already been analysed by Bodmer and Ali (1965) and Dalitz and Rajasekaran (1964) using a three-body model. Bodmer and Ali (1965) and Tang and Herndon (1965) also analysed $^{10}_{\Lambda\Lambda}$Be, taking it to be a four-body system consisting of two α and two Λ particles. Another double hypernucleus, $^{6}_{\Lambda\Lambda}$He, was studied by Ali and Bodmer (1967) with a Λ–Λ–^{4}He model.

More recently Mondal *et al.* (1975) reported an event in nuclear emulsion, identified as evidence of a very heavy hypernucleus. They predicted it to be $^{31}_{\Lambda\Lambda}$Si and calculated its binding energy from range measurements to be 38·2±6·3 MeV. Since the number of double hypernuclei is very small, it seems worth while to

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calculate the Λ–Λ binding energy in ${}^{31}_{\Lambda\Lambda}$Si and compare the result with that of the experimental investigation.

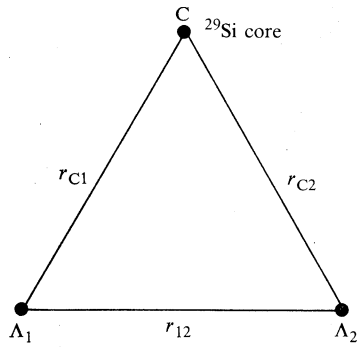

Fig. 1. Triangular coordinate system: r_{C1} and r_{C2} are Λ–core separations; r_{12} is the Λ–Λ separation.

With this aim in view a three-body Λ–Λ–^{29}Si model is assumed for ${}^{31}_{\Lambda\Lambda}$Si. Our calculations are based on the use of the three-body s-state wavefunction of the product form

$$\psi = F(r_{C1})\,F(r_{C2})\,G(r_{12}). \tag{1}$$

Here r_{C1}, r_{C2} and r_{12} are triangular coordinates as shown in Fig. 1. It is essential to include the function $G(r_{12})$ in order to allow for the effect of the Λ–Λ interaction. Convenient analytic forms for the functions F and G are assumed as follows:

$$F(r) = x\exp(-\alpha r^2) + \exp(-\beta r^2), \tag{2a}$$

$$G(r) = y\exp(-\gamma r^2) + \exp(-\delta r^2). \tag{2b}$$

In these functions, assuming that $\alpha \gg \beta$ and $\gamma \gg \delta$, the first terms represent the form of the wavefunction for the two-body systems Λ–^{29}Si and Λ–Λ in the region of close approach, while the second terms are related to the long-range part of the wavefunction.

Our procedure is to determine the function $F(r)$ by a variational method to fit the information on the Λ–^{29}Si system. Since the Λ–Λ system is not bound, the function $G(r)$ is determined by considering a purely attractive potential of shape (Ali and Bodmer 1967; Dalitz and Downs 1958)

$$V_{\Lambda\Lambda}(r_{12}) = -U_{\Lambda\Lambda}(\lambda/\pi)^{3/2}\exp(-\lambda r_{12}^2), \tag{3}$$

where $\lambda = 0{\cdot}935\,\mathrm{fm}^{-2}$ corresponding to the two-pion exchange process, and by arbitrarily adjusting the Λ–Λ potential strength to give a bound state at zero energy. However, a purely attractive potential for the Λ–Λ system is somewhat unrealistic, since the presence of a short-range repulsion in the nucleon–nucleon potential, usually represented by a hard core, suggests that a hard core of similar size may also be present in the Λ–Λ potential (deSwart 1963). Thus, the parameters α and γ provide an excellent approximation, while β and δ provide a good first approximation for the parameters appropriate to the ${}^{31}_{\Lambda\Lambda}$Si system.

2. The Λ–Core Interaction

The Λ–core potential $V_{\Lambda C}$ is obtained by folding a gaussian Λ–N interaction (Dalitz and Downs 1958)

$$V_{\Lambda N}(r) = (1/\pi b^2)^{3/2} \exp(-r^2/b^2) \tag{4}$$

into the core density distribution, where $b = 1{\cdot}03366$ fm is appropriate to the two-pion exchange mechanism. For this purpose, a harmonic oscillator density distribution for the configuration $1s^2\, 1p^4_{3/2}\, 1p^2_{1/2}\, 1d^6_{5/2}\, 2s^1$ of ^{29}Si is calculated to be

$$\rho(r_1) = 4(1/\pi a^2)^{3/2}(\tfrac{1}{8}^{1}_{1} + 3r_1^2/2a^2 + 29r_1^4/30a^4) \exp(-r_1^2/a^2), \tag{5}$$

where $a = 1{\cdot}873$ fm is the oscillator size parameter, chosen so that it yields $\langle r^2\rangle^{1/2} = 3{\cdot}14$ fm for the r.m.s. radius of the ^{29}Si nucleus.

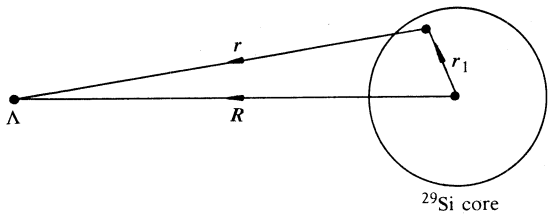

Fig. 2. Triangular coordinate system for Λ–^{29}Si: $\boldsymbol{r}_1$ is the position of a nucleon relative to the centre-of-mass of the ^{29}Si core, $\boldsymbol{r}$ is the Λ–N separation and $\boldsymbol{R}$ is the position of the Λ relative to the centre of the core.

A schematic diagram of the folding-model calculation is shown in Fig. 2. The Λ–^{29}Si potential can be written as

$$V_{\Lambda C}(R) = \int V_{\Lambda N}(\boldsymbol{r}_1 - \boldsymbol{R})\, \rho(r_1)\, \mathrm{d}^3 r_1\,, \tag{6}$$

where $\boldsymbol{r}_1$ and $\boldsymbol{R}$ are related by $\boldsymbol{r}_1 + \boldsymbol{r} = \boldsymbol{R}$. After using equations (4) and (5) in (6) and evaluating the integral, the potential is found to be

$$\begin{aligned} V_{\Lambda C} = {} & -U_{\Lambda C}(1/4\pi a^2 b^2 \lambda^2)^{3/2} \\ & \times \{(1 + 9/4a^2\lambda^2 + 29/8a^4\lambda^4) \\ & \quad + (3/2a^2 b^4 \lambda^4 + 29/6a^4 b^4 \lambda^6) R^2 + (29/30 a^4 b^8 \lambda^8) R^4\} \\ & \times \exp\{-(1/b^2 - 1/b^4\lambda^2) R^2\}, \end{aligned} \tag{7}$$

where $\lambda = (1/a^2 + 1/b^2)^{\frac{1}{2}}$ and $U_{\Lambda C}$ is the potential depth.

3. Two-body Systems

For the Λ–^{29}Si system a variational calculation for $U_{\Lambda C}$ is carried out with the Λ–^{29}Si potential (7) and the wavefunction (2a). The variational principle takes the form

$$U_{\Lambda C} \leqslant (B_\Lambda \langle F \mid F\rangle + \langle F \mid T \mid F\rangle)/\langle F \mid V_{\Lambda C} \mid F\rangle, \tag{8}$$

where B_Λ is the Λ–^{29}Si separation energy. From the curve of B_Λ against $A^{-2/3}$ (Bhaduri *et al.* 1968), the separation energy for Λ in $^{30}_{\Lambda}$Si is found to be about 19·5 MeV. After calculation, the individual terms in (8) take the form

$$\langle F \mid F\rangle = \pi^{3/2}\left(\frac{x^2}{(2\alpha)^{3/2}} + \frac{2x}{(\alpha+\beta)^{3/2}} + \frac{1}{(2\beta)^{3/2}}\right), \tag{9}$$

$$\langle F \mid T \mid F\rangle = \frac{\hbar^2\pi^{3/2}}{\mu}\left(\frac{3\alpha^2x^2}{(2\alpha)^{5/2}} + \frac{6\alpha\beta x}{(\alpha+\beta)^{5/2}} + \frac{3\beta^2}{(2\beta)^{5/2}}\right), \tag{10}$$

$$\begin{aligned}\langle F \mid V_{\Lambda C} \mid F\rangle = \pi^{3/2}\Bigg(&\frac{lx^2}{(2\alpha+p)^{3/2}} + \frac{2lx}{(\alpha+\beta+p)^{3/2}} + \frac{l}{(2\beta+p)^{3/2}} \\ &+ \frac{3mx^2}{2(2\alpha+p)^{5/2}} + \frac{3mx}{(\alpha+\beta+p)^{5/2}} + \frac{3m}{2(2\beta+p)^{5/2}} \\ &+ \frac{15nx^2}{4(2\alpha+p)^{7/2}} + \frac{15nx}{2(\alpha+\beta+p)^{7/2}} + \frac{15n}{4(2\beta+p)^{7/2}}\Bigg),\end{aligned} \tag{11}$$

where l, m, n and p are given by

$$l = 1 + 9/4a^2\lambda^2 + 29/8a^4\lambda^4 = 3{\cdot}32583, \tag{12a}$$

$$m = 3/2a^2b^4\lambda^4 + 29/6a^4b^4\lambda^6 = 0{\cdot}2881\ \text{fm}^{-2}, \tag{12b}$$

$$n = 29/30a^4b^8\lambda^8 = 0{\cdot}00218\ \text{fm}^{-4}, \tag{12c}$$

$$p = 1/b^2 - 1/b^4\lambda^2 = 0{\cdot}88574\ \text{fm}^{-2}. \tag{12d}$$

Table 1. Two-body parameters for the Λ–^{29}Si and Λ–Λ systems
Separation energies: Λ–^{29}Si, 19·5 MeV; Λ–Λ, 0

Parameter	Value	Parameter	Value
	Λ–^{29}Si		Λ–Λ
x	5·275	y	2·9
α	0·6 fm^{-2}	γ	0·4 fm^{-2}
β	0·2 fm^{-2}	δ	0·027 fm^{-2}
$U_{\Lambda C}$	2320·0 MeV fm^3	$U_{\Lambda\Lambda}$	610·8 MeV fm^3

In equation (10), μ denotes the Λ–^{29}Si reduced mass and has a value of 1071·1335 MeV c^{-2}. The optimum parameters α, β, x and the corresponding value of $U_{\Lambda C}$ are given in Table 1.

For the Λ–Λ system, with the gaussian potential (3), we used the variational principle (8) to estimate the potential strength necessary to just bind the Λ–Λ system (i.e. $B = 0$). Values of the parameters y, γ, δ and the corresponding value of $U_{\Lambda\Lambda}$ thus determined are also given in Table 1.

4. Three-body System

The binding energy of ${}^{31}_{\Lambda\Lambda}$Si, where it is considered to be the three-body system Λ–Λ–^{29}Si, is determined by the variational principle as

$$B_{\Lambda\Lambda} \leqslant (\langle\psi \mid T \mid \psi\rangle - U_{\Lambda C}\langle\psi \mid V_{\Lambda_1 C} \mid \psi\rangle - U_{\Lambda C}\langle\psi \mid V_{\Lambda_2 C} \mid \psi\rangle - U_{\Lambda\Lambda}\langle\psi \mid V_{\Lambda\Lambda} \mid \psi\rangle)/\langle\psi \mid \psi\rangle, \tag{13}$$

where the individual terms are given by

$$\begin{aligned}\langle\psi \mid T \mid \psi\rangle = \int \Bigg[& \frac{\hbar^2(M_C + M_\Lambda)}{2M_C M_\Lambda}\left\{\left(\frac{\partial\psi}{\partial r_{C1}}\right)^2 + \left(\frac{\partial\psi}{\partial r_{C2}}\right)^2\right\} \\ & + \frac{\hbar^2}{M_\Lambda}\left(\frac{\partial\psi}{\partial r_{12}}\right)^2 + \frac{\hbar^2}{2M_C}\frac{r_{C1}^2 + r_{C2}^2 - r_{12}^2}{r_{C1} r_{C2}}\frac{\partial\psi}{\partial r_{C1}}\frac{\partial\psi}{\partial r_{C2}} \\ & + \frac{\hbar^2}{2M_\Lambda}\left(\frac{r_{C2}^2 + r_{12}^2 - r_{C1}^2}{r_{C2} r_{12}}\frac{\partial\psi}{\partial r_{C2}}\frac{\partial\psi}{\partial r_{12}}\right. \\ & \left. + \frac{r_{12}^2 + r_{C1}^2 - r_{C2}^2}{r_{12} r_{C1}}\frac{\partial\psi}{\partial r_{12}}\frac{\partial\psi}{\partial r_{C1}}\right)\Bigg] r_{C1} r_{C2} r_{12}\, dr_{C1}\, dr_{C2}\, dr_{12}, \end{aligned} \tag{14}$$

$$\langle\psi \mid V \mid \psi\rangle = \int \psi^* V \psi\, r_{C1} r_{C2} r_{12}\, dr_{C1}\, dr_{C2}\, dr_{12}, \tag{15}$$

$$\langle\psi \mid \psi\rangle = \int |\psi|^2 r_{C1} r_{C2} r_{12}\, dr_{C1}\, dr_{C2}\, dr_{12}. \tag{16}$$

In these integrals the volume element $d\tau = 8\pi^2 r_{C1} r_{C2} r_{12}\, dr_{C1}\, dr_{C2}\, dr_{12}$, as obtained in the triangular coordinate system, has been used.

Table 2. Three-body parameters for the Λ–Λ–^{29}Si system

Variational parameters	y	γ (fm^{-2})	δ (fm^{-2})	$U_{\Lambda\Lambda}$ (MeV fm^3)	x	α (fm^{-2})	β (fm^{-2})	$U_{\Lambda C}$ (MeV fm^3)	$B_{\Lambda\Lambda}$ (MeV)
y, γ, δ	3·5	0·928	0·046	610·8	5·275	0·6	0·2	2320·0	41·54
$y, \gamma, \delta, U_{\Lambda\Lambda}$	2·8	0·469	0·037	356·5	5·275	0·6	0·2	2320·0	39·70

5. Results and Discussion

In the first stage, for computational simplicity, we have in fact kept only the three parameters y, γ and δ free for variation and in the second stage we have taken $U_{\Lambda\Lambda}$ as a free parameter in addition to y, γ and δ. The best values of the parameters

$x, \alpha, \beta, y, \gamma, \delta$ and the maximum values of $B_{\Lambda\Lambda}$ obtained from the two calculations, along with other parameters, are shown in Table 2. The value of the binding energy $B_{\Lambda\Lambda}$ from the second calculation deviates from the first by about 5%. In this way we have derived the maximum values of $B_{\Lambda\Lambda}$ as a function of the strength of the Λ–Λ interaction. Fig. 3 shows the variation of $B_{\Lambda\Lambda}$ with $U_{\Lambda\Lambda}$.

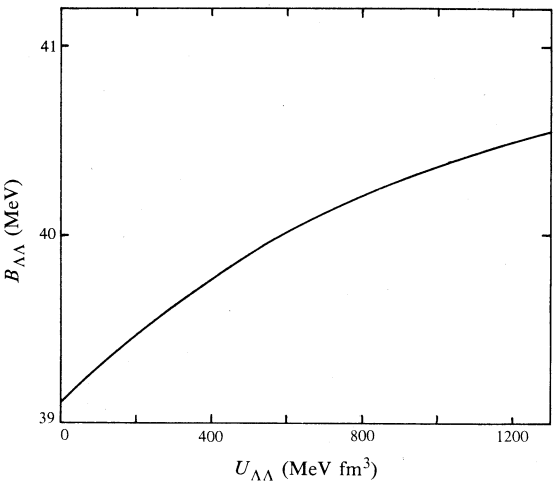

Fig. 3. Plot of $B_{\Lambda\Lambda}$ versus $U_{\Lambda\Lambda}$ for values of the parameters of $x = 5{\cdot}275$, $\alpha = 0{\cdot}6\ \text{fm}^{-2}$, $\beta = 0{\cdot}2\ \text{fm}^{-2}$, $U_{\Lambda c} = 2320\ \text{MeV fm}^3$, $y = 2{\cdot}8$, $\gamma = 0{\cdot}469\ \text{fm}^{-2}$ and $\delta = 0{\cdot}037\ \text{fm}^{-2}$.

If B_Λ is the Λ–^{29}Si separation energy, then for a rigid core the additional binding energy

$$\Delta B_{\Lambda\Lambda}({}^{31}_{\Lambda\Lambda}\text{Si}) = B_{\Lambda\Lambda}({}^{31}_{\Lambda\Lambda}\text{Si}) - 2B_\Lambda({}^{30}_{\Lambda}\text{Si})$$

may be directly related to the strength of the Λ–Λ interaction. In the present case we have $B_{\Lambda\Lambda} = 39{\cdot}7$ MeV, $B_\Lambda = 19{\cdot}5$ MeV and, thus $\Delta B_{\Lambda\Lambda}({}^{31}_{\Lambda\Lambda}\text{Si}) = 0{\cdot}7$ MeV, while the corresponding experimental value is $-0{\cdot}8 \pm 6{\cdot}3$ MeV. Our result for $\Delta B_{\Lambda\Lambda}$ is not inconsistent with the reported value, however, the experimental uncertainties in $B_{\Lambda\Lambda}({}^{31}_{\Lambda\Lambda}\text{Si})$ are rather too large to make any definite conclusion about the dependence of $\Delta B_{\Lambda\Lambda}$ on the mass number. For ${}^{10}_{\Lambda\Lambda}$Be, from the experimental values of Danysz *et al.* (1963), the additional binding energy is $4{\cdot}5 \pm 0{\cdot}5$ MeV, while for ${}^{6}_{\Lambda\Lambda}$He it is $5{\cdot}0 \pm 0{\cdot}6$ MeV, as calculated by Dalitz and Rajasekaran (1964), and $5{\cdot}1 \pm 0{\cdot}6$ MeV as calculated by Ali and Bodmer (1967). Thus one can easily conclude that the additional binding energy is not sensitive to the mass number of the hypernucleus. Since ${}^{31}_{\Lambda\Lambda}$Si is very heavy in comparison with these two hypernuclei, the value of $\Delta B_{\Lambda\Lambda}$ is too small. However, before any definite conclusion can be arrived at, other methods of calculation should be tested.

On the other hand, the present calculation shows that for $U_{\Lambda\Lambda} = 0$ we have $B_{\Lambda\Lambda} = 39{\cdot}1$ MeV with $\Delta B_{\Lambda\Lambda}({}^{31}_{\Lambda\Lambda}\text{Si}) = 0{\cdot}1$ MeV. This is not inconsistent with the observation that if the core is infinitely massive then one has, as expected, the relation $B_{\Lambda\Lambda} = 2B_\Lambda$ for $U_{\Lambda\Lambda} = 0$, i.e. the total binding energy of the three-body system must be just twice the binding energy of a Λ to an infinitely heavy core.

We conclude that, although our present variational calculations on the three-body model of ${}^{31}_{\Lambda\Lambda}$Si are commensurate with the results from the single experimentally observed event, the experimental uncertainties so far are still somewhat too large to extract more detailed information on the Λ–Λ interaction. In any case, more

events, when available, will be helpful in linking information on the lighter hypernuclei with that on heavy hypernuclear matter. Moreover, the four-body model Λ–Λ–n–^{28}Si for ${}^{31}_{\Lambda\Lambda}$Si is still to be tested.

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