

Evolution Equation and Transport Coefficients of Swarms in Initial Relaxation Processes

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Abstract

The problem of a swarm approaching the hydrodynamic regime is studied by using the projection operator method. An evolution equation for the density and the related time-dependent transport coefficient are derived. The effects of the initial condition on the transport characteristics of a swarm are separated from the intrinsic evolution of the swarms, and the difference from the continuity equation with time-dependent transport coefficients introduced by Tagashira *et al.* (1977, 1978) is discussed. To illustrate this method, calculations on the relaxation model collision operator have been carried out. The results are found to agree with the analysis by Robson (1975).

1. Introduction

Research on the space and time evolution of an isolated group (swarm) of charged particles in a neutral gas in the presence of a uniform electric field is of importance for defining and understanding the transport characteristics for times shorter than the relaxation time. The charged particles are assumed to be injected at a point in space with some initial distribution of velocities and then to approach a steady state distribution (i.e. the hydrodynamic regime).

The problem of a swarm approaching the hydrodynamic regime has been previously studied by solving the initial value problem of a velocity distribution function (see e.g. Skullerud 1974, 1977) or by solving a continuity equation with time-dependent transport coefficients (Tagashira *et al.* 1977, 1978). Time-dependent transport coefficients have been obtained by direct numerical analysis of the time-resolved Boltzmann equation (Kitamori *et al.* 1980) and by Monte Carlo simulation (McIntosh 1974; Braglia 1977; Lin and Bardsley 1977; Braglia and Baiocchi 1978). The aim of the present paper is to give an alternative analysis of swarm behaviour in the non-hydrodynamic regime by deriving the evolution of the density distribution in space and time. A projection operator which acts on the velocity distribution function is used to derive the precise evolution equation from the Boltzmann equation in Section 2*a*.

The evolution equation given here shows that the evolution characteristics in the short time region are essentially non-Markovian processes (equation 13), and the time derivative of the number density of the swarm $\partial_t n(r, t)$ should be expressed as a power series in the convolution integral of the density gradients with a time-dependent tensorial function (equation 20). The evolution equation derived in this paper also

shows the effect of the initial distribution of velocities on the transport characteristics of a swarm by separating them from the intrinsic evolution (equation 20).

The relation between the continuity equation with time-dependent transport coefficients (equation 5) and the new evolution equation is discussed in Section 3*b*. Generalised forms for the moment of the density of swarm and for the transport coefficient are given by means of a Laplace transformation (equations 18 and 27), in which a dependence on the initial velocity distribution is distinguished from the intrinsic components.

The theoretical analysis given here is applicable to the real problem for electron and ion swarms. To illustrate this analysis, calculations for the relaxation model of the collision operator with an arbitrary isotropic velocity distribution after collision are carried out (Section 3*a*). In the case of the Bogoliubov–Green–Kirkwood (BGK) model of the collision operator, calculations are found to give agreement with the analysis of Robson (1975). The effects of reactions on the evolution characteristics of swarms are also discussed in Section 3*c*.

2. Derivation of the Evolution Equation for the Number Density

(a) Boltzmann Equation and Continuity Equation

The velocity distribution function $f(v, r, t)$ at position r and time t must satisfy the Boltzmann equation

$$(\partial_t + \Gamma)f(v, r, t) = 0; \quad (1a)$$

$$\Gamma = v \cdot \nabla_r + a \cdot \nabla_v + J, \quad (1b)$$

$$a = (e/m)E, \quad Jf(v, r, t) = -(\partial_t f(v, r, t))_{\text{coll}}, \quad (1c, d)$$

where m and e are the mass and charge of the particles respectively.

The number density of particles at r is written as

$$n(r, t) = \int f(v, r, t) dv. \quad (2)$$

The present aim is to solve the initial value problem of equations (1) and (2) for a given initial velocity distribution and to give a proper expression for the evolution of $n(r, t)$ in the initial relaxation processes.

Previous methods have been based on the continuity equation (see e.g. Tagashira *et al.* 1977). The continuity equation is obtained by integrating equation (1) with respect to v :

$$\partial_t n(r, t) + \nabla \cdot j(r, t) = S(r, t); \quad (3a)$$

$$j(r, t) = V(r, t) n(r, t), \quad S(r, t) = (\partial_t n(r, t))_{\text{coll}}. \quad (3b, c)$$

Equation (3a) states that the changes in $n(r, t)$ are due to the convective particle current $V(r, t) n(r, t)$ and the production term $S(r, t)$.

When a long time has elapsed the effect of the initial velocity distribution vanishes and the hydrodynamic regime is established. The distribution function becomes a functional of $n(r, t)$ and the particle flow $j(r, t)$ can be expressed as a power series

in the gradient operator with constant coefficients. Equation (3a) then takes the form (Kumar and Robson 1973)

$$\left(\partial_t - \sum_{j=0}^{\infty} \omega^{(j)} \odot (-\nabla)^j \right) n(r, t) = 0. \quad (4)$$

The constants $\omega^{(j)}$ are the symmetric tensorial transport coefficients of rank j and \odot denotes a j -fold scalar product. In order to express the development in time before the hydrodynamic regime has been established, Tagashira *et al.* (1977, 1978) transformed the continuity equation (4) into a power series of the gradient operator but with time-dependent coefficients:

$$\left(\partial_t - \sum_{j=0}^{\infty} \omega^{(j)}(t) \odot (-\nabla)^j \right) n(r, t) = 0. \quad (5)$$

Kumar (1981) discussed a treatment of the problem. Further, an explicit analytic solution of the initial value problem for a swarm with the BGK model collision operator has been given by Robson (1975).

(b) Projection Operator

In order to derive the evolution equation and the time-dependent transport coefficients which describe the development of a swarm exactly at all times we use a 'projection operator' method. The properties of this operator are briefly introduced in this subsection.

The projection operator p acts on the phase-space distribution and is defined by (see e.g. Messiah 1961)

$$pf(v, r, t) = \phi_0(v) \int \psi(v') f(v', r, t) dv', \quad (6)$$

where $\phi_0(v)$ and $\psi(v)$ are functions of the variable v only and satisfy

$$\int \psi(v) \phi_0(v) dv = 1. \quad (7)$$

From these definitions, it can be shown that the operator p satisfies the relation $p^2 = p$. The operator defined as $p' = 1 - p$ satisfies $p'^2 = p'$ and $p'p = pp' = 0$. Using the operator p and p' the velocity distribution function $f(v, r, t)$ can be written as

$$f(v, r, t) = pf(v, r, t) + p'f(v, r, t). \quad (8)$$

If the function $\psi(v)$ is chosen to be unity, then the projected function pf is given by

$$pf(v, r, t) = \phi_0(v) \int f(v, r, t) dv = \phi_0(v) n(r, t). \quad (9)$$

The function $\phi_0(v)$ may be chosen to be the velocity distribution function in the steady state case under a constant external acceleration of $a = (e/m)E$.

(c) *General Expression of the Evolution Equation*

In order to simplify the discussion, it is assumed that no particles are produced by collision, i.e. $\int Jf(v, r, t) dv = 0$, and therefore

$$\Gamma_0 \phi_0(v) = 0, \quad (10a)$$

where

$$\Gamma_0 = a \cdot \nabla_v + J, \quad (10b)$$

$$p\Gamma_0 = \Gamma_0 p = 0. \quad (11)$$

Applying p and p' to both sides of the Boltzmann equation we have

$$\partial_t pf = -(p\Gamma pf + p\Gamma p'f), \quad \partial_t p'f = -(p'\Gamma pf + p'\Gamma p'f), \quad (12a, b)$$

where $\Gamma = \Gamma_0 + \Gamma_1$ and $\Gamma_1 = v \cdot \nabla_r$.

By solving equation (12b) for $p'f$ with the initial condition $f(v, r, t=0)$ and eliminating $p'f$ from equation (12a), the generalised evolution equation for the particle density $n(r, t)$ is obtained as (Zwanzig 1964; Mori 1965)

$$\begin{aligned} \partial_t \phi_0(v) n(r, t) = & -p\Gamma_1 \phi_0(v) n(r, t) \\ & + p\Gamma_1 \int_0^t d\tau e^{-(t-\tau)p'\Gamma} p'\Gamma \phi_0(v) n(r, \tau) \\ & - p\Gamma_1 e^{-t p'\Gamma} \{f(v, r, t=0) - \phi_0(v) n(r, t=0)\}, \end{aligned} \quad (13)$$

where $n(r, t=0)$ is the initial number density of particles. The second term of the right-hand side of (13) shows the characteristics of non-Markovian evolution processes (e.g. Alder and Alley 1978, 1981), and the third term expresses the memory of the initial velocity distribution. In the discussion that follows, we assume an initial velocity distribution of the form

$$f(v, r, t=0) = f_0(v) n(r, t=0); \quad n(r, t=0) = \delta(r). \quad (14a, b)$$

(A more general form of the initial distribution of velocities is discussed at the end of this subsection.) If the initial velocity distribution function $f_0(v)$ is chosen to be the steady state distribution $\phi_0(v)$, the third term vanishes and no memory of the initial velocity distribution exists in the evolution characteristics of the number density.

By introducing the Fourier and Laplace transforms of $n(r, t)$

$$\hat{N}(k, s) = \int_0^\infty dt \int dr e^{-ik \cdot r - st} n(r, t), \quad \hat{N}_0(k) = \int dr e^{-ik \cdot r} n(r, t=0), \quad (15a, b)$$

and using relation (11), the evolution equation (13) can be transformed into

$$\begin{aligned} \left\{ s + p\Gamma_k - p\Gamma_k \left(\frac{1}{s + p'\Gamma_k} \right) p'\Gamma_k \right\} \phi_0(v) \hat{N}(k, s) \\ = \left\{ \phi_0(v) - p\Gamma_k \left(\frac{1}{s + p'\Gamma_k} \right) p'f_0(v) \right\} \hat{N}_0(k), \end{aligned} \quad (16a)$$

where

$$\Gamma_k = \Gamma_0 + i k \cdot v, \quad p' \Gamma_k = i k \cdot p' v + \Gamma_0 = i k \cdot (v - p v) + \Gamma_0. \quad (16b, c)$$

Since the operator $(s + p' \Gamma_k)^{-1}$ can be written as the polynomial power series

$$\frac{1}{s + p' \Gamma_k} = \sum_{n=0}^{\infty} (-i k)^n \left(\frac{1}{s + \Gamma_0} \right) \left\{ (p' v) \left(\frac{1}{s + \Gamma_0} \right) \right\}^n, \quad (17)$$

equation (16a) becomes

$$\left(s - \sum_{j=1}^{\infty} (-i k)^j \hat{\Omega}^{(j)}(s) \right) \hat{N}(k, s) = \left(1 + \sum_{j=1}^{\infty} (-i k)^j \hat{\Theta}^{(j)}(s) \right) \hat{N}_0(k), \quad (18a)$$

where

$$\hat{\Omega}^{(1)}(s) = \int dv v \phi_0(v) = \langle v \rangle, \quad (18b)$$

$$\hat{\Omega}^{(2)}(s) = \int dv v \left(\frac{1}{s + \Gamma_0} \right) (v - \langle v \rangle) \phi_0(v), \quad (18c)$$

$$\hat{\Omega}^{(j)}(s) = \int dv v \left(\frac{1}{s + \Gamma_0} \right) \left\{ (p' v) \left(\frac{1}{s + \Gamma_0} \right) \right\}^{j-2} (v - \langle v \rangle) \phi_0(v), \quad (j \geq 3); \quad (18d)$$

$$\hat{\Theta}^{(1)}(s) = \int dv v \left(\frac{1}{s + \Gamma_0} \right) \{ f_0(v) - \phi_0(v) \}, \quad (18e)$$

$$\hat{\Theta}^{(j)}(s) = \int dv v \left(\frac{1}{s + \Gamma_0} \right) \left\{ (p' v) \left(\frac{1}{s + \Gamma_0} \right) \right\}^{j-1} \{ f_0(v) - \phi_0(v) \}, \quad (j \geq 2). \quad (18f)$$

Applying inverse transformations, the density in space and time is obtained as

$$n(r, t) = (2\pi)^{-3} \int ds \int dk e^{i k \cdot r + s t} \hat{N}(k, s). \quad (19)$$

Therefore the evolution equation for the density, valid for all times after the injection of the particles is given by

$$\begin{aligned} \partial_t n(r, t) = & \sum_{j=1}^{\infty} \int_0^t d\tau \Omega^{(j)}(t - \tau) \odot (-\nabla)^j n(r, \tau) \\ & + \sum_{j=1}^{\infty} \Theta^{(j)}(t) \odot (-\nabla)^j n(r, t=0), \end{aligned} \quad (20a)$$

where $\mathbf{\Omega}^{(j)}(t)$ and $\mathbf{\Theta}^{(j)}(t)$ are tensorial functions of rank j

$$\mathbf{\Omega}^{(j)}(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \hat{\mathbf{\Omega}}^{(j)}(s) ds, \quad (20b)$$

$$\mathbf{\Theta}^{(j)}(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \hat{\mathbf{\Theta}}^{(j)}(s) ds, \quad (20c)$$

and where c is the axis of conversion. This is a more general form of the expression by Alley and Alder (1979, equation 6). Equation (20a) can be applied when external forces are present and should be compared with the continuity equation with time-dependent transport coefficients (equation 5). Both $\mathbf{\Omega}^{(j)}(t)$ and $\mathbf{\Theta}^{(j)}(t)$ can be derived for a given operator J , and quantities such as the drift of the centroid and the increase of the mean square displacement of particles from the centroid may be obtained exactly.

Equation (20a) shows that the evolution of a swarm consists of two parts. From equations (18), it can be seen that the function $\mathbf{\Omega}^{(j)}(t)$ is independent of the initial velocity distribution and is given only by the steady state distribution $\phi_0(v)$. Therefore, the first part of the right-hand side of (20a) shows intrinsic characteristics of evolution not affected by the initial velocity distribution. This part of the time evolution of the density of the swarm is a convolution integral between the time-dependent functions $\mathbf{\Omega}^{(j)}(t)$ and the gradients of the density.

As time elapses after the injection of particles, the time evolution operator $\exp(-tp'\Gamma)$ [or in the Fourier-Laplace transformation form $(s+p'\Gamma_k)^{-1}$] approaches zero, and it is clear that both $\mathbf{\Omega}^{(j)}(t)$ and $\mathbf{\Theta}^{(j)}(t)$ tend to zero. Therefore for large times, the first part of (20a) becomes equivalent to the conventional continuity equation with constant transport coefficients, i.e. equation (4), thus establishing the hydrodynamic regime:

$$\partial_t n(r, t) \approx \sum_{j=1}^{\infty} \left(\int_{-\infty}^t \mathbf{\Omega}^{(j)}(t-\tau) d\tau \right) \odot (-\nabla)^j n(r, t). \quad (21)$$

The constant tensorial coefficients $\omega^{(j)}$ are given approximately by

$$\omega^{(j)} \approx \int_{-\infty}^t \mathbf{\Omega}^{(j)}(t-\tau) d\tau = \int_0^{\infty} \mathbf{\Omega}^{(j)}(\tau) d\tau. \quad (22)$$

The second part of (20a) and the right-hand side of (18a) express the memory effects of the velocity distribution of the injected particles. As shown in these equations, the function $\mathbf{\Theta}^{(j)}(t)$ depends on the difference between the velocity distribution functions corresponding to the initial and steady state. If the initial velocity distribution $f_0(v)$ is chosen to be the same as that of the steady state, then $\mathbf{\Theta}^{(j)}(t)$ becomes zero.

Equation (14a) is not the most general form of the initial distribution. A more general expression is

$$f(v, r, t=0) = \sum_{i=0}^{\infty} f_i(v) \odot (-\nabla)^i n(r, t=0), \quad (23)$$

which is related to the solution by Tagashira *et al.* (1978, equation 4) for the Boltzmann equation (1a). We can repeat the analysis for this distribution and show

that the intrinsic function $\Omega^{(j)}(t)$ is independent of the function $f_i(v)$, while $\Theta^{(j)}(t)$ is dependent on the higher order expansion terms $f_i(v)$ ($i \geq 1$) for $j \geq 2$.

Thus, even with the more general form for the initial distribution, the analysis presented in this subsection leads to the separation of the intrinsic evolution from the effects of the initial distribution.

3. Moment of the Number Density and Time-dependent Transport Coefficients

(a) Centroid and Mean Square Displacement

From equations (18), one can derive general formulas which express exactly the position of the centroid $\langle r \rangle(t)$ and the mean square displacement of the pulse $\langle R^2 \rangle(t)$. These quantities are denoted by

$$\langle r \rangle(t) = \int r n(r, t) dt / n_0, \quad \langle r^2 \rangle(t) = \int r^2 n(r, t) dt / n_0, \quad (24a, b)$$

$$\langle R^2 \rangle(t) = \langle r^2 \rangle(t) - \{\langle r \rangle(t)\}^2, \quad (25)$$

respectively, where the total number of particles $n_0 = \int n(r, t) dr$ and $R = r - \langle r \rangle(t)$ is the position of the particles relative to the centroid, and may be determined by an idealised time-of-flight experiment.

If the initial pulse is described by a delta-function in space, the j th moment of the number density $\langle r^j \rangle(t)$ and its time derivative can be derived from the relation

$$n_0 \langle \hat{r}^j \rangle(s) = \frac{1}{(-i)^j} \left(\frac{\partial}{\partial k} \right)^j \hat{N}(k, s) |_{k \rightarrow 0}, \quad (26)$$

with

$$\begin{aligned} \langle r^j \rangle(t) &= \mathcal{L}^{-1} \langle \hat{r}^j \rangle(s) \\ &= (j!) \left\{ \sum_{h=0}^{j-1} \left(\int_0^t \Omega^{(j-h)}(\tau) \int_0^{t-\tau} \frac{\langle r^h \rangle(t')}{h!} dt' d\tau \right) + \int_0^t \Theta^{(j)}(\tau) d\tau \right\}. \end{aligned} \quad (27)$$

$$\begin{aligned} \frac{d}{dt} \langle r^j \rangle(t) &= \mathcal{L}^{-1} s \langle \hat{r}^j \rangle(s) \\ &= (j!) \left\{ \sum_{h=0}^{j-1} \left(\int_0^t \Omega^{(j-h)}(t-\tau) \frac{\langle r^h \rangle(\tau)}{h!} d\tau \right) + \Theta^{(j)}(t) \right\}, \end{aligned} \quad (28)$$

where $\langle \hat{r}^j \rangle(s)$ is the Laplace transformation of the j th moment of the number density $\langle r^j \rangle(t)$ and the operators \mathcal{L} and \mathcal{L}^{-1} represent the Laplace and inverse Laplace transformation respectively.

The motion of the centroid $\langle r \rangle(t)$ and the second moment of the number density $\langle r^2 \rangle(t)$ are given by

$$\langle r \rangle(t) = \int_0^t \Omega^{(1)}(\tau)(t-\tau) d\tau + \int_0^t \Theta^{(1)}(\tau) d\tau = \langle v \rangle t + \int_0^t \Theta^{(1)}(\tau) d\tau, \quad (29)$$

$$\langle r^2 \rangle(t) = 2 \int_0^t \Omega^{(2)}(\tau)(t-\tau) d\tau + 2 \int_0^t \Theta^{(2)}(\tau) d\tau + 2\langle v \rangle \int_0^t \langle r \rangle(\tau) d\tau. \quad (30)$$

Therefore, the mean square displacement of the pulse is

$$\begin{aligned} \langle R^2 \rangle(t) = & 2 \int_0^t \Omega^{(2)}(\tau)(t-\tau) d\tau \\ & + 2 \int_0^t \Theta^{(2)}(\tau) d\tau - 2\langle v \rangle \int_0^t \tau \Theta^{(1)}(\tau) d\tau - \left(\int_0^t \Theta^{(1)}(\tau) d\tau \right)^2, \end{aligned} \quad (31)$$

where the first term is the component independent of the initial velocity distribution, and represents intrinsic diffusion and corresponds to the expression for the mean square displacement in terms of the time correlation function (Kubo 1966).

The function $\Omega^{(2)}(t)$ is identified as the autocorrelation function of the random velocity v (Skullerud 1974; Braglia 1980; Kumar *et al.* 1980):

$$\Omega^{(2)}(t) = \langle v(0) v(t) \rangle. \quad (32)$$

For t long enough after a swarm is initiated, the mean square displacement of the pulse can be simply written as

$$\langle R^2 \rangle(t) \rightarrow 2(Dt + A), \quad (33)$$

where

$$D = \int_0^\infty \Omega^{(2)}(\tau) d\tau, \quad (34a)$$

$$A = \int_0^\infty \tau \Omega^{(2)}(\tau) d\tau + \int_0^\infty \Theta^{(2)}(\tau) d\tau - \frac{1}{2} \left(\int_0^\infty \Theta^{(1)}(\tau) d\tau \right)^2. \quad (34b)$$

The diffusion coefficient can be also obtained from equation (18c) as a limiting value of $s \rightarrow 0$ as follows:

$$D - \lim_{s \rightarrow 0} \hat{\Omega}^{(2)}(s) = \int dv v(a \cdot \nabla_v + J)^{-1} (v - \langle v \rangle) \phi_0(v). \quad (35)$$

The drift velocity and the diffusion coefficient at time t are described by

$$W_r(t) = \frac{d}{dt} \langle r \rangle(t) = \int_0^t \Omega^{(1)}(\tau) d\tau + \Theta^{(1)}(t) = \langle v \rangle + \Theta^{(1)}(t), \quad (36)$$

$$\begin{aligned} D(t) = \frac{1}{2} \frac{d}{dt} \langle R^2 \rangle(t) = & \int_0^t \Omega^{(2)}(\tau) d\tau + \Theta^{(2)}(t) \\ & - \Theta^{(1)}(t) \left(\int_0^t \Theta^{(1)}(\tau) d\tau + \langle v \rangle t \right). \end{aligned} \quad (37)$$

Now we consider a relaxation model of the collision operator given by

$$Jf(v, r, t) = \nu \left(1 - \phi_s(v) \int dv \right) f(v, r, t), \quad (38a)$$

where $\phi_s(v)$ is an arbitrary isotropic velocity distribution function satisfying the relations

$$\int \phi_s(v) dv = 1, \quad \int v \phi_s(v) dv = 0. \quad (38b, c)$$

In this case the drift velocity $\langle v \rangle$ and the mean square velocity in the steady state case are given from (10) as

$$\langle v \rangle = a\nu^{-1}, \quad \langle v^2 \rangle = 2\langle v \rangle^2 + \int dv v^2 \phi_s(v), \quad (39a, b)$$

and we can transform equations (18) into the real time domain by using

$$\frac{1}{s + \Gamma_0} = \frac{1}{s} - \frac{\Gamma_0}{s^2} + \frac{(\Gamma_0)^2}{s^3} - \dots, \quad (40a)$$

$$\int dv v (\Gamma_0)^{n+1} h(v) = \nu \int dv v (\Gamma_0)^n h(v) \quad (n \geq 1), \quad (40b)$$

$$\begin{aligned} \int dv v^2 (\Gamma_0)^{n+1} h(v) &= \nu \int dv v^2 (\Gamma_0)^n h(v) \\ &\quad - 2a\nu^{n-1} \int dv v (\Gamma_0) h(v), \end{aligned} \quad (40c)$$

\vdots

$$\int dv v (\Gamma_0) h(v) = \nu \int dv v h(v) - a \int dv h(v), \quad (40d)$$

$$\begin{aligned} \int dv v^2 (\Gamma_0) h(v) &= \nu \int dv v^2 h(v) - (\langle v^2 \rangle - 2\langle v \rangle^2) \nu \int dv h(v) \\ &\quad - 2a \int dv v h(v), \end{aligned} \quad (40e)$$

for an arbitrary function $h(v)$ satisfying $h(\infty) \equiv 0$.

Then, the functions $\Omega^{(j)}(t)$ and $\Theta^{(j)}(t)$ in (18) and (27) can be easily obtained:

$$\Omega^{(2)}(t) = (\langle v^2 \rangle - \langle v \rangle^2) e^{-\nu t}, \quad (41)$$

$$\Theta^{(1)}(t) = (\langle v \rangle_0 - \langle v \rangle) e^{-\nu t}, \quad (42)$$

$$\begin{aligned} \Theta^{(2)}(t) &= \{ \langle v^2 \rangle_0 - \langle v^2 \rangle - \langle v \rangle \langle v \rangle_0 \} t e^{-\nu t} \\ &\quad + a(\langle v \rangle_0 - \langle v \rangle) t^2 e^{-\nu t}. \end{aligned} \quad (43)$$

Consequently, the centroid and the second moment of the particles and the time-dependent drift velocity and diffusion coefficient are given by

$$\langle r \rangle(t) = \langle v \rangle t + (\langle v \rangle_0 - \langle v \rangle) \nu^{-1} (1 - e^{-\nu t}), \quad (44)$$

$$\begin{aligned} \langle r^2 \rangle(t) = & \langle v \rangle^2 t^2 \\ & + 2\{2\langle v^2 \rangle - \langle v^2 \rangle_0 + \langle v \rangle \langle v \rangle_0 - 3\langle v \rangle^2 - 2a\nu^{-1}(\langle v \rangle_0 - \langle v \rangle)\} \nu^{-1} t \\ & + 2\{\langle v^2 \rangle_0 - 2\langle v^2 \rangle + 3\langle v \rangle^2 - 2\langle v \rangle \langle v \rangle_0 + 2a\nu^{-1}(\langle v \rangle_0 - \langle v \rangle)\} \nu^{-2} (1 - e^{-\nu t}) \\ & - 2\{\langle v^2 \rangle_0 - \langle v^2 \rangle - \langle v \rangle \langle v \rangle_0 + \langle v \rangle^2 + 2a\nu^{-1}(\langle v \rangle_0 - \langle v \rangle)\} \nu^{-1} t (1 - e^{-\nu t}) \\ & - 2a\nu^{-1}(\langle v \rangle_0 - \langle v \rangle) t^2 e^{-\nu t}, \end{aligned} \quad (45)$$

$$W_r(t) = \langle v \rangle + (\langle v \rangle_0 - \langle v \rangle) e^{-\nu t}, \quad (46)$$

$$\begin{aligned} D(t) = & (\langle v^2 \rangle - \langle v \rangle^2) \nu^{-1} \\ & + 2(\langle v \rangle_0 \langle v \rangle - \langle v^2 \rangle_0 - \langle v^2 \rangle) \nu^{-1} e^{-\nu t} \\ & + \{\langle v^2 \rangle_0 - \langle v^2 \rangle - 2\langle v \rangle(\langle v \rangle_0 - \langle v \rangle)\} t e^{-\nu t} \\ & + a(\langle v \rangle_0 - \langle v \rangle) t^2 e^{-\nu t} + (\langle v \rangle_0 - \langle v \rangle)^2 \nu^{-1} e^{-2\nu t}, \end{aligned} \quad (47)$$

where

$$\langle v \rangle_0 = \int v f_0(v) dv, \quad \langle v^2 \rangle_0 = \int v^2 f_0(v) dv$$

are respectively the initial drift velocity and the initial mean square velocity.

For times much shorter than ν^{-1} , immediately after the swarm is released, the centroid (equation 44) and the mean square displacement can be expressed as

$$\langle r \rangle(t) \approx \langle v \rangle_0 t + \frac{1}{2} a t^2, \quad \langle R^2 \rangle(t) \approx (\langle v^2 \rangle_0 - \langle v \rangle_0^2) t^2. \quad (48a, b)$$

These equations show that before many collisions have occurred, the swarm exhibits behaviour similar to that of charged particles in the absence of neutral molecules; this has been pointed out by Robson (1975).

For the BGK model collision operator, the function $\phi_s(v)$ is given by a Maxwellian of temperature T as

$$\phi_s(v) = (\alpha^2/2\pi)^{\frac{3}{2}} \exp(-\frac{1}{2}\alpha^2|v|^2), \quad (49a)$$

where

$$\alpha^2 = m/kT. \quad (49b)$$

The mean square velocity $\langle v^2 \rangle$ in the steady state case is obtained from equations (39) as

$$\langle v^2 \rangle = 2aav^{-2} + \alpha^{-2}[\mathbf{1}], \quad (50)$$

where $[\mathbf{1}]$ is the unit tensor.

When the swarm is initially injected with the same form as equation (49a), but with different temperature, i.e. $\langle v \rangle_0 = 0$ and $\langle v^2 \rangle_0 = (\alpha')^{-2}[\mathbf{1}]$, the present expressions for the centroid and the mean square displacement at time t , equations (44) and (45), agree with Robson (1975) (his equation 13).

(b) Comparison with Time-dependent Transport Coefficients

In the expression for the time-dependent transport coefficients of Kitamori *et al.* (1980), the effects of the initial velocity distribution on the coefficients are not separated from the intrinsic time-dependent component. Equation (20a), however, shows clearly the dependence on the initial velocity distribution in the second term on the right-hand side.

When the continuity equation is obtained by integrating the Boltzmann equation (1a) with respect to velocity v , the statistical information concerning the random velocity v present in the initial velocity distribution and the development of the swarm is not explicitly represented. On the other hand, in the present derivation of the evolution equation (20a) and the time-dependent parameters by the use of projection operators, we distinguish in the velocity distribution function a 'relevant' part denoted by $pf(v, r, t)$ and an 'irrelevant' part denoted by $p'f(v, r, t)$, as shown in equation (8), and obtain a generalised equation of evolution for the 'relevant' part, $pf(v, r, t) = \phi_0(v) n(r, t)$ (see equations 12 and 13). This 'relevant' part contains all the information which is necessary for the calculation of the number density at any time and we shall see that (13) is equivalent to the original Boltzmann equation, containing the same statistical information as (1a). This is the reason the projection operator method is useful in expressing the short time development of swarms in the initial relaxation processes.

Equations (5) and (20a) are, respectively, the expansion of $\partial_t n(r, t)$ from the time-dependent transport coefficient method and from the present method using projection operators. Equating the expansion terms of both equations and using

$$f(t-\tau) = e^{-\tau \partial_t} f(t) = \sum_{h=0}^{\infty} \frac{(-\tau)^h}{h!} (\partial_t)^h f(t) \quad (51)$$

gives the relation

$$\begin{aligned} \sum_{j=1}^{\infty} \omega^{(j)}(t) \odot (-\nabla)^j n(r, t) &= \sum_{j=1}^{\infty} \sum_{h=0}^{\infty} \left(\int_0^t \frac{(-\tau)^h}{h!} \Omega^{(j)}(\tau) d\tau \right) \odot (-\nabla)^j (\partial_t)^h n(r, t) \\ &+ \sum_{j=1}^{\infty} \Theta^{(j)}(t) \odot (-\nabla)^j n(r, t=0). \end{aligned} \quad (52)$$

A relationship between the coefficients of these expansions can be derived from the time derivative of the j th moment of the number density from equation (5),

$$\frac{d}{dt} \langle r^j \rangle(t) = (j!) \left(\sum_{h=0}^{j-1} \omega^{(j-h)}(t) \frac{\langle r^h \rangle(t)}{h!} \right), \quad (53)$$

and the equivalent equation by the projection operator method from the right-hand side of (52), which is equivalent to (28). Equating both expressions for $(d/dt) \langle r^j \rangle(t)$, the following relation between the coefficients can be obtained:

$$\begin{aligned} \sum_{h=0}^{j-1} \omega^{(j-h)}(t) \frac{\langle r^h \rangle(t)}{h!} &= \sum_{h=0}^{j-1} \sum_{n=0}^{\infty} \left(\int_0^t \frac{(-\tau)^n}{n!} \Omega^{(j-h)}(\tau) d\tau \right) \\ &\times \left(\frac{d}{dt} \right)^n \frac{\langle r^h \rangle(t)}{h!} + \Theta^{(j)}(t). \end{aligned} \quad (54)$$

In the intrinsic case where $\Theta^{(j)}(t) = 0$ and $\langle r_i \rangle(t) = \langle v \rangle t$ (i.e. there are no effects present due to the initial distribution), the time-dependent transport coefficients can be obtained from the function $\Omega^{(j)}(t)$. Using equation (54) with $j = 1, 2$ and 3 , and solving for $\omega_i^{(j)}(t)$ gives

$$\omega_i^{(1)}(t) = \int_0^t \Omega^{(1)}(\tau) d\tau = \langle v \rangle, \quad \omega_i^{(2)}(t) = \int_0^t \Omega^{(2)}(\tau) d\tau, \quad (55a, b)$$

$$\omega_i^{(3)}(t) = \int_0^t \Omega^{(3)}(\tau) d\tau - \int_0^t \Omega^{(1)}(\tau) d\tau \int_0^t \tau \Omega^{(2)}(\tau) d\tau, \quad (55c)$$

where the subscript i denotes the intrinsic case. Higher order transport coefficients $\omega_i^{(j)}(t)$ ($j \geq 4$) are obtained from equation (54) in a similar manner.

(c) Effects of Reactions

In order to make the analysis simple, the discussion in the previous sections was for no source term. We now discuss the case where ionisation cannot be neglected. Here the function $\phi_0(v)$ does not satisfy equation (10a) due to the presence of ionisation, and as a result the operators $p\Gamma_0$ and $\Gamma_0 p$ in (11) are no longer zero. The operator $p\Gamma_1$ on the right-hand side of (13) must be replaced by

$$p\Gamma = p\Gamma_1 + pJ, \quad (56)$$

and the projection operator $p = \phi_0(v) \int dv$ should be defined by a new function $\phi_0(v)$ satisfying the relations

$$(\Gamma_0 + R)\phi_0(v) = 0, \quad R = - \int J\phi_0(v) dv, \quad (57a, b)$$

where R is an ionisation frequency.

The generalised evolution equation including the source term can then be given in the same form as (20a) but including terms with $j = 0$. The new functions $\Omega^{(0)}(t)$ and $\Theta^{(0)}(t)$ which express the intrinsic reaction term and the effect of the

initial velocity distribution respectively in the evolution equation are given by Laplace transform expressions as

$$\hat{\Omega}^{(0)}(s) = R, \quad (58a)$$

$$\hat{\Theta}^{(0)}(s) = - \int dv J \left(\frac{1}{s + \Gamma_0 - pJ} \right) \{f_0(v) - \phi_0(v)\}. \quad (58b)$$

The other functions $\hat{\Omega}^{(j)}(s)$ and $\hat{\Theta}^{(j)}(s)$ can also be obtained as

$$\hat{\Omega}^{(1)}(s) = \langle v \rangle - \int dv J \left(\frac{1}{s + \Gamma_0 - pJ} \right) (v - \langle v \rangle) \phi_0(v), \quad (59a)$$

⋮

$$\begin{aligned} \hat{\Omega}^{(j)}(s) = & \int dv v \left(\frac{1}{s + \Gamma_0 - pJ} \right) \left\{ (p'v) \left(\frac{1}{s + \Gamma_0 - pJ} \right) \right\}^{j-2} (v - \langle v \rangle) \phi_0(v) \\ & - \int dv J \left(\frac{1}{s + \Gamma_0 - pJ} \right) \left\{ (p'v) \left(\frac{1}{s + \Gamma_0 - pJ} \right) \right\}^{j-1} (v - \langle v \rangle) \phi_0(v), \end{aligned} \quad (59b)$$

$$\begin{aligned} \hat{\Theta}^{(1)}(s) = & \int dv v \left(\frac{1}{s + \Gamma_0 - pJ} \right) \{f_0(v) - \phi_0(v)\} \\ & - \int dv J \left(\frac{1}{s + \Gamma_0 - pJ} \right) (p'v) \left(\frac{1}{s + \Gamma_0 - pJ} \right) \{f_0(v) - \phi_0(v)\}. \end{aligned} \quad (59c)$$

$$\begin{aligned} \hat{\Theta}^{(j)}(s) = & \int dv v \left(\frac{1}{s + \Gamma_0 - pJ} \right) \left\{ (p'v) \left(\frac{1}{s + \Gamma_0 - pJ} \right) \right\}^{j-1} \{f_0(v) - \phi_0(v)\} \\ & - \int dv J \left(\frac{1}{s + \Gamma_0 - pJ} \right) \left\{ (p'v) \left(\frac{1}{s + \Gamma_0 - pJ} \right) \right\}^j \{f_0(v) - \phi_0(v)\}, \end{aligned} \quad (j \geq 2). \quad (59d)$$

These equations show the effects of the birth of new particles on the evolution characteristics of the swarm.

If the operator $(\Gamma_0 - pJ)$ satisfies the following relation for an arbitrary function $h(v, t)$,

$$\int dv J(\Gamma_0 - pJ) h(v, t) = \nu_r \int dv J h(v, t), \quad (60a)$$

where

$$h(v, t) = g(v, t) - \phi_0(v), \quad \int g(v, t) dv = 1, \quad (60b, c)$$

then the time-dependent ionisation frequency $R(t)$ can be approximately given as

$$R(t) = R + (R_0 - R)e^{-\nu_r t}, \quad (61a)$$

where

$$R_0 = - \int J f_0(v) dv \quad (61b)$$

is the ionisation frequency at the initial time ($t = 0$). This result is a consequence of the fact that under the conditions of equations (60) the mean value of the velocity distribution function taken over the whole group of the particles, $g(v, t) = \int f(v, r, t) dr / n(t)$, can be presented by the Boltzmann equation with the collision term of a relaxation model as follows:

$$\partial_t g(v, t) = -\Gamma_0 g(v, t) = -\nu_r \{g(v, t) - \phi_0(v)\} + \phi_0(v) R(t). \quad (62)$$

The second term on the right-hand side of (62) represents ionisation and indicates that new particles just after ionisation have the velocity distribution $\phi_0(v)$ of the steady state case. Equation (61a) is an approximate form of the time-dependent ionisation frequency derived under the condition $\nu_r \gg R$.

Equations (58), (59) and (61) can be used to explain the time-dependent characteristic and the initial velocity dependence of the transport coefficients found by studies using Monte Carlo simulation (McIntosh 1974) and the numerical analysis of the Boltzmann equation (Kitamori *et al.* 1978, 1980).

4. Concluding Remarks

A general description of swarm evolution has been derived which includes all the processes that occur from the time of the injection of particles into the gas to the beginning of the hydrodynamic regime. The intrinsic time-dependent transport coefficients and the effects of the initial velocity distribution on the swarm evolution can be analytically calculated from the theory given here.

It should be noted that although the conventional continuity equation (4) and the continuity equation with time-dependent coefficients (5) appear very similar, their effects on the evolution of the swarm are quite different. The expansion terms of equation (5) or, equivalently, those in the first part of the right-hand side of (20a) can be expressed as expansions in the time derivatives of the number density, as shown in (52). Therefore, it is apparent that the exact evolution equation in its differential form should involve terms in $(\partial_t)^j n(r, t)$ with $j \geq 2$ and would not be the typical parabolic partial differential equation of the conventional diffusion equation, even if the source term and higher order diffusion terms were omitted.

The theory given here is for the idealised time-of-flight experiment with no boundaries. But in a real time-of-flight experiment, as dealt with for example in the study by Wagner *et al.* (1967), boundary effects due to finite enclosures and detecting electrodes lead to non-hydrodynamic behaviour. However, an experiment in which the mean of photon flux is measured (Blevin *et al.* 1976) should be less affected by the boundaries. The Fourier transformation method used here would not normally be directly applicable to experiment without some modification beginning with the Boltzmann equation. Such an analytical model of the Boltzmann equation which takes finite boundaries into account has been given by Kumar (1984).

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