Concomitants of Dual Generalized Order Statistics from Bivariate Burr II Distribution

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Abstract
Dual generalized order statistics is a common approach to enable descending ordered random variables like reverse order statistics and lower record values. In this paper probability density function of single concomitant and joint probability density function of two concomitants of dual generalized order statistics from bivariate Burr II distribution are obtained and expressions for moment generating function and cumulant generating function are derived. Also the expressions for mean, variance and covariance are given. Further, results are deduced for the reverse order statistics and lower record values.

Keywords: Dual generalized order statistics, Burr II distribution, Moment and Cumulant generating functions.

1. Introduction
Burkschat et al. (2003) introduced the concept of the dual generalized order statistics (dgos) to enable a common approach to descending ordered random variables like reverse order statistics and lower record values.

Suppose \( X_d(1,n,m,k), X_d(2,n,m,k), \ldots, X_d(n,n,m,k) \) \((k \geq 1, m \) is a real number \( \geq -1)\), are \( n \) dgos from an absolutely continuous \( w.r.t. \) Lebesgue measure distribution function \( (df) \) \( F(x) \) and probability density function \( (pdf) f(x) \). Their joint pdf can be written as

\[
k \prod_{i=1}^{n-1} \gamma_j = k - (n - j) (m + 1), \quad j = 1, 2, \ldots, n - 1, \quad \text{on the cone} \quad F^{-1}(1) > x_1 \geq x_2 \geq \cdots \geq x_n > F^{-1}(0).
\]

If \( m = 0, k = 1, \) then \( X_d(r,n,m,k) \) reduces to \((n-r+1)\) reverse order statistics \( X_{n-r+1} \) from the sample \( X_1, X_2, \ldots, X_n \) and when \( m = -1, \) then \( X_d(r,n,m,k) \) reduces to \( k \) order record values. For more details of order statistics and record values, one may refer to David and Nagaraja (2003) and Ahsanullah (2004), respectively.

In view of (1), the pdf of \( X_d(r,n,m,k) \) is

\[
f_{r,n,m,k}(x) = \frac{C_{r-1}}{(r-1)!} [F(x)]^{r-1} f(x) \gamma_{m-1}(F(x))
\]

where \( \gamma_{j} = m + 1 \) and corresponding df is

\[
h_{m}(x) = \begin{cases} \frac{1}{m+1} (1-x)^{m+1}, & m \neq -1 \\ \log x, & m = -1 \\
\end{cases}
\]

and \( g_{m}(x) = h_{m}(x) - h_{m}(1), \quad x \in (0,1). \)

Burr II distribution which is also known as generalized Gumbel bivariate logistic distribution and discussed by Satterthwaite and Hutchinson (1978). The pdf of bivariate Burr II distribution is given as

\[
f(x, y) = \frac{v (v+1)e^{-x}e^{-y}}{(1 + e^{-x} + e^{-y})^{v+2}}, \quad -\infty < x, y < \infty
\]

and corresponding df is

\[
F(x, y) = \frac{1}{(1 + e^{-x} + e^{-y})^v}, \quad -\infty < x, y < \infty
\]

The conditional pdf of \( Y \) given \( X \) is

\[
f(y | x) = \frac{(v+1)e^{-x}(1 + e^{-x})^{v+1}}{(1 + e^{-x} + e^{-y})^{v+2}}, \quad -\infty < y, x < \infty
\]
and the marginal pdf of $X$ is
\[
    f(x) = \frac{ve^{-x}}{(1 + e^{-x})^{v+1}}, \quad -\infty < x < \infty \quad (7)
\]
and corresponding marginal df is
\[
    F(x) = \frac{1}{(1 + e^{-x})^{v}}, \quad -\infty < x < \infty. \quad (8)
\]

Concomitants of order statistics have wide applications in the field such as selection procedure, ocean engineering, inference problems, prediction analysis etc. For detailed survey one may refer to Castillo (1988), David (1996), Do and Hall (1992), Gross (1973), O'Connell and David (1976), Yang (1981a & b) and Yoe and David (1984) and references therein.

Let $(X_i, Y_i), i=1,2,\ldots,n$ be the $n$ pairs of independent random variables from some bivariate population with distribution function $F(x,y)$. If we arrange the $X$ - variates in descending order as $X_d(1,n,m,k) \geq X_d(2,n,m,k) \geq \ldots \geq X_d(n,n,m,k)$ then $Y$ - variates paired (not necessarily in descending order ) with these dual generalized ordered statistics are called the concomitants of dual generalized order statistics and are denoted by $Y_{d[1,n,m,k]}, Y_{d[2,n,m,k]}, \ldots, Y_{d[n,n,m,k]}$.

The pdf of $Y_{d[r,n,m,k]}$, the $r^{th}$ concomitant of $dgos$, is given as
\[
    g_{d[r,n,m,k]}(y) = \int_{-\infty}^\infty f(y \mid x) f_{r,n,m,k}(x) \, dx \quad (9)
\]
and the joint pdf of $Y_{d[r,n,m,k]}$ and $Y_{d[s,n,m,k]}$ \(1 \leq r < s \leq n\) is
\[
    g_{d[r,s,n,m,k]}(y_1, y_2) = \int_{-\infty}^\infty \int_{-\infty}^\infty f(y_1 \mid x_1) f(y_2 \mid x_2) \\
    \times f_{r,s,n,m,k}(x_1, x_2) \, dx_2 \, dx_1. \quad (10)
\]

Ahsanullah and Beg (2006) derived the expression for concomitants of $gos$ for Gumbel’s bivariate exponential distribution whereas Beg and Ahsanullah (2008) obtained the concomitants of $gos$ for Farlie-Gumbel-Morgenstern distributions and established some recurrence relations for the concomitants of $gos$.

Das et al. (2012) carried out the comparative study on concomitant of order statistics and record values for weighted inverse Gaussian distribution. Further, Tahtashebi and Behboodian (2012) obtained the Shannon’s entropy for the concomitants of $gos$ in Farlie-Gumbel-Morgenstern family. An excellent review of work on concomitants of order statistics is available in David and Nagaraja (1998).

Here in this paper pdf of $r^{th}$, $1 \leq r \leq n$ and the joint pdf of $r^{th}$ and $s^{th}$, $1 \leq r < s \leq n$, concomitants of $dgos$ from bivariate Burr II distribution are obtained. Further, moment generating function (mgf) and cumulant generating function (cgf) are studied and expressions for mean, variance and covariance derived.

2. Moment Generating Function of $Y_{d[r,n,m,k]}$

Before deriving the expression for mgf of $Y_{d[r,n,m,k]}$, we shall obtain the pdf of $Y_{d[r,n,m,k]}$.

**Lemma 2.1:** For the bivariate Burr II distribution with pdf as given in (4), the pdf of $r^{th}$ concomitant of $dgos$, in view of (9) and (6) is,
\[
    g_{d[r,n,m,k]}(y) = \frac{C_r}{(r-1)! (m+1)^{r-1}} \, v \, (v+1) e^{-y} \\
    \times \sum_{i=0}^{r-1} (-1)^i \left( \frac{r-1}{i} \right) \frac{1}{v \gamma_{r-i} + 1} \\
    \times e^{(v+2) \gamma_{r-i} + 1 + (v \gamma_{r-i} + 2)} \, , \, m \neq -1
\]
where,
\[
    z \mathbb{F}_1 \left[ \begin{array}{c}
        a, b \\
        c
    \end{array} \right] = \sum_{p=0}^{\infty} \frac{(-1)^p \, a \, b \, c \, \gamma^p \, \mathbb{F}_1 \left[ \begin{array}{c}
        1 \\
        p+1
    \end{array} \right]}{p!}
\]
is conditionally convergent for $|z|=1$, $z \neq 1$, if $-1 < Re(w) \leq 0$ and
\[
    g_{d[r,n-1,k]}(y) = (v+1) e^{-y} \sum_{p=0}^{\infty} \frac{(v+2) \, (v \, e^{-y} \, \gamma)^p \, \mathbb{F}_1 \left[ \begin{array}{c}
        1 \\
        p+1 \, v \gamma
    \end{array} \right]}{p!} \, , \, m = -1
\]
**Proof:** we have
\[
    g_{d[r,n,m,k]}(y) = \frac{C_r}{(r-1)! (m+1)^{r-1}} \, v \, (v+1) e^{-y} \sum_{i=0}^{r-1} (-1)^i \left( \frac{r-1}{i} \right)
\]
\[ \times \int_{-\infty}^{\infty} \frac{e^{-x}}{(1 + e^{x} + e^{-y})^{v+2}} \frac{1}{(1 + e^{-x})^{v+y-i}} \, dx. \] (13)

Let \( t = (1 + e^{-x})^{-1} \), then the R.H.S. of (13) reduces to

\[ = \frac{C_{r-1}}{(r-1)!(m+1)^{1-r}} v (v+1) e^{-x} \sum_{i=0}^{r-1} (-1)^{i} \binom{r-1}{i} \int_{0}^{t} t^{v+y-i} (1 + t e^{-y})^{-v-y-2} \, dt. \] (14)

Since,

\[ (1 + z)^{-a} = \sum_{p=0}^{\infty} \frac{(-1)^{p} (a)_{p} z^{p}}{p!}, \] (15)

where \((a)_{p} = \frac{\Gamma(a + p)}{\Gamma(a)} ; a \neq 0, -1, -2, \ldots\)

and \((\lambda + m) = \frac{\lambda(\lambda + 1)}{m} \). (16)


Thus in view of (15) and (16), (11) can be established. Expression (12) can be obtained by simplifying (11) and taking \( m \to -1 \).

Now moment generating function of \( Y_{d[r,n,m,k]} \) is given by

\[ M_{d[r,n,m,k]}(t) = \frac{C_{r-1}}{(r-1)!(m+1)^{1-r}} v (v+1) \]
\[ \times \sum_{i=0}^{r-1} (-1)^{i} \binom{r-1}{i} \frac{1}{v+y-i+1} \]
\[ \int_{0}^{\infty} e^{y} \, y_{1}^{r-1} F_{1} \left[ (v+2), (v+y-i+1) \right] ; -e^{-y} \, y_{1}^{r-1} \, dy. \] (17)

Let \( z = e^{-y} \), then R.H.S. of (17) reduces to

\[ = \frac{C_{r-1}}{(r-1)!(m+1)^{1-r}} v (v+1) \sum_{i=0}^{r-1} (-1)^{i} \binom{r-1}{i} \frac{1}{v+y-i+1} \]
\[ \times \int_{0}^{\infty} z^{y-i} \, z \, y_{1}^{r-1} F_{1} \left[ (v+y-i+1), (v+y-i+2) \right] ; -z \, dz. \] (18)

Now using the relation, given by Prudinov et al. (1986) as

\[ \int_{0}^{\infty} x^{p-1} F_{1} \left[ a, b ; c \right] ; -\eta \, x \, dx = \frac{(\eta)^{-p} \Gamma(c) \Gamma(p) \Gamma(a-p) \Gamma(b-p)}{\Gamma(a) \Gamma(b) \Gamma(c-p)}, \] (19)

we get

\[ M_{d[r,n,m,k]}(t) = \frac{C_{r-1}}{(r-1)!(m+1)^{1-r}} v (v+1) \]
\[ \times \sum_{i=0}^{r-1} (-1)^{i} \binom{r-1}{i} B \left[ k \frac{1}{m+1} + \frac{t}{v(m+1)} + (n-r)+i, 1 \right], \] (20)

Since \( \sum_{a=0}^{b} (-1)^{a} a \binom{b}{a} B(a+k, c) = B(k, c+b) \), thus (20) becomes

\[ M_{d[r,n,m,k]}(t) = \frac{\Gamma(1-t) \Gamma(v+t+1)}{\Gamma(v+1)} \frac{1}{\prod_{i=1}^{r} \left( 1 + \frac{t}{v^{2+i}} \right)}. \] (21)

Cumulant generating function of \( Y_{d[r,n,m,k]} \) is given as

\[ K_{d[r,n,m,k]} = \ln \Gamma(1-t) + \ln \Gamma(v+t+1) - \ln \Gamma(v+1) \]
\[ - \sum_{i=1}^{r} \ln \left( 1 + \frac{t}{v^{2+i}} \right). \]

Since,

\[ E(Y_{d[r,n,m,k]}) = \mu_{1}[r,n,m,k] = \frac{d}{dt} K_{d[r,n,m,k]}(t) \] and

\[ V(Y_{d[r,n,m,k]}) = \mu_{2}[r,n,m,k] = \frac{d^{2}}{dt^{2}} K_{d[r,n,m,k]}(t) \]

at \( t = 0 \),

Thus,

\[ \mu_{1}[r,n,m,k] = \psi(v+1) - \psi(1) - \frac{1}{v} \sum_{i=1}^{r} \frac{1}{\gamma_{i}} = \frac{\sum_{i=1}^{r} \frac{1}{\gamma_{i}} - 1}{v} \sum_{i=1}^{r} \frac{1}{\gamma_{i}} \] (22)

\[ \mu_{2}[r,n,m,k] = \frac{\pi^{2}}{3} - \sum_{i=1}^{r} \frac{1}{\gamma_{i}^{2}} + \frac{1}{v} \sum_{i=1}^{r} \frac{1}{\gamma_{i}^{2}}. \] (23)
We have Andrews (1985)

\[
\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \psi(1) + \sum_{n=0}^{\infty} \left[ \frac{1}{n+1} - \frac{1}{n+x} \right], \quad x > 0
\]

(24)

is known as digamma function and

\[
\psi(n+1) = \psi(1) + \sum_{k=1}^{n} \frac{1}{k}; \quad n = 1, 2, \ldots
\]

\[
\psi(x+n) = \psi(x) + \sum_{k=0}^{n-1} \frac{1}{x+k}.
\]

**Remark 2.1:** At \( m = 0 \) and \( k = 1 \) in (22) and (23), we get mean and variance of concomitants of order statistics from bivariate Burr II distribution as obtained by Begum and Khan (1997).

Mean is 

\[
\mu_{1(r,n)} = \sum_{i=1}^{r-1} \frac{1}{i} - \frac{1}{r} \sum_{j=i}^{r} \frac{1}{v j} (n-i+1)
\]

and

Variance is 

\[
\mu_{2(r,n)} = \frac{\pi^2}{3} - \sum_{i=1}^{r-1} \frac{1}{i^2} + \int_{i=1}^{r} \frac{1}{v(j-n+1)^2}
\]

\[
= \frac{\pi^2}{3} - \sum_{i=1}^{r-1} \frac{1}{i^2} + \frac{1}{v^2} \sum_{j=r}^{n-1} \frac{1}{j^2}.
\]

**Remark 2.2:** At \( m \to -1 \) in (22) and (23), we get mean and variance of concomitants of \( k \)th lower record values.

\[
\mu_{1[r,n,-1,k]} = \sum_{i=1}^{r} \frac{1}{i} \frac{v}{vk}
\]

\[
\mu_{2[r,n,-1,k]} = \frac{\pi^2}{3} - \sum_{i=1}^{r-1} \frac{1}{i^2} + \frac{r}{(vk)^2}.
\]

3. Joint Moment Generating Function of \( Y_{d[r,n,m,k]} \) and \( Y_{d[s,n,m,k]} \)

Here, we shall first obtain the joint pdf of \( Y_{d[r,n,m,k]} \) and \( Y_{d[s,n,m,k]} \).

**Lemma 3.1:** For Burr II distribution with \( df \) as given in (4), the joint pdf of \( r \)th and \( s \)th concomitants of \( dgos \) is given as

\[
S_{d[r,s,n,m,k]}(Y_1, Y_2) = \frac{C_{s-1}}{(r-1)(s-r-1)(m+1)^{s-r}} v^2 (v+1)^2 e^{-v_1} e^{-v_2}
\]

\[
\times \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \left( \begin{array}{c} r-1 \\ i \\ j \\ j \end{array} \right) \frac{1}{v_{r-j}^i + 1} \frac{1}{v_{r-i}^j + 2} \left[ (v_{r-j}^i + 2) : (v+2) (v_{r-j}^i + 1) ; (v+2) ; v_{r-i}^j + 3) ; (v_{r-i}^j + 2) ; v_{r-i}^j + 3 \right]
\]

\[
\times F_{1/2;1}^{1/2;1}(m, n; x; y)
\]

where,

\[
p_{p} = \left[ (a_p) ; (b_p) ; (c_k) \right] \quad ; x, y
\]

\[
p_{l;mc} = \left[ (\alpha_i) ; (\beta_j) ; (\gamma_m) \right]
\]

\[
= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \prod_{j=1}^{m} \frac{1}{v_{j}^r} y^s
\]

is known as Kampé de Fériet function (Srivastava and Karlson, 1985).

**Proof:** We have

\[
S_{d[r,s,n,m,k]}(Y_1, Y_2) = \frac{C_{s-1} (v+1)^2 e^{-v_1} e^{-v_2}}{(r-1)(s-r-1)(m+1)^{s-r}}
\]

\[
\times \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \left( \begin{array}{c} r-1 \\ i \\ j \end{array} \right) \frac{1}{v_{r-j-i}^i + 1} \frac{1}{v_{r-i-j}^j + 2} \left[ (v_{r-j-i}^i + 2) : (v+2) (v_{r-j-i}^i + 1) ; (v+2) ; v_{r-i-j}^j + 3) ; (v_{r-i-j}^j + 2) ; v_{r-i-j}^j + 3 \right]
\]

\[
\times F_{1/2;1}^{1/2;1}(m, n; x; y)
\]

where,

\[
p_{p} = \left[ (a_p) ; (b_p) ; (c_k) \right] \quad ; x, y
\]

\[
p_{l;mc} = \left[ (\alpha_i) ; (\beta_j) ; (\gamma_m) \right]
\]

\[
= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \prod_{j=1}^{m} \frac{1}{v_{j}^r} y^s
\]

\[
m = -1
\]

(26)

**Proof:** We have

\[
S_{d[r,s,n,m,k]}(Y_1, Y_2) = \frac{C_{s-1} (v+1)^2 e^{-v_1} e^{-v_2}}{(r-1)(s-r-1)(m+1)^{s-r}}
\]

\[
\times \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \left( \begin{array}{c} r-1 \\ i \\ j \end{array} \right) \frac{1}{v_{r-j-i}^i + 1} \frac{1}{v_{r-i-j}^j + 2} \left[ (v_{r-j-i}^i + 2) : (v+2) (v_{r-j-i}^i + 1) ; (v+2) ; v_{r-i-j}^j + 3) ; (v_{r-i-j}^j + 2) ; v_{r-i-j}^j + 3 \right]
\]

\[
\times F_{1/2;1}^{1/2;1}(m, n; x; y)
\]

(27)

where,

\[
I(x_1, x_2) = \int_{v_1}^{\infty} \frac{e^{-v_2}}{(1+e^{-v_1})(v-r-e^{-v_1})(m+1)^{v-r+1}} \times \int_{v_2}^{\infty} \frac{1}{(1+e^{-v_1}+e^{-v_2})^{v+2}} I(x_1, y_2) dx_1
\]

(28)
If we put \( t = (1 + e^{-y_2})^{-1} \), then the R.H.S. of (28) reduces to
\[
I(x_1, y_2) = \frac{(1 - e^{-y_1})^{-1}}{0} \int_{y_{y-j}} (1 + t e^{-y_2})^{-(v+2)} dt.
\]
Using (15) and after simplification, we get
\[
I(x_1, y_2) = \sum_{p=0}^{\infty} \frac{(v+2)_p}{p!} \frac{(-e^{-y_2})^p}{v_{y-j}^p} \int_{0} (1 + t e^{-y_2})^{-(v+2)} dt
\]
\[
= \sum_{p=0}^{\infty} \frac{(v+2)_p}{p!} \frac{1}{v_{y-j}^p + p + 1}
\]
\[
\times \frac{1}{(1 + e^{-y_1})^{v_{y-j}^p + p + 1}} (29)
\]
Now putting the value of (29) in (27), we get
\[
\gamma_{d_{r,n,m,k}}(x_1, y_2) = \frac{C_{s-1}}{(r-1)(s-r-1)(m+1)^{v-2}}
\]
\[
\times (v+2)^2 (v+1)^2 e^{-y_2} v_{y-j}^p
\]
\[
\times \sum_{i=0}^{r-1} \sum_{j=0}^{s-1} (-1)^{i+j} \left( \frac{r-1}{i} \right) \left( \frac{s-r-1}{j} \right) \sum_{p=0}^{\infty} \frac{(v+2)_p}{p!} (-e^{-y_2})^p
\]
\[
\times \frac{1}{v_{y-j}^p + p + 1} e^{-y_1} (1 + e^{-y_1})^{v_{y-j}^p + p + 1}
\]
\[
\times \frac{1}{(1 + e^{-y_1})^{v_{y-j}^p + p + 1}} (30)
\]
Setting \( z = (1 + e^{-y_1})^{-1} \), and using the relation (15), we get
\[
\gamma_{d_{r,n,m,k}}(x_1, y_2) = \frac{C_{s-1}}{(r-1)(s-r-1)(m+1)^{v-2}}
\]
\[
\times (v+2)^2 (v+1)^2 e^{-y_1} e^{-y_2}
\]
\[
\times \sum_{i=0}^{r-1} \sum_{j=0}^{s-1} (-1)^{i+j} \left( \frac{r-1}{i} \right) \left( \frac{s-r-1}{j} \right) \sum_{p=0}^{\infty} \frac{(v+2)_p}{p!} (-e^{-y_2})^p
\]
\[
\times \frac{1}{v_{y-j}^p + p + 1} v_{y-j} e^{-y_1} (1 + e^{-y_1})^{v_{y-j}^p + p + 1}
\]
\[
\times \frac{1}{(1 + e^{-y_1})^{v_{y-j}^p + p + 1}} (31)
\]
Nothing that
\[
(\lambda + \eta) = \frac{\lambda (\lambda + 1)}{\eta} \quad \text{and} \quad (\lambda + \eta + n) = \frac{\lambda (\lambda + 1)_{p+n}}{\eta_{p+n}}
\]
(Srivastava and Karlson, 1985)
and using in (31), we get the result as given in (25). (26) can be obtained by simplifying (25) and taking \( m \to -1 \).

The joint moment generating function of \( Y_{d[r,n,m,k]} \) and \( Y_{d[s,n,m,k]} \) in view of (25) is given as
\[
M_{d_{r,n,m,k}}(t_1,t_2) = \frac{C_{s-1}}{(r-1)(s-r-1)(m+1)^{v-2}}
\]
\[
\times \sum_{i=0}^{r-1} \sum_{j=0}^{s-1} (-1)^{i+j} \left( \frac{r-1}{i} \right) \left( \frac{s-r-1}{j} \right) \left( \frac{1}{v_{y-j}^p + 1} v_{y-j} + 2 \right)
\]
\[
\times \left[ (v_{y-j} + 2)(v+2), (v_{y-j} + 1)(v+2); (v_{y-j} + 2); \right]
\]
\[
\times \left[ (v_{y-j} + 3)(v+2); \right]
\]
\[
dt_1 dt_2 \quad (32)
\]
Let \( e^{-y_1} = z_1 \) and \( e^{-y_2} = z_2 \), then
\[
\gamma_{d_{r,n,m,k}}(x_1, y_2) = \frac{C_{s-1}}{(r-1)(s-r-1)(m+1)^{v-2}}
\]
\[
\times \sum_{i=0}^{r-1} \sum_{j=0}^{s-1} (-1)^{i+j} \left( \frac{r-1}{i} \right) \left( \frac{s-r-1}{j} \right) \left( \frac{1}{v_{y-j}^p + 1} v_{y-j} + 2 \right)
\]
\[
\times \sum_{p=0}^{\infty} \frac{(v_{y-j} + 2)_p (v+2)_p}{(v_{y-j} + 3)_p} d\eta_1 d\eta_2 \quad (33)
\]
Now using relation \((\lambda)_m = (\lambda)_m (\lambda + m)_n\) as given by Srivastava and Karlson (1985), we have
\[
M_{d_{r,n,m,k}}(t_1,t_2) = \frac{C_{s-1}}{(r-1)(s-r-1)(m+1)^{v-2}}
\]
\[
\times \sum_{i=0}^{r-1} \sum_{j=0}^{s-1} (-1)^{i+j} \left( \frac{r-1}{i} \right) \left( \frac{s-r-1}{j} \right) \left( \frac{1}{v_{y-j}^p + 1} v_{y-j} + 2 \right)
\]
\[
\times \left[ (v_{y-j} + 2)_p (v+2)_p (v_{y-j} + 1)_p (v+2)_p \right]
\]
\[
\times \left[ (v_{y-j} + 3)_p (v+2)_p \right]
\]
\[
\times \left[ (v_{y-j} + 2)_p (v+2)_p \right]
\]
\[
dt_1 dt_2 \quad (34)
\]
Now on application of (19) in (34) and after simplification, we get

\[
\frac{C_{r-1} v^2 (v+1)}{(r-1)!(s-r-1)!(m+1)^{s-2}}
\]

\[
\times \sum_{j=0}^{r-1} x^{-1} \sum_{j=0}^{s-r-1} (-1)^{s-r-j} \binom{r-1}{i} \binom{s-r-1}{j} \Gamma(1-t_1) \Gamma(v+t_1+1) \Gamma(1-t_2) \Gamma(v+t_2+1)
\]

\[
\times \prod_{i=1}^{r} \frac{v^{-r_i^2-1} F_2 \left[ \begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2 \end{array} \right] - \varphi d \varphi }{v^{r_i^2}}
\]

(35)

Now using Prudnikov et al. (1986)

\[
\frac{1}{v^{r_i^2}} \frac{1}{F_2 \left[ \begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2 \end{array} \right] - \varphi d \varphi }{v^{r_i^2}}
\]

\[
\frac{1}{v^{r_i^2}} \frac{1}{F_2 \left[ \begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2 \end{array} \right] - \varphi d \varphi }{v^{r_i^2}}
\]

\[
\frac{1}{v^{r_i^2}} \frac{1}{F_2 \left[ \begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2 \end{array} \right] - \varphi d \varphi }{v^{r_i^2}}
\]

and using the relation (24), we get

\[
\text{Cov}[Y_{d[r,n,m,k]}, Y_{d[s,n,m,k]}] = \frac{1}{v^2} \sum_{i=1}^{r} \frac{1}{v_i^2} \sum_{j=1}^{s} \frac{1}{v_j^2}
\]

(38)

Remark 3.1: Set \( m = 0, k = 1 \) and replace \( n - r + 1 \) by \( s \) and \( n - s + 1 \) by \( r \) in (38), we get covariance between concomitant of order statistics from bivariate Burr II distribution as

\[
\text{Cov}[Y_{d[r,n,m,k]}, Y_{d[s,n,m,k]}] = \frac{1}{v^2} \sum_{i=1}^{r} \frac{1}{v_i^2} \sum_{j=1}^{s} \frac{1}{v_j^2}
\]

\[
= \frac{1}{v^2} \sum_{i=1}^{r} \frac{1}{v_i^2} \sum_{j=1}^{s} \frac{1}{v_j^2}
\]

(39)

This result was also obtained by Begum and Khan (1997).

Remark 3.2: As \( m \to -1 \) in (39), we get covariance of concomitant of \( k^{th} \) record values from bivariate Burr II distribution as

\[
\text{Cov}[Y_{d[r,n-1,m,k]}, Y_{d[s,n-1,m,k]}] = \frac{r}{(vk)^2}
\]

4. Conclusion

In this paper, we have obtained the marginal and joint moment generating function of concomitants of \( dgos \) from bivariate Burr II distribution. A good application of this setup is the use of mgf of concomitants of \( dgos \) for computing the moments of any order of concomitant of order statistics, record values, sequential order statistics etc.

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