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# Concomitants of Dual Generalized Order Statistics from Bivariate Burr II Distribution

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## Abstract

*Dual generalized order statistics is a common approach to enable descending ordered random variables like reverse order statistics and lower record values. In this paper probability density function of single concomitant and joint probability density function of two concomitants of dual generalized order statistics from bivariate Burr II distribution are obtained and expressions for moment generating function and cumulant generating function are derived. Also the expressions for mean, variance and covariance are given. Further, results are deduced for the reverse order statistics and lower record values.*

***Keywords:*** *Dual generalized order statistics, Burr II distribution, Moment and Cumulant generating functions.*

## 1. Introduction

Burkschat *et al.* (2003) introduced the concept of the dual generalized order statistics (*dgos*) to enable a common approach to descending ordered random variables like reverse order statistics and lower record values.

Suppose $X_d(1,n,m,k), X_d(2,n,m,k),\ldots,X_d(n,n,m,k)$ ( $k \ge 1, m$ is a real number $\ge -1$), are $n$ *dgos* from an absolutely continuous (*w.r.t.* Lebesgue measure) distribution function $(df)\, F(x)$ and probability density function $(pdf)\, f(x)$. Their joint $pdf$ can be written as

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[F(x_i)]^m f(x_i)\right)[F(x_n)]^{k-1} f(x_n) \quad (1)$$

where $\gamma_j = k+(n-j)(m+1),\ j=1,2,\ldots,n-1$, on the cone $F^{-1}(1) > x_1 \ge x_2 \ge \ldots \ge x_n > F^{-1}(0)$.

If $m=0, k=1$, then $X_d(r,n,m,k)$ reduces to $(n-r+1)^{th}$ reverse order statistics $X_{n-r+1:n}$ from the sample $X_1, X_2,\ldots,X_n$ and when $m=-1$, then $X_d(r,n,m,k)$ reduces to $k^{th}$ lower record values. For more details of order statistics and record values, one may reffer to David and Nagaraja (2003) and Ahsanullah (2004), respectively.

In view of (1), the $pdf$ of $X_d(r,n,m,k)$ is

$$f_{r,n,m,k}(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) \quad (2)$$

and joint $pdf$ of $X_d(r,n,m,k)$ and $X_d(s,n,m,k)$, $1 \le r < s \le n$ is

$$f_{r,s,n,m,k}(x,y)$$
$$= \frac{C_{s-1}}{(r-1)!\,(s-r-1)!}[F(x)]^m f(x)\, g_m^{r-1}(F(x))$$
$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s - 1} f(y),\ x > y, \quad (3)$$

where

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1}, & m \ne -1 \\ -\log x, & m = -1 \end{cases}$$

and $g_m(x) = h_m(x) - h_m(1),\ x \in [0,1)$.

Burr II distribution which is also known as generalized Gumbel bivariate logistic distribution and discussed by Satterthwaite and Hutchinson (1978).

The $pdf$ of bivariate Burr II distribution is given as

$$f(x,y) = \frac{v(v+1)e^{-x}e^{-y}}{(1+e^{-x}+e^{-y})^{v+2}},\ -\infty < x,\ y < \infty \quad (4)$$

and corresponding $df$ is

$$F(x,y) = \frac{1}{(1+e^{-x}+e^{-y})^{v}},\ -\infty < x,\ y < \infty. \quad (5)$$

The conditional $pdf$ of $Y$ given $X$ is

$$f(y\,|\,x) = \frac{(v+1)\,e^{-y}(1+e^{-x})^{v+1}}{(1+e^{-x}+e^{-y})^{v+2}},\ -\infty < y < \infty \quad (6)$$

and the marginal $pdf$ of $X$ is

$$f(x) = \frac{ve^{-x}}{(1+e^{-x})^{v+1}}, \ -\infty < x < \infty \qquad (7)$$

and corresponding marginal $df$ is

$$F(x) = \frac{1}{(1+e^{-x})^{v}}, \ -\infty < x < \infty. \qquad (8)$$

Concomitants of order statistics have wide applications in the fileld such as selection procedure, ocean engineering, inference problems, prediction analysis etc. For detailed survey one may refer to Castillo (1988), David (1996), Do and Hall (1992), Gross (1973), O'Connell and David (1976), Yang (1981a & b) and Yoe and David (1984) and references therein.

Let $(X_i, Y_i), i = 1,2,\ldots,n,$ be the $n$ pairs of independent random variables from some bivariate population with distribution function $F(x, y)$. If we arrange the $X-$variates in descending order as $X_d(1,n,m,k) \geq X_d(2,n,m,k) \geq \ldots \geq X_d(n,n,m,k)$ then $Y-$variates paired (not necessarily in descending order ) with these dual generalized ordered statistics are called the concomitants of dual generalized order statistics and are denoted by $Y_{d[1,n,m,k]}, Y_{d[2,n,m,k]}, \ldots, Y_{d[n,n,m,k]}$.

The $pdf$ of $Y_{d[r,n,m,k]}$, the $r^{th}$ concomitant of $dgos$ is given as

$$g_{d[r,n,m,k]}(y) = \int_{-\infty}^{\infty} f(y \mid x)\, f_{r,n,m,k}(x)\, dx \qquad (9)$$

and the joint $pdf$ of $Y_{d[r,n,m,k]}$ and $Y_{d[s,n,m,k]}$ $1 \leq r < s \leq n$ is

$$g_{d[r,s,n,m,k]}(y_1, y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_1} f(y_1 \mid x_1)\, f(y_2 \mid x_2) \times f_{r,s,n,m,k}(x_1, x_2)\, dx_2\, dx_1. \qquad (10)$$

Ahsanullah and Beg (2006) derived the expression for concomitants of $gos$ for Gumbel's bivariate exponential distribution whereas Beg and Ahsanullah (2008) obtained the concomitants of $gos$ for Farlie-Gumbel-Morgenstern distributions and established some recurrence relations for the concomitants of $gos$. Das *et al*. (2012) carried out the comparative study on concomitant of order statistics and record values for weighted inverse Gaussian distribution. Further, Tahmasebi and Behboodian (2012) obtained the Shannon's entropy for the concomitants of $gos$ in Farlie-Gumbel-Morgenstern family. An excellent review of work on concomitants of order statistics is available in David and Nagaraja (1998).

Here in this paper $pdf$ of $r^{th}$, $1 \leq r \leq n$ and the joint $pdf$ of $r^{th}$ and $s^{th}$, $1 \leq r < s \leq n$, concomitants of $dgos$ from bivariate Burr II distribution are obtained. Further, moment generating function $(mgf)$ and cumulant generating function $(cgf)$ are studied and expressions for mean, variance and covariance derived.

## 2. Moment Generating Function of $Y_{d[r,n,m,k]}$

Before deriving the expression for $mgf$ of $Y_{[r,n,m,k]}$, we shall obtain the $pdf$ of $Y_{d[r,n,m,k]}$.

**Lemma 2.1:** For the bivariate Burr II distribution with $pdf$ as given in (4), the $pdf$ of $r^{th}$ concomitant of $dgos$, in view of (9) and (6) is,

$$g_{d[r,n,m,k]}(y) = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\, v(v+1)e^{-y} \times \sum_{i=0}^{r-1}(-1)^i \binom{r-1}{i} \frac{1}{v\gamma_{r-i}+1} \times {}_2F_1\left[\begin{matrix}(v+2), & (v\gamma_{r-i}+1) \\ (v\gamma_{r-i}+2) & \end{matrix}; -e^{-y}\right], \quad m \neq -1 \qquad (11)$$

where,

$${}_2F_1\left[\begin{matrix}a, & b \\ c & \end{matrix}; -z\right] = \sum_{p=0}^{\infty} \frac{(-1)^p\, (a)_p (b)_p}{(c)_p} \frac{z^p}{p!}$$

is conditionally convergent for $|z| = 1, z \neq 1,$ if $-1 < Re(w) \leq 0$

and

$$g_{d[r,n,-1,k]}(y) = (v+1)e^{-y} \sum_{p=0}^{\infty} \frac{(v+2)_p (-e^{-y})^p}{p!} \times \frac{1}{\left(1+\frac{p+1}{vk}\right)^r}, \ m = -1 \qquad (12)$$

**Proof :** we have

$$g_{d[r,n,m,k]}(y) = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\, v(v+1)e^{-y} \sum_{i=0}^{r-1}(-1)^i \binom{r-1}{i}$$

$$\times \int_{-\infty}^{\infty} \frac{e^{-x}}{(1+e^{-x}+e^{-y})^{v+2}} \frac{1}{(1+e^{-x})^{v\gamma_{r-i}-v}} dx. \qquad (13)$$

Let $t=(1+e^{-x})^{-1}$, then the *R.H.S.* of (13) reduces to

$$= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} v(v+1) e^{-y} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i}$$

$$\times \int_0^1 t^{v\gamma_{r-i}} (1+t\,e^{-y})^{-(v+2)}\, dt. \qquad (14)$$

Since,

$$(1+z)^{-a} = \sum_{p=0}^{\infty} \frac{(-1)^p (a)_p z^p}{p!}, \qquad (15)$$

where $(a)_p = \dfrac{\Gamma(a+p)}{\Gamma(a)}; a \neq 0,-1,-2,\ldots$

and $(\lambda+m) = \dfrac{\lambda(\lambda+1)_m}{(\lambda)_m}$ . (16)

See Srivastava and Karlson (1985).

Thus in view of (15) and (16), (11) can be established. Expression (12) can be obtained by simplifying (11) and taking $m \to -1$.

Now moment generating function of $Y_{d[r,n,m,k]}$ is given by

$$M_{d[r,n,m,k]}(t) = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} v(v+1)$$

$$\times \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{v\gamma_{r-i}+1}$$

$$\times \int_{-\infty}^{\infty} e^{ty}\, {}_2F_1 \begin{bmatrix} (v+2), & (v\gamma_{r-i}+1) & \\ & & ; -e^{-y} \\ (v\gamma_{r-i}+2) & & \end{bmatrix} e^{-y} dy. \qquad (17)$$

Let $z=e^{-y}$, then *R.H.S.* of (17) reduces to

$$= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} v(v+1) \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{v\gamma_{r-i}+1}$$

$$\times \int_0^{\infty} z^{-t}\, {}_2F_1 \begin{bmatrix} (v+2), & (v\gamma_{r-i}+1) & \\ & & ; -z \\ (v\gamma_{r-i}+2) & & \end{bmatrix} dz. \qquad (18)$$

Now using the relation, given by Prudinov *et al.* (1986) as

$$\int_0^{\infty} x^{p-1}\, {}_2F_1 \begin{bmatrix} a, & b & \\ & & ; -\eta\, x \\ c & & \end{bmatrix} dx$$

$$= \frac{(\eta)^{-p}\, \Gamma(c)\,\Gamma(p)\,\Gamma(a-p)\,\Gamma(b-p)}{\Gamma(a)\,\Gamma(b)\,\Gamma(c-p)},$$

$$[0 < Re\, p < Re\, a, Re\, b; |\arg \eta| < \pi], \qquad (19)$$

we get

$$M_{d[r,n,m,k]}(t) = \frac{C_{r-1}}{(r-1)!(m+1)^r} \frac{\Gamma(1-t)\,\Gamma(v+t+1)}{\Gamma(v+1)}$$

$$\times \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} B\left(\frac{k}{m+1} + \frac{t}{v(m+1)} + (n-r) + i,\ 1\right). \qquad (20)$$

Since $\sum_{a=0}^{b} (-1)^a \binom{b}{a} B(a+k,c) = B(k,c+b)$, thus (20) becomes

$$M_{d[r,n,m,k]}(t) = \frac{\Gamma(1-t)\,\Gamma(v+t+1)}{\Gamma(v+1)} \frac{1}{\prod_{i=1}^{r}\left(1+\dfrac{t}{v\,\gamma_i}\right)}. \qquad (21)$$

Cumulant generating function of $Y_{d[r,n,m,k]}$ is given as

$$K_{d[r,n,m,k]} = \ln\Gamma(1-t) + \ln\Gamma(v+t+1) - \ln\Gamma(v+1)$$

$$- \sum_{i=1}^{r} \ln\left(1+\frac{t}{v\gamma_i}\right).$$

Since,

$E(Y_{d[r,n,m,k]}) = \mu_{1[r,n,m,k]} = \dfrac{d}{dt} K_{d[r,n,m,k]}(t)$ and

$V(Y_{d[r,n,m,k]}) = \mu_{2[r,n,m,k]} = \dfrac{d^2}{dt^2} K_{d[r,n,m,k]}(t)$

at $t=0$

Thus,

$$\mu_{1[r,n,m,k]} = \psi(v+1) - \psi(1) - \frac{1}{v}\sum_{i=1}^{r}\frac{1}{\gamma_i} = \sum_{i=1}^{v}\frac{1}{i} - \frac{1}{v}\sum_{i=1}^{r}\frac{1}{\gamma_i} \qquad (22)$$

$$\mu_{2[r,n,m,k]} = \frac{\pi^2}{3} - \sum_{i=1}^{v}\frac{1}{i^2} + \frac{1}{v^2}\sum_{i=1}^{r}\frac{1}{\gamma_i^2}. \qquad (23)$$

We have Andrews (1985)

$$\psi(x)=\frac{d}{dx}\ln\Gamma(x)=\psi(1)+\sum_{n=0}^{\infty}\left[\frac{1}{n+1}-\frac{1}{n+x}\right],\quad x>0 \tag{24}$$

is known as digamma function and

$$\psi(n+1)=\psi(1)+\sum_{k=1}^{n}\frac{1}{k}\ ;\ n=1,2,\ldots$$

$$\psi(x+n)=\psi(x)+\sum_{k=0}^{n-1}\frac{1}{x+k}.$$

**Remark 2.1:** At $m=0$ and $k=1$ in (22) and (23), we get mean and variance of concomitants of order statistics from bivariate Burr II distribution as obtained by Begum and Khan (1997).

$$\text{Mean}=\mu_{1[r:n]}=\sum_{i=1}^{v}\frac{1}{i}-\frac{1}{v}\sum_{i=1}^{r}\frac{1}{(n-i+1)}$$

$$=\sum_{i=1}^{v}\frac{1}{i}-\frac{1}{v}\sum_{j=r}^{n}\frac{1}{j}$$

and

$$\text{Variance}=\mu_{2[r:n]}=\frac{\pi^2}{3}-\sum_{i=1}^{v}\frac{1}{i^2}+\frac{1}{v^2}\sum_{i=1}^{r}\frac{1}{(n-i+1)^2}$$
$$=\frac{\pi^2}{3}-\sum_{i=1}^{v}\frac{1}{i^2}+\frac{1}{v^2}\sum_{j=r}^{n}\frac{1}{j^2}.$$

**Remark 2.2:** At $m\to -1$ in (22) and (23), we get mean and variance of concomitants of $k^{th}$ lower record values.

$$\mu_{1[r,n,-1,k]}=\sum_{i=1}^{v}\frac{1}{i}-\frac{r}{vk}.$$

$$\mu_{2[r,n,-1,k]}=\frac{\pi^2}{3}-\sum_{i=1}^{v}\frac{1}{i^2}+\frac{r}{(vk)^2}.$$

## 3. Joint Moment Generating Function of $Y_{d[r,n,m,k]}$ and $Y_{d[s,n,m,k]}$

Here, we shall first obtain the joint $pdf$ of $Y_{d[r,n,m,k]}$ and $Y_{d[s,n,m,k]}$.

**Lemma 3.1:** For Burr II distribution with $df$ as given in (4), the joint $pdf$ of $r^{th}$ and $s^{th}$ concomitants of $dgos$ is given as

$$g_{d[r,s,n,m,k]}(y_1,y_2)$$
$$=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}v^2(v+1)^2e^{-y_1}e^{-y_2}$$
$$\times\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}\frac{1}{v\gamma_{s-j}+1}\frac{1}{v\gamma_{r-i}+2}$$
$$\times F^{1:2;1}_{1:1;0}\left[\begin{matrix}(v\gamma_{r-i}+2):\ (v+2)(v\gamma_{s-j}+1);\ (v+2);\\ ;-e^{-y_1},-e^{-y_2}\\ (v\gamma_{r-i}+3):\ (v\gamma_{s-j}+2);\ \_;\end{matrix}\right],$$
$$m\neq -1 \tag{25}$$

where,

$$F^{p:q;k}_{l:m;n}\left[\begin{matrix}(a_p):\ (b_q);\ (c_k)\\ ;\ x,y\\ (\alpha_l):\ (\beta_m);\ (\gamma_n)\end{matrix}\right]$$
$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}$$

is known as Kampé de Fériet function (Srivastava and Karlson, 1985).

$$g_{d[r,s,n,-1,k]}(y_1,y_2)=(v+1)^2e^{-y_1}e^{-y_2}$$
$$\times\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\frac{(v+2)_p(-e^{-y_2})^p}{p!}\frac{(v+2)_l(-e^{-y_1})^l}{l!}$$
$$\times\frac{1}{\left(1+\frac{p+l+2}{vk}\right)^r}\frac{1}{\left(1+\frac{p+1}{vk}\right)^{s-r}},-\infty<y_1,y_2<\infty,$$
$$m=-1 \tag{26}$$

**Proof :** We have

$$g_{d[r,s,n,m,k]}(y_1,y_2)=\frac{C_{s-1}v^2(v+1)^2e^{-y_1}e^{-y_2}}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$
$$\times\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}$$
$$\times\int_{-\infty}^{\infty}\frac{e^{-x_1}}{(1+e^{-x_1})^{v(s-r+i-j)(m+1)-v}}$$
$$\times\frac{1}{(1+e^{-x_1}+e^{-y_1})^{v+2}}I(x_1,y_2)\,dx_1 \tag{27}$$

where,

$$I(x_1,y_2)=\int_{-\infty}^{x_1}\frac{e^{-x_2}}{(1+e^{-x_2})^{v\gamma_{s-j}-v}}\frac{1}{(1+e^{-x_2}+e^{-y_2})^{v+2}}\,dx_2. \tag{28}$$

If we put $t=(1+e^{-x_2})^{-1}$, then the *R.H.S*. of (28) reduces to

$$I(x_1,y_2)=\int_0^{(1+e^{-x_1})^{-1}} t^{\nu\gamma_{s-j}}\,(1+t\,e^{-y_2})^{-(\nu+2)}\,dt.$$

Using (15) and after simplification, we get

$$I(x_1,y_2)=\sum_{p=0}^{\infty}\frac{(\nu+2)_p\,(-e^{-y_2})^p}{p!}\int_0^{(1+e^{-x_1})^{-1}} t^{\nu\gamma_{s-j}+p}\,dt$$

$$=\sum_{p=0}^{\infty}\frac{(\nu+2)_p\,(-e^{-y_2})^p}{p!}\frac{1}{\nu\gamma_{s-j}+p+1}$$

$$\times\frac{1}{(1+e^{-x_1})^{\nu\gamma_{s-j}+p+1}} \qquad (29)$$

Now putting the value of (29) in (27), we get

$$g_{d[r,s,n,m,k]}(y_1,y_2)=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$

$$\times\nu^2(\nu+1)^2 e^{-y_1}e^{-y_2}$$

$$\times\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}\sum_{p=0}^{\infty}\frac{(\nu+2)_p\,(-e^{-y_2})^p}{p!}$$

$$\times\frac{1}{\nu\gamma_{s-j}+p+1}\int_{-\infty}^{\infty}\frac{e^{-x_1}}{(1+e^{-x_1})^{\nu\gamma_{r-i}-\nu+p+1}}.$$

$$\times\frac{1}{(1+e^{-x_1}+e^{-y_1})^{\nu+2}}\,dx_1 \qquad (30)$$

Setting $z=(1+e^{-x_1})^{-1}$, and using the relation (15), we get

$$=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\nu^2(\nu+1)^2 e^{-y_1}e^{-y_2}$$

$$\times\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}\sum_{p=0}^{\infty}\frac{(\nu+2)_p\,(-e^{-y_2})^p}{p!}$$

$$\times\sum_{l=0}^{\infty}\frac{(\nu+2)_l\,(-e^{-y_1})^l}{l!}\frac{1}{\nu\gamma_{s-j}+p+1}\frac{1}{\nu\gamma_{r-i}+p+l+2}.$$

$$(31)$$

Nothing that

$$(\lambda+\eta)=\frac{\lambda(\lambda+1)_\eta}{(\lambda)_\eta}\quad\text{and}\quad(\lambda+\eta+n)=\frac{\lambda(\lambda+1)_{\eta+n}}{(\lambda)_{\eta+n}}$$

(Srivastava and Karlson, 1985)

and using in (31), we get the result as given in (25). (26) can be obtained by simplifying (25) and taking $m\to-1$.

The joint moment generating function of $Y_{d[r,n,m,k]}$ and $Y_{d[s,n,m,k]}$ in view of (25) is given as

$$M_{d[r,s,n,m,k]}(t_1,t_2)=\frac{C_{s-1}\,\nu^2(\nu+1)^2}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$

$$\times\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}$$

$$\times\frac{1}{\nu\gamma_{s-j}+1}\frac{1}{\nu\gamma_{r-i}+2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{t_1y_1}e^{t_2y_2}e^{-y_1}e^{-y_2}$$

$$\times F^{1:2;1}_{1:1;0}\left[\begin{matrix}(\nu\gamma_{r-i}+2):(\nu+2),(\nu\gamma_{s-j}+1);(\nu+2);\\ ;\,-e^{-y_1},-e^{-y_2}\\ (\nu\gamma_{r-i}+3):(\nu\gamma_{s-j}+2);\end{matrix}\right].$$

$$dy_1dy_2 \qquad (32)$$

Let $e^{-y_1}=z_1$ and $e^{-y_2}=z_2$, then

$$=\frac{C_{s-1}\,\nu^2(\nu+1)^2}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$

$$\times\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}$$

$$\times\frac{1}{\nu\gamma_{s-j}+1}\frac{1}{\nu\gamma_{r-i}+2}\int_0^{\infty}\int_0^{\infty}z_1^{-t_1}z_2^{-t_2}$$

$$\times\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\frac{(\nu\gamma_{r-i}+2)_{p+l}(\nu+2)_p}{(\nu\gamma_{r-i}+3)_{p+l}}$$

$$\times\frac{(\nu\gamma_{s-j}+1)_p\,(\nu+2)_l}{(\nu\gamma_{s-j}+2)_p}\frac{(-z_1)^l}{l!}\frac{(-z_2)^p}{p!}\,dz_1dz_2. \qquad (33)$$

Now using relation $(\lambda)_{m+n}=(\lambda)_m(\lambda+m)_n$ as given by Srivastava and Karlson (1985), we have

$$M_{d[r,s,n,m,k]}(t_1,t_2)=\frac{C_{s-1}\,\nu^2(\nu+1)^2}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$

$$\times\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}\frac{1}{\nu\gamma_{s-j}+1}\frac{1}{\nu\gamma_{r-i}+2}$$

$$\times\left\{\int_0^{\infty}z_2^{-t_2}\sum_{p=0}^{\infty}\frac{(\nu\gamma_{r-i}+2)_p(\nu+2)_p}{(\nu\gamma_{r-i}+3)_p}\frac{(\nu\gamma_{s-j}+1)_p}{(\nu\gamma_{s-j}+2)_p}\frac{(-z_2)^p}{p!}\right.$$

$$\left.\times\left[\int_0^{\infty}z_1^{-t_1}\,{}_2F_1\left[\begin{matrix}(\nu\gamma_{r-i}+2+p),\;(\nu+2)\\ ;\,-z_1\\ (\nu\gamma_{r-i}+3+p)\end{matrix}\right]dz_1\right]dz_2\right\}.$$

$$(34)$$

Now on application of (19) in (34) and after simplification, we get

$$= \frac{C_{s-1} v^2 (v+1)}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$

$$\times \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j}$$

$$\times \frac{1}{v\gamma_{s-j}+1} \frac{1}{v\gamma_{r-i}+t_1+1} \frac{\Gamma(1-t_1)\,\Gamma(v+t_1+1)}{\Gamma(v+1)}$$

$$\times \left[ \int_0^\infty z_2^{-t_2} \, {}_3F_2 \left[ \begin{matrix} (v+2),\,(v\gamma_{r-i}+t_1+1),(v\gamma_{s-j}+1) \\ ;-z_2 \\ (v\gamma_{r-i}+t_1+2),\,(v\gamma_{s-j}+2) \end{matrix} \right] dz_2 \right]. \quad (35)$$

Now using Prudnikov *et al.* (1986)

$$\int_0^\infty x^{s-1} {}_3F_2 \left[ \begin{matrix} (a_1),\ (a_2),\ (a_3) \\ ;\ -x \\ (b_1),\ (b_2) \end{matrix} \right] dx\,,$$

$$= \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(s)\Gamma(a_1-s)\Gamma(a_2-s)\Gamma(a_3-s)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(b_1-s)\Gamma(b_2-s)}\,,$$

$[0 < Re\, s < Re\, a_j;\ j = 1,2,3]$ and simplifying, we get

$$M_{d[r,s,n,m,k]}(t_1,t_2)$$

$$= \frac{\Gamma(1-t_1)\,\Gamma(1-t_2)\,\Gamma(v+t_1+1)\Gamma(v+t_2+1)}{\Gamma(v+1)\Gamma(v+1)}$$

$$\times \frac{1}{\prod_{i=1}^{r}(1+\frac{t_1+t_2}{v\gamma_i})} \frac{1}{\prod_{i=r+1}^{s}(1+\frac{t_2}{v\gamma_i})}. \quad (36)$$

Cumulant generating function of two concomitant $Y_{d[r,n,m,k]}$ and $Y_{d[s,n,m,k}$ is given by

$$K_{d[r,s,n,m,k]}(t_1,t_2) = \ln\Gamma(1-t_1) + \ln\Gamma(1-t_2)$$

$$+\ln\Gamma(v+1+t_1) + \ln\Gamma(v+1+t_2) - 2\ln\Gamma(v+1)$$

$$-\sum_{i=1}^{r} \ln\left(1+\frac{t_1}{v\gamma_i}+\frac{t_2}{v\gamma_i}\right) - \sum_{i=r+1}^{s} \ln\left(1+\frac{t_2}{v\gamma_i}\right). \quad (37)$$

Noting that,

$$Cov[Y_{d[r,n,m,k]}, Y_{d[s,n,m,k]}] = \frac{d^2}{dt_1 dt_2} K_{[r,s,n,m,k]}(t_1,t_2)$$

$at\ t_1 = 0, t_2 = 0.$

and using the relation (24), we get

$$Cov[Y_{d[r,n,m,k]}, Y_{d[s,n,m,k]}] = \frac{1}{v^2} \sum_{i=1}^{r} \frac{1}{\gamma_i^2}. \quad (38)$$

**Remark 3.1:** Set $m = 0, k = 1$ and replace $n-r+1$ by $s$ and $n-s+1$ by $r$ in (38), we get covariance between concomitant of order statistics from bivariate Burr II distribution as

$$Cov[Y_{[r:n]}, Y_{[s:n]}] = \frac{1}{v^2} \sum_{i=1}^{r} \frac{1}{(n+1-i)^2}$$

$$= \frac{1}{v^2} \sum_{l=s}^{n} \frac{1}{l^2}.$$

This result was also obtained by Begum and Khan (1997).

**Remark 3.2:** As $m \to -1$ in (39), we get covariance of concomitant of $k^{th}$ record values from bivariate Burr II distribution as

$$Cov[Y_{d[r,n,-1,k]}, Y_{d[s,n,-1,k]}] = \frac{r}{(vk)^2}.$$

## 4. Conclusion

In this paper, we have obtained the marginal and joint moment generating function of concomitants of *dgos* from bivariate Burr II distribution. A good application of this setup is the use of *mgf* of concomitants of *dgos* for computing the moments of any order of concomitant of order statistics, record values, sequential order statistics etc.

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