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Generalisation of the transient field solution for a thin layer of finite thickness

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ABSTRACT

We derive a closed-form solution for the transient magnetic field developed by a magnetic source located above a thin conductive layer. The solution extends the solution found by J.C. Maxwell for a conductive thin sheet. In addition to Maxwell's expression, the new solution explicitly depends on the thickness of the conductive layer. It can be used in a rapid inversion algorithm. The new solution allows an interpreter to estimate the thickness and conductivity of the conductive layer in addition to its conductance.

INTRODUCTION

The transient solution found by Maxwell (1891) for a planar homogeneous thin sheet surrounded by an insulator and energised by a magnetic dipole is still widely used for interpretation of airborne electromagnetic (AEM) data. In this solution, the dipole moment is zero up to some moment ($t=0$), then it changes to a non-zero value and remains constant afterwards. The change in the primary magnetic field induces an electric current in the conductor. In turn, the induced current develops a secondary magnetic field. The secondary field above the conductive layer is equivalent to the field of another magnetic dipole located on the other side of the thin sheet. The (imaginary) dipole at $t=0$ coincides with a mirror image of the source. As time increases, the imaginary dipole recedes from the conductive layer at the speed of

$$v_S = \frac{2}{\mu_0 S} \ , \tag{1}$$

where S is the conductance of the layer, and $\mu_0 = 4\pi \cdot 10^{-7}$ is the magnetic permeability of free space.

The simplicity of Maxwell's solution is one of its benefits. However, the thin sheet model is of limited utility in interpretation of AEM data. A more practical model consists a conductive layer of finite thickness (slab) rather than a thin sheet. Immediately after the step-on change of the external source, the induced current is concentrated at the upper surface of the slab. The induced current decays and redistributes itself deeper into the layer as time progresses. At later times, the electric field is almost uniformly distributed across the layer. From this moment, the response of the layer becomes identical to the response of a thin sheet with the conductance

$$S = h\sigma \ , \tag{2}$$

where h and σ are the layer thickness and conductivity, respectively. Thus, the thin sheet solution becomes applicable to a slab model at a late stage of the transient process.

Interpretation of AEM data often involves a numerical simulation of a slab model response. The solution can be significantly accelerated when the analytic properties of the response function in the frequency domain are known (Goldman and Fitterman, 1987). A number of fast approximate solutions are also used to estimate the response (e.g., Macnae and Lamontagne, 1987). Nevertheless, the simplicity and beauty of Maxwell's solution is a good incentive for finding a closed-form solution applicable to the slab model.

THEORY

Consider a horizontal layer of thickness h and conductivity σ. The medium outside the layer is assumed to be non-conductive. A cartesian coordinate system, with the OX- and OY-axes directed along the surface of the layer and OZ-axis pointing downwards is used in our consideration. We start in the frequency domain using $exp(-\iota\omega t)$ as the time factor.

As shown by Vasseur and Weidelt (1977), an electromagnetic field in a 1-D medium can be described using two scalar potentials. The potentials define the toroidal and poloidal modes of the electromagnetic field. Electric currents of the toroidal mode are confined to horizontal planes. The poloidal currents cross the layers comprising the stratified structure.

In a stratified earth's model energized by an external current flowing above the earth, the poloidal mode is not excited. Therefore, the electromagnetic field can be expressed in terms of the potential of the toroidal mode V. In particular, in the insulator

$$\boldsymbol{H} = -\nabla \frac{\partial V}{\partial n} \ , \tag{3}$$

where $\boldsymbol{n} = -\boldsymbol{e}_z$ is an upward unit vector. The potential satisfies the Laplace's equation

$$\nabla^2 V = 0 \ . \tag{4}$$

Above the conductive layer, the magnetic field can also be represented as a sum of the primary and secondary fields. For a magnetic dipole

$$\boldsymbol{M} = \boldsymbol{M}_\tau + M_n \boldsymbol{n} \ , \tag{5}$$

where $\boldsymbol{M}_\tau$ and M_n are the horizontal and vertical components of the magnetic moment, the primary magnetic field created by the source is

$$\boldsymbol{H}^p(r,z) = \nabla \left[\boldsymbol{M} \nabla \frac{1}{4\pi\sqrt{r^2+(z-z_0)^2}} \right] \tag{6}$$

In this expression, the dipole is assumed to be located at the point $x=y=0$, $z=z_0$ (the dipole altitude is $-z_0$) and a step current turned on at $t=0$. Thus,

$$\frac{\partial V^p}{\partial n}(r,z) = -M\nabla\frac{1}{4\pi\sqrt{r^2+(z-z_0)^2}} \quad . \tag{7}$$

In the Fourier domain this equation implies that for $z_0<z<0$

$$\frac{\partial V^p}{\partial n}(r,z) = \mathcal{F}^{-1}\left[kV^p(k,0)e^{-kz}\right] \ , \tag{8}$$

where

$$kV^p(k,0) = -\frac{1}{2}\left[M_n + \iota\frac{k}{k}M_\tau\right]e^{kz_0} \ , \tag{9}$$

and $\mathcal{F}^{-1}$ denotes an inverse Fourier transform.

An arbitrary function $f(r,z)$ and its Fourier image $f(k,z)$ are related by

$$f(r,z) = \mathcal{F}^{-1}[f(k,z)] = \int_{\mathbb{R}^2} f(k,z)e^{\iota k,r}\frac{ds_k}{(2\pi)^2} \ , \tag{10}$$

where $r=xe_x+ye_y$ and $k=k_xe_x+k_ye_y$ are the horizontal radius vector and the wave number, respectively.

The secondary magnetic field caused by the currents induced in the conductive layer can be calculated using the response function

$$\gamma_k(\omega) = \frac{V^s(k,0,\omega)}{V^p(k,0,\omega)} \ , \tag{11}$$

which specifies the relative effect of the currents flowing inside the earth to that of the external source. A similar parameter is often used in the theory of global electromagnetic induction in the earth (Schmidt, 1918).

Considering the slab model as a 1-D structure, it is easy to see that

$$\gamma_k(\omega) = -\frac{\kappa_0^2\tanh[\kappa_k h]/\kappa_k}{2k+[2k^2+\kappa_0^2]\tanh[\kappa_k h]/\kappa_k} \tag{12}$$

where

$$\kappa_k(\omega) = \sqrt{k^2-\iota\omega\mu_0\sigma} \ . \tag{13}$$

The conductive layer can be considered as a thin layer if the inequality

$$|\kappa_0(\omega)|\,h = h\sqrt{\omega\mu_0\sigma} \ll 1 \tag{14}$$

is valid for the essential part of the frequency spectrum. Another restriction

$$kh \ll 1 \tag{15}$$

should be imposed on the spatial frequency of the observed response. For the model under consideration, these two conditions are not independent, because

$$\omega\mu_0\sigma h^2 = \omega\mu_0 Sh \sim \frac{h}{v_S t} \ , \tag{16}$$

$$kh \sim \frac{h}{|z|+|z_0|+v_s t} < \frac{h}{v_S t} \ ,$$

where $|z_0|$ are $|z|$ are the source and receiver altitudes, respectively, and r is a horizontal separation between the source and receiver.

For a thin layer, equation (12) specifying the "internal-to-external" response function reduces to

$$\gamma_k(\omega) \approx \frac{\iota\omega\mu_0 S e^{-\frac{2}{3}kh}}{2k\left(1+\frac{1}{3}kh\right)-\iota\omega\mu_0 S} \ , \tag{17}$$

where S is the layer conductance.

From equation (11), the time-domain response can be calculated as

$$V^s(k,z,t) = -V^p(k,0)\,e^{kz} \times \tag{18}$$

$$\int_{-\infty}^{+\infty}\gamma_k(\omega)\,e^{-\iota\omega t}\frac{d\omega}{2\pi\iota\omega} \ ,$$

where, for an external source working in the step-on mode,

$$V^p(k,0,\omega) = -\frac{V^p(k,0)}{\iota\omega} \ . \tag{19}$$

The integral in equation (18) can be calculated using the residue theorem. For the response function (17), the secondary field potential

$$V^s(k,z,t) = -V^p(k,0)\times \tag{20}$$

$$\left(1-k^2\frac{hv_S t}{3}\right)e^{k\left(z-\frac{2}{3}h-v_S t\right)} \ .$$

The vertical derivative of the secondary potential

$$\frac{\partial V^s}{\partial n}(r,z,t) = \tag{21}$$

$$\left(1-\frac{hv_S t}{3}\frac{\partial^2}{\partial n^2}\right)\mathcal{F}\left[kV^p(k,0)\,e^{k\left(z-\frac{2}{3}h-v_S t\right)}\right] \ .$$

From equations (3) and (9), the secondary magnetic field at $t>0$ is

$$H^s(r,z,t) = \left(1-\frac{hv_S t}{3}\frac{\partial^2}{\partial n^2}\right)\times \tag{22}$$

$$\nabla\left[M^*\nabla\frac{1}{4\pi\sqrt{r^2+\left(z+z_0-\frac{2}{3}h-v_S t\right)^2}}\right] \ ,$$

where

$$M^* = M_\tau - M_n n \tag{23}$$

is a mirror image of the source magnetic dipole with respect to the earth's surface.

If the terms depending on the layer thickness h are ignored, equation (22) coincides to the field of the image dipole that at $t=0+$ is located at $x=y=0$, $z=-z_0$ and recedes downwards at $t>0$ with speed v_S. In this approximation, equation (22) coincides with Maxwell's solution.

From equations (14) and (16), the valid time range of this solution is restricted by the condition

$$\sqrt{\frac{h}{v_S t}} \ll 1 \ . \tag{24}$$

Expressions (6) for the primary magnetic fields can also presented in an alternative form

$$H^p(r,z) = \hat{\mathfrak{C}}(r,z-z_0)\, M \ , \tag{25}$$

where matrix elements of the 3×3 matrix $\mathfrak{C}$ are

$$\mathfrak{C}_{xx}(r,z) = \frac{3\xi^2-1}{4\pi R^3} \ , \tag{26}$$

$$\mathfrak{C}_{yy}(r,z) = \frac{3\eta^2-1}{4\pi R^3} \ ,$$

$$\mathfrak{C}_{nn}(r,z) = \frac{2-3(\xi^2+\eta^2)}{4\pi R^3} \ ,$$

$$\mathfrak{C}_{xy}(r,z) = \mathfrak{C}_{yx}(r,z) = \frac{3\xi\eta}{4\pi R^3} \ ,$$

$$\mathfrak{C}_{xn}(r,z) = \mathfrak{C}_{nx}(r,z) = -\frac{3\xi\zeta}{4\pi R^3} \ ,$$

$$\mathfrak{C}_{yn}(r,z) = \mathfrak{C}_{ny}(r,z) = -\frac{3\eta\zeta}{4\pi R^3} \ ,$$

and $R^2 = x^2 + y^2 + z^2$, $\xi = x/R$, $\eta = y/R$, $\zeta = z/R$.

Similarly,

$$H^s(r,z,t) = \tag{27}$$

$$\left(1-\frac{h v_s t}{3}\frac{\partial^2}{\partial n^2}\right)\hat{\mathfrak{C}}\left(r, z+z_0-\frac{2}{3}h-v_s t\right) M^* \ ,$$

and

$$\frac{\partial^2 \mathfrak{C}_{xx}}{\partial n^2}(r,z) = 3\,\frac{35\xi^2\zeta^2+5\eta^2-4}{4\pi R^5} \ , \tag{28}$$

$$\frac{\partial^2 \mathfrak{C}_{yy}}{\partial n^2}(r,z) = 3\,\frac{35\eta^2\zeta^2+5\xi^2-4}{4\pi R^5} \ ,$$

$$\frac{\partial^2 \mathfrak{C}_{nn}}{\partial n^2}(r,z) = 3\,\frac{8-40(\xi^2+\eta^2)+35(\xi^2+\eta^2)^2}{4\pi R^5} \ ,$$

$$\frac{\partial^2 \mathfrak{C}_{xy}}{\partial n^2}(r,z) = 15\xi\eta\,\frac{6-7\xi\eta}{4\pi R^5} \ ,$$

$$\frac{\partial^2 \mathfrak{C}_{xn}}{\partial n^2}(r,z) = -15\xi\zeta\,\frac{4-7(\xi^2+\eta^2)}{4\pi R^5} \ ,$$

$$\frac{\partial^2 \mathfrak{C}_{yn}}{\partial n^2}(r,z) = -15\eta\zeta\,\frac{4-7(\xi^2+\eta^2)}{4\pi R^5} \ ,$$

To analyse the benefits of the improved asymptotic solution, we consider a conductive slab of thickness of 50 m. The primary field is generated by a vertical magnetic dipole at an altitude of 120 m. The vertical magnetic field is measured by a receiver at an attitude of 60 m. The horizontal separation between the source and receiver is 100 m which is a typical configuration of an airborne acquisition system. We compare an "exact" expression for the slab response H_z^L with two asymptotic solutions. The exact numerical response is calculated using a double Fourier-Bessel transform and expression (12) for the slab response in the Fourier domain. One of the asymptotic solutions (H_z^S) represents the model response calculated using Maxwell's analytic solution for an infinitely thin conductive sheet.

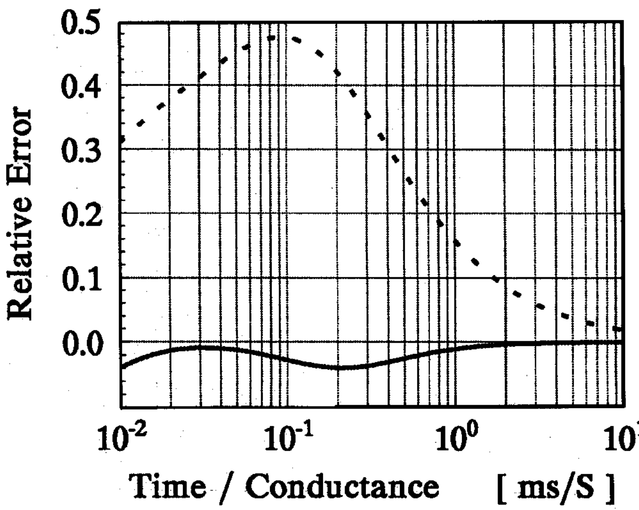

Figure 1. Errors of improved (solid) and classic (dashed) solutions: layer thickness 50 m. Altitudes: transmitter 120 m, receiver 60 m; separation 100 m.

Another asymptotic solution (H_z^{Sh}) accounts for the layer thickness.

In Figure 1, the dashed curve represents the relative error $H_z^S / H_z^L - 1$ of the classical thin sheet solution. The solid curve shows the error $H_z^{Sh} / H_z^L - 1$ of the new asymptotic expression (22), which accounts for the layer thickness.

The improved solution is more accurate at all time delays after turning on the source. The error of the improved solution does exceed 4%. Further numerical experiments show that the maximum error of the new solution decreases approximately as the second power of the layer thickness.

CONCLUSIONS

An analytic expression derived for a slab of finite thickness preserves the advantage of the thin sheet solution by J.C. Maxwell because it is also expressed in terms of elementary functions. The model response described by such a solution can be rapidly calculated. It can be used as the basis for rapid inversion of airborne data. The improved solution is much more accurate than the thin sheet solution and is applicable over a wider time range.

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