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Supplementary Material

Deep bed filtration and formation damage by particles with distributed properties

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Supplementary Material

1. Supplementary Material A. Governing Equations for Deep Bed Filtration.

Governing equations to describe deep bed filtration accounting for permeability decline is defined through equations (A.1) to (A.3) describing mass balance, particle capture and Darcy's law respectively.

$$\phi\frac{\partial c}{\partial t}+U\frac{\partial c}{\partial x}=-\frac{\partial s}{\partial t} \qquad (\ .1)$$

$$\frac{\partial s}{\partial t}=\lambda\left(1-\frac{s}{s_m}\right)cU \qquad (\ .2)$$

$$U=-\frac{k_0}{\mu(1+\beta s)}\frac{\partial p}{\partial x} \qquad (\ .3)$$

Here ϕ stands for the porosity, U is the Darcy's flow velocity, c and s are the suspended and retained particles concentrations respectively, s_m is the maximum retained concentration, λ is the filtration coefficient which is defined as the probability of particle capture per unit length of its trajectory, p is the pressure, k_0 is the initial permeability, β is the formation damage coefficient, μ is the carrier fluid's viscosity, t is time and x is the cartesian coordinate.

The process of continuous injection with constant concentration into a clean medium defines the initial and boundary conditions to be as described in equation (A.4).

$$t=0:\ c=s=0;\quad x=0:\ c=c^0 \qquad (\ .4)$$

2. Supplementary Material B. Exact Analytical Solution for 1D Clean Bed Filtration

Starting from dimensionless parameters and variables (B.1), the following conversion is implemented on governing equations (.1) to (.3) and initial and boundary conditions (.4) to obtain equations (B.2) to (B.4).

$$x\to\frac{x}{L},t\to\frac{Ut}{\phi L},c\to\frac{c}{c^0},s\to\frac{s}{\phi c^0},p\to\frac{kp}{UL\mu},\lambda\to\lambda L,s_m=\frac{s_m}{\phi c^0},f(c)\to\frac{Lf(c)}{c^0} \qquad (\ .5)$$

$$\frac{\partial c}{\partial t}+\frac{\partial c}{\partial x}=-\frac{\partial s}{\partial t} \qquad (\ .6)$$

$$\frac{\partial s}{\partial t}=h(s)f(c) \qquad (\ .7)$$

$$1=-\frac{1}{1+\beta\phi c^{0}s}\frac{\partial p}{\partial x} \tag{.8}$$

$h(s)$ and $f(c)$ are called filtration function and suspension function respectively. Here, $h(s)$ is defined as blocking (Langmuir's) function (B.5). $f(c)$ can be defined in various ways according to the physics of the problem. In classical deep bed filtration theory, $f(c)$ is equal to λc as demonstrated in Supplementary Material A.

$$h(s)=\left(1-\frac{s}{s_m}\right) \tag{.9}$$

The dimensionless initial and boundary conditions are defined in equations (B.6) and (B.7):

$$t=0: c=s=0$$

(.10)

$$x=0: c=1,\ p=\frac{kp_0}{U\mu L}$$

(.11)

To solve this system of equations with the mentioned initial and boundary conditions, first a new independent variable is introduced.

$$\tau=t-x \tag{.12}$$

Equations (B.9) and (B.10) represent the new form of equations (.6) and (.7) in the new reference system and accounting for blocking (Langmuir's) function (.9).

$$\frac{\partial c}{\partial x}=-\left(1-\frac{s}{s_m}\right)f(c) \tag{.13}$$

$$\frac{\partial s}{\partial \tau}=\left(1-\frac{s}{s_m}\right)f(c) \tag{.14}$$

Another way of representing equations (.13) and (.14) is by equation (B.11):

$$\frac{\partial c}{\partial x}=-\frac{\partial s}{\partial \tau} \tag{.15}$$

Also, the dimensionless initial and boundary conditions in the new reference system are defined with equations (B.12) and (B.13).

$$\tau = -x:\ c = s = 0 \tag{.16}$$

$$\tau = t:\ c = 1 \tag{.17}$$

Figure 1 shows different zones in initial and Lagrangian coordinates.

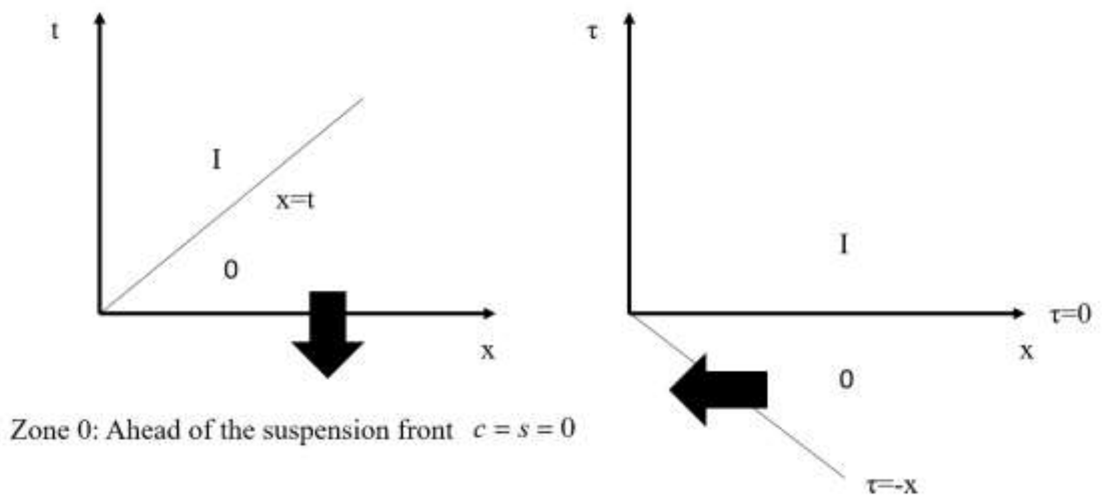

Figure 1:Introducing different zones in initial and Lagrangian coordinates.

Suspended and retained particles concentrations ahead of the front are zero.

$$c = s = 0 \tag{.18}$$

Suspension concentration behind the font can be obtained substituting $s = 0$ into equation (.13).

$$\frac{\partial c}{\partial x} = -f\left(c\right) \tag{.19}$$

Rearranging equation (.19) yields:

$$\int_{c^-(x)}^{1} \frac{du}{f\left(u\right)} = x \tag{.20}$$

$$\varphi(1) - \varphi\left(c^-\left(x,0\right)\right) = x,\quad \varphi'\left(c\right) = \frac{1}{f\left(c\right)} \tag{.21}$$

Expressing retained concentration in equation (.13) results in:

$$\frac{1}{f(c)}\frac{\partial c}{\partial x} + 1 = \frac{s}{s_m} \tag{.22}$$

Also, from equation (B.19) we conclude that:

$$\frac{\partial\varphi(c)}{\partial x}=\frac{1}{f(c)}\frac{\partial c}{\partial x}=\frac{\partial\varphi(c)}{\partial c}\frac{\partial c}{\partial x} \tag{ .23}$$

From equations (.22) and (.23), equation (B.20) is obtained.

$$s(x,\tau)=s_m\left[\frac{1}{f(c)}\frac{\partial c}{\partial x}+1\right]=s_m\left[\frac{\partial\varphi(c)}{\partial x}+1\right] \tag{ .24}$$

Substituting equation (.24) into equation (.15):

$$\frac{\partial}{\partial\tau}\left(s_m\frac{\partial\varphi(c)}{\partial x}\right)+\frac{\partial c}{\partial x}=0 \tag{ .25}$$

Changing the order of derivatives in equation (.25) and integrating in x, accounting for boundary conditions yields:

$$s_m\frac{\partial\varphi(c)}{\partial\tau}+c=1 \tag{ .26}$$

Rearranging equation (.26):

$$\frac{\partial c}{(1-c)\,f(c)}=\frac{\partial\tau}{s_m} \tag{ .27}$$

Taking integral of both sides of equation (.27):

$$\int_{c^-(x)}^{c}\frac{du}{(1-u)\,f(u)}=\frac{\tau}{s_m} \tag{ .28}$$

Taking derivative in *x* of both sides of equation (.28) leads to:

$$\frac{1}{(1-c)\,f(c)}\frac{\partial c}{\partial x}-\frac{1}{\left(1-c^-(x,0)\right)f\left(c^-(x,0)\right)}\frac{dc^-(x,0)}{dx}=0 \tag{ .29}$$

From substituting equation (.24) into equation (.29) we can determine $s(x,\tau)$.

$$\frac{s(x,\tau)-s_m}{1-c(x,\tau)}=\frac{-s_m}{\left(1-c(x,0)\right)} \tag{ .30}$$

$$s(x,t)=s_m\left[\frac{c(x,t)-c^-(x,x)}{1-c^-(x,x)}\right] \tag{ .31}$$

Pressure drop across the core can be determined using equation (B.28).

$$\Delta p(t) = 1 + \beta\phi c^0 \int_0^1 s(u,t)\,du \qquad (.32)$$

Here, β is the formation damage coefficient and c^0 is the initial concentration of the injected particles. The dimensionless pressure drop or impedance is defined with the following equation.

$$J(t) = \frac{\Delta p(t)}{\Delta p(t=0)} \qquad (.33)$$

3. Supplementary Material C. Analytical Models for Different Suspension Functions

Traditional (Classical) suspension function:

The traditional suspension function is defined using equation (C.1).

$$f(c) = \lambda c \qquad (.34)$$

By substituting equation (.34) into equation (.20) we obtain equation (C.2).

$$\int_{c^-(x)}^{1} \frac{du}{\lambda u} = x \qquad (.35)$$

Suspended concentration just behind the front can be calculated using equation (.35).

$$c^-(x) = e^{-x\lambda} \qquad (.36)$$

Substituting equation (.34) into equation (.28) yields equation (C.4).

$$\int_{c^-(x)}^{c} \frac{du}{(1-u)\lambda u} = \frac{\tau}{s_m} \qquad (.37)$$

Using the expression obtained for the suspended concentration just behind the front (.36) and substituting it in equation (.37) we can derive the exact solution for the suspended concentration behind the front.

$$c = \frac{1}{1 - e^{-\lambda\frac{\tau}{s_m}} + e^{\left(-\lambda\frac{\tau}{s_m} + \lambda x\right)}}$$

(.38)

Substituting equations (.38) and (.36) into equation (.31) leads to finding the expression for the retained particles.

$$s(x,t)=s_m\frac{\left(1-e^{-\lambda\frac{\tau}{s_m}}\right)}{\left(1-e^{-\lambda\frac{\tau}{s_m}}+e^{\left(-\lambda\frac{\tau}{s_m}+\lambda x\right)}\right)}$$

(.39)

Quadratic suspension function:

The quadratic suspension function is defined using equation (C.7).

$$f(c)=\lambda c+\lambda bc^2$$

(.40)

By substituting equation (.40) into equation (.20) we obtain equation (C.8).

$$\int_{c^-(x)}^{1}\frac{du}{\lambda u+\lambda bu^2}=x$$

(.41)

Suspended concentration just behind the front can be calculated using equation (.41).

$$c^-(x)=\frac{1}{e^{\lambda x}(1+b)-b} \qquad (.42)$$

Substituting equation (.40) into equation (.28) yields equation (C.10).

$$\int_{c^-(x)}^{c}\frac{du}{(1-u)(\lambda u+\lambda bu^2)}=\frac{\tau}{s_m} \qquad (.43)$$

The suspended concentration can be expressed implicitly using equation (C.11).

$$\frac{-1}{\lambda+\lambda b}\ln(1-u)\big|_{c^-(x)}^{c}+\frac{1}{\lambda}\ln(u)\big|_{c^-(x)}^{c}-\frac{b}{\lambda(1+b)}\ln\left(\frac{1}{b}+u\right)\big|_{c^-(x)}^{c}=\frac{\tau}{s_m} \qquad (.44)$$

Equation (.31) leads to finding the expression for the retained particles.

Asymptotic suspension function:

Supplementary Material D explains derivations of the asymptotic suspension functions.

The asymptotic suspension function used for matching in this paper is defined using equation (C.12). Here, subscripts 1 and 2 stand for the first and the second population of colloids respectively. ε represents a small value.

$$f\left(c\right)=\lambda_1(\varepsilon c^{\frac{\lambda_1}{\lambda_2}})+\lambda_2(c-\varepsilon c^{\frac{\lambda_1}{\lambda_2}}) \quad (.45)$$

By substituting equation (.45) into equation (.20) we obtain equation (C.13).

$$\int_{c^-(x)}^{1}\frac{du}{\lambda_1(\varepsilon u^{\frac{\lambda_1}{\lambda_2}})+\lambda_2(u-\varepsilon u^{\frac{\lambda_1}{\lambda_2}})}=x \quad (.46)$$

The suspended concentration just behind the front can be calculated from equation (.46). Equation (C.14) is obtained by substituting equation (.45) into equation (.28); which gives us the suspended concentration behind the front.

$$\int_{c^-(x)}^{c}\frac{du}{\left(1-u\right)\left(\lambda_1(\varepsilon u^{\frac{\lambda_1}{\lambda_2}})+\lambda_2(u-\varepsilon u^{\frac{\lambda_1}{\lambda_2}})\right)}=\frac{\tau}{s_m} \quad (.47)$$

Equation (.31) leads to finding the expression for the retained particles.

4. Supplementary Material D. Derivations of the Asymptotic Suspension Functions

In this section we aim to explain how to derive asymptotic forms of suspension functions for a binary system. As explained in the paper, here, we have six different asymptotic suspension functions, corresponding to different assumptions and orders of magnitudes of expansions. Here, the derivation of the suspension function is explained for the first case. To obtain the rest of them the same mathematical procedure must be followed. Table 1 and

Table 2 show the summary of calculations in all cases.

For the first case of asymptotic formulations, we have the assumptions of $c_1^0=\varepsilon$ and first order expansion. For a binary system, we have the following system of equations (Supplementary Material, Section E) for the suspended particles concentration:

$$c_1+c_2=c,\ \frac{c_1}{c_1^{\ 0}}=\left(\frac{c_2}{c_2^{\ 0}}\right)^{\frac{\lambda_1}{\lambda_2}}$$

(.48)

Using asymptotic expansions up to first order for each suspension concentration, we have:

$$c_1=x_0+\varepsilon x_1,\ c_2=y_0+\varepsilon y_1 \quad (.49)$$

Substitting equations (.48) into equations (.49) we obtain equations (D.3) and (D.4):

$$x_0 + \varepsilon x_1 + y_0 + \varepsilon y_1 = c$$

(.50)

$$\frac{x_0 + \varepsilon x_1}{c_1^{\,0}} = \left(\frac{y_0 + \varepsilon y_1}{c_2^{\,0}} \right)^{\frac{\lambda_1}{\lambda_2}}$$

(.51)

When $c_1^0 = \varepsilon$ equation (.51) becomes:

$$\frac{x_0 + \varepsilon x_1}{\varepsilon} = \left(\frac{y_0 + \varepsilon y_1}{1-\varepsilon} \right)^{\frac{\lambda_1}{\lambda_2}} \qquad (.52)$$

Taylor expansion up to first order for the right-hand side of equation (.52) gives us:

$$g = \left(\frac{y_0 + \varepsilon y_1}{1-\varepsilon} \right)^{\frac{\lambda_1}{\lambda_2}} \qquad (.53)$$

$$g\,|_{\varepsilon=0} = y_0^{\frac{\lambda_1}{\lambda_2}}$$

(.54)

$$\frac{dg}{d\varepsilon}\,|_{\varepsilon=0} = \frac{\lambda_1}{\lambda_2}\, y_0^{\frac{\lambda_1}{\lambda_2}-1} (y_0 + y_1)$$

(.55)

$$g = g\,|_{\varepsilon=0} + (\varepsilon - 0)\frac{dg}{d\varepsilon}\,|_{\varepsilon=0}$$

(.56)

$$g = y_0^{\frac{\lambda_1}{\lambda_2}} + \varepsilon \frac{\lambda_1}{\lambda_2}\, y_0^{\frac{\lambda_1}{\lambda_2}-1} (y_0 + y_1)$$

(.57)

Substituting equation (.57) into equation (.52) we obtain equation (D.11):

$$\frac{x_0 + \varepsilon x_1}{\varepsilon} = y_0^{\frac{\lambda_1}{\lambda_2}} + \varepsilon \frac{\lambda_1}{\lambda_2}\, y_0^{\frac{\lambda_1}{\lambda_2}-1} (y_0 + y_1) \qquad (.58)$$

Rearranging equation (.58) yeilds:

$$x_0+\varepsilon x_1=\varepsilon y_0^{\frac{\lambda_1}{\lambda_2}}+\varepsilon^2\frac{\lambda_1}{\lambda_2}y_0^{\frac{\lambda_1}{\lambda_2}-1}(y_0+y_1)$$

(.59)

The system of equations that needs to be solved is:

$$x_0+\varepsilon x_1+y_0+\varepsilon y_1=c$$
$$x_0+\varepsilon x_1=\varepsilon y_0^{\frac{\lambda_1}{\lambda_2}}+\varepsilon^2\frac{\lambda_1}{\lambda_2}y_0^{\frac{\lambda_1}{\lambda_2}-1}(y_0+y_1)$$

(.60)

Finding the terms corresponding to the powers of ε and successively setting them to zero gives us:

$$x_0=0$$
$$x_1=c^{\frac{\lambda_1}{\lambda_2}}$$

(.61)

$$y_0=c$$
$$y_1=-c^{\frac{\lambda_1}{\lambda_2}}$$

(.62)

Substituting equations (.61) and (.62) into equations (.49) we obtain:

$$c_1=\varepsilon c^{\frac{\lambda_1}{\lambda_2}}$$

-

$$c_2=c-\varepsilon c^{\frac{\lambda_1}{\lambda_2}}$$

(.63)

Since in a binary system, the suspension function is defined as:

$$f(c)=\lambda_1c_1+\lambda_2c_2$$

(.64)

Substituting equations (.63) in equation (.64) yields:

$$f(c)=\lambda_1(\varepsilon c^{\frac{\lambda_1}{\lambda_2}})+\lambda_2(c-\varepsilon c^{\frac{\lambda_1}{\lambda_2}})$$

(.65)

Tables 1 and 2 show a summary of the calculations for each case.

Table 1: Summary of the calculations of the suspended concentration for each population for all cases.

Case	c_1	c_2
1	$\varepsilon c^{\frac{\lambda_1}{\lambda_2}}$	$c-\varepsilon c^{\frac{\lambda_1}{\lambda_2}}$
2	$\varepsilon c^{\frac{\lambda_1}{\lambda_2}}+\varepsilon^2\frac{\lambda_1}{\lambda_2}c^{\frac{\lambda_1}{\lambda_2}}(1-c^{\frac{\lambda_1}{\lambda_2}-1})$	$c-\varepsilon c^{\frac{\lambda_1}{\lambda_2}}-\varepsilon^2\frac{\lambda_1}{\lambda_2}c^{\frac{\lambda_1}{\lambda_2}}(1-c^{\frac{\lambda_1}{\lambda_2}-1})$
3	$c-\varepsilon c^{\frac{\lambda_2}{\lambda_1}}$	$\varepsilon c^{\frac{\lambda_2}{\lambda_1}}$
4	$c-\varepsilon c^{\frac{\lambda_2}{\lambda_1}}-\varepsilon^2\frac{\lambda_2}{\lambda_1}c^{\frac{\lambda_2}{\lambda_1}}(1-c^{\frac{\lambda_2}{\lambda_1}-1})$	$\varepsilon c^{\frac{\lambda_2}{\lambda_1}}+\varepsilon^2\frac{\lambda_2}{\lambda_1}c^{\frac{\lambda_2}{\lambda_1}}(1-c^{\frac{\lambda_2}{\lambda_1}-1})$
5	$\frac{c_1^0}{c_2^0}\frac{c}{1+\frac{c_1^0}{c_2^0}}+\varepsilon\frac{c\ln(\frac{c}{c_1^0+c_2^0})}{(c_1^0+c_2^0)(\frac{1}{c_1^0}+\frac{1}{c_2^0})}$	$\frac{c}{1+\frac{c_1^0}{c_2^0}}-\varepsilon\frac{c\ln(\frac{c}{c_1^0+c_2^0})}{(c_1^0+c_2^0)(\frac{1}{c_1^0}+\frac{1}{c_2^0})}$
6	$\frac{c_1^0}{c_2^0}\frac{c}{1+\frac{c_1^0}{c_2^0}}+\varepsilon\frac{c\ln(\frac{c}{c_1^0+c_2^0})}{(c_1^0+c_2^0)(\frac{1}{c_1^0}+\frac{1}{c_2^0})}+$ $\varepsilon^2\frac{(\ln(\frac{A}{c_2^0})(\frac{B}{c_2^0}+(A\frac{\ln(\frac{A}{c_2^0})}{c_2^0}))+\frac{2B}{c_2^0}+\frac{2A\ln(\frac{A}{c_2^0})}{c_2^0}+\frac{B\ln(\frac{A}{c_2^0})}{c_2^0})}{(\frac{1}{c_1^0}+\frac{2}{c_2^0})}$	$\frac{c}{1+\frac{c_1^0}{c_2^0}}-\varepsilon\frac{c\ln(\frac{c}{c_1^0+c_2^0})}{(c_1^0+c_2^0)(\frac{1}{c_1^0}+\frac{1}{c_2^0})}-$ $\varepsilon^2\frac{(\ln(\frac{A}{c_2^0})(\frac{B}{c_2^0}+(A\frac{\ln(\frac{A}{c_2^0})}{c_2^0}))+\frac{2B}{c_2^0}+\frac{2A\ln(\frac{A}{c_2^0})}{c_2^0}+\frac{B\ln(\frac{A}{c_2^0})}{c_2^0})}{(\frac{1}{c_1^0}+\frac{2}{c_2^0})}$

Table 2: Summary of the calculations of the suspension function for all cases.

Case	$f(c)$
1	$\lambda_1(\varepsilon c^{\frac{\lambda_1}{\lambda_2}})+\lambda_2(c-\varepsilon c^{\frac{\lambda_1}{\lambda_2}})$
2	$\lambda_1(\varepsilon c^{\frac{\lambda_1}{\lambda_2}}+\varepsilon^2\frac{\lambda_1}{\lambda_2}c^{\frac{\lambda_1}{\lambda_2}}(1-c^{\frac{\lambda_1}{\lambda_2}-1}))+\lambda_2(c-\varepsilon c^{\frac{\lambda_1}{\lambda_2}}-\varepsilon^2\frac{\lambda_1}{\lambda_2}c^{\frac{\lambda_1}{\lambda_2}}(1-c^{\frac{\lambda_1}{\lambda_2}-1}))$
3	$\lambda_1(c-\varepsilon c^{\frac{\lambda_2}{\lambda_1}})+\lambda_2(\varepsilon c^{\frac{\lambda_2}{\lambda_1}})$
4	$\lambda_1(c-\varepsilon c^{\frac{\lambda_2}{\lambda_1}}-\varepsilon^2\frac{\lambda_2}{\lambda_1}c^{\frac{\lambda_2}{\lambda_1}}(1-c^{\frac{\lambda_2}{\lambda_1}-1}))+\lambda_2(\varepsilon c^{\frac{\lambda_2}{\lambda_1}}+\varepsilon^2\frac{\lambda_2}{\lambda_1}c^{\frac{\lambda_2}{\lambda_1}}(1-c^{\frac{\lambda_2}{\lambda_1}-1}))$

5	$\lambda_1\left(\frac{c_1^0}{c_2^0}\frac{c}{1+\frac{c_1^0}{c_2^0}}+\varepsilon\frac{c\ln\left(\frac{c}{c_1^0+c_2^0}\right)}{(c_1^0+c_2^0)\left(\frac{1}{c_1^0}+\frac{1}{c_2^0}\right)}\right)+\lambda_2\left(\frac{c}{1+\frac{c_1^0}{c_2^0}}-\varepsilon\frac{c\ln\left(\frac{c}{c_1^0+c_2^0}\right)}{(c_1^0+c_2^0)\left(\frac{1}{c_1^0}+\frac{1}{c_2^0}\right)}\right)$
6	$\lambda_1\left(\frac{c_1^0}{c_2^0}\frac{c}{1+\frac{c_1^0}{c_2^0}}+\varepsilon\frac{c\ln\left(\frac{c}{c_1^0+c_2^0}\right)}{(c_1^0+c_2^0)\left(\frac{1}{c_1^0}+\frac{1}{c_2^0}\right)}+\varepsilon^2\frac{\left(\ln\left(\frac{A}{c_2^0}\right)\left(\frac{B}{c_2^0}+\left(A\frac{\ln\left(\frac{A}{c_2^0}\right)}{c_2^0}\right)\right)+\frac{2B}{c_2^0}+\frac{2A\ln\left(\frac{A}{c_2^0}\right)}{c_2^0}+\frac{B\ln\left(\frac{A}{c_2^0}\right)}{c_2^0}\right)}{\left(\frac{1}{c_1^0}+\frac{2}{c_2^0}\right)}\right)+$ $\lambda_2\left(\frac{c}{1+\frac{c_1^0}{c_2^0}}-\varepsilon\frac{c\ln\left(\frac{c}{c_1^0+c_2^0}\right)}{(c_1^0+c_2^0)\left(\frac{1}{c_1^0}+\frac{1}{c_2^0}\right)}-\varepsilon^2\frac{\left(\ln\left(\frac{A}{c_2^0}\right)\left(\frac{B}{c_2^0}+\left(A\frac{\ln\left(\frac{A}{c_2^0}\right)}{c_2^0}\right)\right)+\frac{2B}{c_2^0}+\frac{2A\ln\left(\frac{A}{c_2^0}\right)}{c_2^0}+\frac{B\ln\left(\frac{A}{c_2^0}\right)}{c_2^0}\right)}{\left(\frac{1}{c_1^0}+\frac{2}{c_2^0}\right)}\right)$

* $A=\frac{c}{1+\frac{c_1^0}{c_2^0}}$

$B=-\frac{c\ln\left(\frac{c}{c_1^0+c_2^0}\right)}{(c_1^0+c_2^0)\left(\frac{1}{c_1^0}+\frac{1}{c_2^0}\right)}$

5. Supplementary Material E. Averaging of Multicomponent Colloidal Flow with Size-Distributed Particles

We start with governing equations for multiple populations:

$$\frac{\partial}{\partial t}\left(\phi\beta\overset{:}{C}_k+\overset{:}{S}_k\right)+U\frac{\partial}{\partial x}\left(\alpha\overset{:}{C}_k\right)=0$$

(.66)

$$\frac{\partial\overset{:}{S}_k}{\partial t}=h\left(\sum_{i=1}^{m}\overset{:}{B}_i\overset{:}{S}_i\right)F_k\left(\overset{:}{C}_k\right)\alpha U,\ \ k=1,2,...,N$$

(.67)

In order to obtain the dimensionless form of equations, we use the following dimensionless parameters and variables:

$$x=\frac{\overset{:}{x}}{L};t=\frac{U\overset{:}{t}}{\phi L};c^0=\sum_{i=1}^{m}C_i^0;C_k=\frac{\overset{:}{C}_k}{c^0};S_k=\frac{\overset{:}{S}_k}{\phi c^0};\lambda_k=\overset{:}{\lambda}_k L;$$

$$B_k=\phi c^0\overset{:}{B}_k;A=\frac{\overset{:}{A}}{\phi c^0};k=1,2,...,N$$

(.68)

Substituting parameters (.68) into system (.66) and (.67) we obtain the dimensionless governing system:

$$\frac{\partial}{\partial t}\left(\beta C_k + S_k\right) + \frac{\partial}{\partial x}\left(\alpha C_k\right) = 0$$

(.69)

$$\frac{\partial S_k}{\partial t} = h\left(\sum_{i=1}^{m} B_i S_i\right) F_k\left(C_k\right)\alpha,\ k = 1,2,...,N$$

(.70)

Also, the initial conditions and boundary conditions are:

$$t = 0 : C_k = S_k = 0$$
$$x = 0 : C_k = C_k^0$$

(.71)

The next step is upscaling the system for total concentrations. In order for this purpose to achieve, we substitute equation (.70) into equation (.69) which yields:

$$\beta \frac{\partial C_k}{\partial t} + \alpha \frac{\partial C_k}{\partial x} = -h\left(\sum_{i=1}^{m} B_i S_i\right) F_k\left(C_k\right)\alpha$$ -

(.72)

Equations (E.8) show the characteristic form of first order partial differential equations.

$$\frac{dt}{dx} = \frac{\beta}{\alpha},\ \frac{dC_k}{dx} = -h\left(\sum_{i=1}^{m} B_i S_i\right) F_k\left(C_k\right)$$

(.73)

Rewriting equation (.73) in another form yields:

$$\frac{dG_k\left(C_k\right)}{dx} = -h\left(\sum_{i=1}^{m} B_i S_i\right);\ G_k\left(C_k\right) = \int_{C_k^o}^{C_k} \frac{du}{F_k(u)},\ k = 1,2,...,N$$

(.74)

$G_k(C_k(x,t))$ is independent of the colloid population since the right-hand side of equation (.74) is the same for all k=1,2…N. In particular, we have:

$$G_k\left(C_k\right) = G_1(C_1)$$

(.75)

The total concentrations for the suspended and retained particles and occupied area are defined as:

$$c=\sum_{k=1}^{N}C_k,\ s=\sum_{k=1}^{N}S_k,\ b=\sum_{k=1}^{N}B_kS_k$$

(.76)

Now, we can express the individual concentrations from equation (.75)

$$C_k=G_k^{-1}\left[G_1(C_1)\right]$$

(.77)

Substituting relationships (.77) into equations (.76) and expressing total concentration c via C_1 yields:

$$c=\sum_{k=1}^{N}G_k^{-1}\left[G_1(C_1)\right],\ C_1=g_1(c),\ g_1=\left[\sum_{k=1}^{N}G_k^{-1}(G_1)\right]^{-1} \qquad (.78)$$

Substituting equations (.78) into equations (.77) allows for expressing each individual component C_1 as a function of total suspension concentration.

$$C_k=G_k^{-1}\left[G_1(g_1(c))\right]=g_k(c) \qquad (.79)$$

Substituting equations (.76) into equation (.69):

$$\frac{\partial}{\partial t}\left(\beta c+s\right)+\frac{\partial(\alpha c)}{\partial x}=0 \qquad (.80)$$

Then we first multiply equation (.70) by B_k, then considering equation (.79) and summing the results we can find kinetics equation for site occupation:

$$\frac{\partial b}{\partial t}=h(b)d(c)\alpha,\ d(c)=\sum_{k=1}^{N}B_kF_k(g_k(c))$$

(.81)

Here, $d(c)$ is called the occupation function. Adding equations (.70) for k=1,2…N leads to the kinetics equation for retention rate.

$$\frac{\partial s}{\partial t}=h(b)f(c)\alpha,\ f(c)=\sum_{k=1}^{N}F_k(g_k(c))$$

(.82)

Assuming a binary system with linear suspension functions:

$$F_k(C_k) = \lambda_k C_k,\ k = 1, 2$$

(.83)

Formulae (.75), (.77), and (.78) for N=2 become:

$$G_k(C_k) = \ln\left[\frac{C_k}{C_k^0}\right]^{\frac{1}{\lambda_k}},\ C_2 = C_2^0\left[\frac{C_1}{C_1^0}\right]^{\frac{\lambda_2}{\lambda_1}},\ \ c = C_1 + C_2^0\left[\frac{C_1}{C_1^0}\right]^{\frac{\lambda_2}{\lambda_1}},$$

$$n = \frac{\lambda_1}{\lambda_2};\ k = 1, 2$$

(.84)

Therefore, for suspension and occupation functions we have:

$$f(c) = \lambda_1 g_1(c) + \lambda_2 C_2^0\left[\frac{g_1(c)}{C_1^0}\right]^{\frac{\lambda_2}{\lambda_1}}$$

(.85)

$$d(c) = B_1\left[\lambda_1 g_1(c) + \varepsilon\lambda_2 C_2^0\left[\frac{g_1(c)}{C_1^0}\right]^{\frac{\lambda_2}{\lambda_1}}\right],\ \varepsilon = \frac{B_2}{B_1}$$

(.86)

We can downscale the system by expressing individual concentrations from total concentration in equations (.70).

$$C_k(x,t) = C_k^0\left[\frac{g_1(c(x,t))}{C_1^0}\right]^{\frac{\lambda_k}{\lambda_1}},\ \ S_k(x,t) = \alpha\int_0^t \left(1 - b(x,y)\right)F_k(C_k(x,y))dy$$

(.87)

For the case of Langmuir's filtration blocking function, we have:

$$h(b) = 1 - b$$

(.88)

Changing coordinates to Lagrangian coordinate τ:

$$\tau = t - \frac{\overset{\circ}{\beta}}{\alpha}x$$

(.89)

System (.80)-(.82) in coordinates (x, τ) becomes:

$$\frac{\partial c}{\partial x} = -(1-b)\, f(c)$$
(.90)

$$\frac{\partial s}{\partial \tau} = (1-b)\, f(c)\alpha$$
(.91)

$$\frac{\partial b}{\partial \tau} = (1-b) d(c)\alpha \qquad (.92)$$

From equation (.90) and along the characteristic line $b=0$, we obtain:

$$\int\limits_{c^{-}\left(x,-\frac{\beta}{\alpha}x\right)}^{1} \frac{dy}{f(y)} = x$$
(.93)

expressing occupied vacancy b from equation (.90) and substituting it into equation (.92) and integrating the obtained equation in x yields the expression for suspended concentration:

$$\int\limits_{c^{-}\left(x,-\frac{\beta}{\alpha}x\right)}^{c} \frac{dy}{\alpha f(y) w(y)} = \tau$$
(.94)

$$w(c) = \int\limits_{c}^{1} \frac{d(y)}{f(y)} dy$$
(.95)

To obtain occupied concentration, first we take x-derivative of both sides of equation (.95). Then we substitute gradient of suspended concentration into equation (.90) which yields:

$$b(x,\tau) = 1 - \frac{w(c)}{w(c^{-})}$$
(.96)

The retained concentration can also be expressed from suspended concentration:

$$s(x,\tau) = \frac{c(x,\tau) - c^{-}(x)}{w(c^{-})}$$
(.97)