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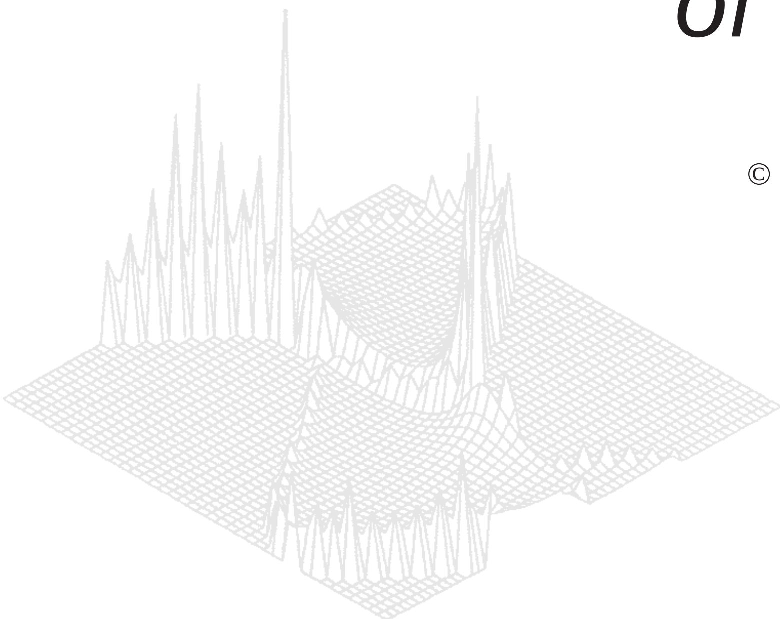

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Behaviour of the Kramer Radiating Star

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Abstract

We study the behaviour of the model for a radiating star proposed by Kramer. The evolution of the model is governed by a second order nonlinear differential equation. The general solution of this equation is expressed in terms of elementary and special functions. This completes the solution of the Einstein field equations for the interior of the star. The model matches smoothly to the Vaidya exterior solution and the condition $p = qB$ is satisfied at the boundary. We briefly study the thermodynamics of the model and indicate the difficulty in specifying the temperature explicitly.

1. Introduction

The problem of a spherically symmetric star undergoing gravitational collapse because of heat dissipation has generated tremendous interest in recent years. An interior radiating solution has to be matched to the exterior Vaidya solution (1951, 1953). The junction conditions at the surface of the spherically symmetric radiating star (Santos 1985) have to be satisfied for a realistic model. A number of models, with different conditions and properties, have been proposed by de Oliveira *et al.* (1986, 1987, 1988), Bonnor *et al.* (1989) and Grammenos (1995), amongst others. Such models are of significance in the description of relativistic astrophysical processes. It is important to generate models in which the gravitational potentials are given explicitly so that it is possible to study physical features of the radiating star.

In this paper we study the model proposed by Kramer (1992) in which a nonstatic model is generated from a static model by allowing certain parameters to become functions of time. The interior static model is taken to be the interior Schwarzschild solution in isotropic coordinates. Those aspects of the Kramer model relevant to this paper are briefly reviewed, and we show that the evolution of the model is governed by a nonlinear second order differential equation. We completely integrate the nonlinear equation in terms of elementary and special functions so that the gravitational behaviour of the Kramer (1992) model is completely specified. A remarkable feature of this model is that all the thermodynamical variables and gravitational potentials are given in terms of only one time-dependent variable $y = y(t)$ and we are in a position to completely determine the analytic behaviour of y. This is in contrast to many other radiating

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models which are reducible to a differential equation which is not integrable in closed form. The physical properties of the model are briefly discussed, in particular the temperature. We comment on the viability of the solution in the description of physical processes.

2. The Kramer Model

The exterior gravitational field of a spherically symmetric radiating star is taken to be the Vaidya (1951, 1953) solution. In coordinates (v, r', θ, ϕ) this exterior line element is given by

$$ds^2 = -\left(1 - \frac{2m(v)}{r'}\right) dv^2 - 2dvdr' + r'^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) . \tag{1}$$

The quantity $m(v)$ denotes the total mass as measured by an observer at infinity. We require $(dm/dv) \leq 0$ as the mass of the star is decreasing because of the energy being carried away in the form of radiation. The interior spacetime in coordinates (t, r, θ, ϕ) was taken to be

$$\begin{aligned} ds^2 =& -A^2(t,r)dt^2 + B^2(t,r)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \\ =& -\frac{(1+2yr^2-2y-r^2y^2)^2}{(1+y)^2(1+yr^2)^2}dt^2 + \frac{(1+y)^6}{(1+yr^2)^2}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \end{aligned} \tag{2}$$

by Kramer (1992). The line element (2) was generated by writing the Schwarzschild interior solution in isotropic coordinates, and then allowing the constants in the metric functions to become functions of time, i.e. $y = y(t)$. The quantities y and m are related by

$$\frac{m}{2r_0} = y + \ln^2\left(\frac{1-y}{1-y_0}\right) .$$

From this equation it is clear that when $y = y_0$ we have

$$y = \frac{m}{2r_0} ,$$

where r_0 is a constant and m is the mass parameter in the interior Schwarzschild solution. We should point out that the line element (2) belongs to the general class of conformally flat solutions with heat flow derived by Maiti (1982) and Banerjee *et al.* (1989).

It is a simple matter to demonstrate that the condition of pressure isotropy

$$\frac{A_{rr}}{A} + \frac{B_{rr}}{B} = \left(2\frac{B_r}{B} + \frac{1}{r}\right)\left(\frac{A_r}{A} + \frac{B_r}{B}\right)$$

is automatically satisfied as it does not contain any time derivatives. (In fact the isotropy condition is always satisfied for a known static perfect fluid solution where the parameters in that solution now become functions of time.) Then the remaining Einstein field equations yield

$$\mu = \frac{12y}{(1+y)^6} + 3\left(\frac{dy}{dt}\right)^2 \left(\frac{2yr^2 - r^2 + 3}{1 + 2yr^2 - 2y - y^2r^2}\right)^2, \tag{3}$$

$$q = -\frac{4r(1+yr^2)^2}{(1+y)^4(1+2yr^2-2y-y^2r^2)^2}\frac{dy}{dt}, \tag{4}$$

$$\begin{aligned} p = {} & \frac{12(1-r^2)y^2}{(1+y)^6(1+2yr^2-2y-y^2r^2)} - \frac{2(1+y)(1+yr^2)(2yr^2-r^2+3)}{(1+2yr^2-2y-y^2r^2)^2}\frac{d^2y}{dt^2} \\ & - \left[\frac{4[3(1+yr^2)^2 - r^2(1+y)(2yr^2-r^2+3)] + (2yr^2-r^2+3)^2}{(1+2yr^2-2y-y^2r^2)^2}\right]\left(\frac{dy}{dt}\right)^2 \\ & - \left[\frac{2(3y^2r^4 - y^2r^2 + 4yr^2 - r^2 + 3)(2yr^2-r^2+3)}{(1+2yr^2-2y-y^2r^2)^3}\right]\left(\frac{dy}{dt}\right)^2, \end{aligned} \tag{5}$$

where μ is the energy density, q is the magnitude of the heat flow, and p is the isotropic pressure. The gravitational and matter variables depend only on the quantity y. It remains to determine $y = y(t)$: the junction conditions govern the behaviour of the function y.

The junction conditions for matching two line elements continuously across a spherically symmetric spacelike hypersurface Σ, corresponding to the surface of the star, was first derived by Santos (1985). For the line elements (1) and (2) the junction conditions become

$$\frac{1 + 2yr_\Sigma{}^2 - 2y - r_\Sigma{}^2y^2}{(1+y)(1+yr_\Sigma{}^2)}dt = \left(1 - \frac{2m}{r'_\Sigma} + 2\frac{dr'_\Sigma}{dv}\right)^{\frac{1}{2}} dv\,, \tag{6}$$

$$r_\Sigma \frac{(1+y)^3}{1+yr_\Sigma{}^2} = r'_\Sigma(v)\,, \tag{7}$$

$$p_\Sigma = \left[q\frac{(1+y)^3}{1+yr^2}\right]_\Sigma, \tag{8}$$

$$\begin{aligned} m(v) = {} & \left[\frac{r^3}{2}\frac{(1+y)^9}{(1+yr^2)^3}\frac{(3-r^2+2r^2y)^2}{[1+2y(r^2-1)-r^2y^2]^2}\left(\frac{dy}{dt}\right)^2 \right. \\ & \left. + \frac{2r^3y(1+y)^3}{(1+yr^2)^2}\right]_\Sigma. \end{aligned} \tag{9}$$

From equations (4), (5) and (8) we obtain a second order differential equation that determines the behaviour of y. Using (4), (5) and the junction condition

(8) we obtain the result

$$\frac{d^2y}{dt^2} + \frac{2(2-y)}{(1+y)(1-y)}\left(\frac{dy}{dt}\right)^2 = \frac{1}{(1+y)^3}\frac{dy}{dt}\,, \tag{10}$$

which is a nonlinear differential equation. The evolution of the model proposed by Kramer (1992) has been reduced to the differential equation (10).

3. Gravitational Behaviour

We observe that (10) is an autonomous equation and can be easily reduced to a first order equation as it does not contain the independent variable t explicitly. Reducing the order and integrating we obtain

$$\frac{dy}{dt} = -\frac{1-y}{(1+y)^3}\ln\frac{1-y}{1-y_0}\,, \tag{11}$$

where $\ln(1-y_0)$ is a constant of integration. Thus (11) is the most general first integral admitted by the nonlinear equation (10); the solution (11) was also found by Kramer (1992).

At first sight it seems that it is not possible to complete the integration of (11) in closed form. However, in (11) the variables y and t are separable. We can write (11) as

$$\int \frac{(1+y)^3}{(1-y)}\frac{dy}{\ln[(1-y)/(1-y_0)]} = -t - y_1\,.$$

The above integral may be evaluated if we make the substitution $\zeta = (1-y)/(1-y_0)$. This generates the solution

$$\begin{aligned} t + y_1 = {} & 8\ln\left(\ln\frac{1-y}{1-y_0}\right) - 12(1-y_0)\mathrm{Li}\left(\frac{1-y}{1-y_0}\right) \\ & + 6(1-y_0)^2\mathrm{Li}\left(\frac{1-y}{1-y_0}\right)^2 - (1-y_0)^3\mathrm{Li}\left(\frac{1-y}{1-y_0}\right)^3\,, \end{aligned} \tag{12}$$

where y_1 is a constant of integration. The general solution of (10) depends on elementary functions and the special function Li (Lebedev 1972; Gradshteyn and Ryzhik 1994) which is defined by

$$\mathrm{Li}(x) = \int_0^x \frac{dt}{\ln t}\,,$$

where the integral is a Cauchy principal value. From the definition of the logarithmic integral it follows that

$$\mathrm{Li}(y^{n+1}) = \int_0^y \frac{t^n}{\ln t}\,dt\,.$$

It is interesting to note that the special function Li arising in (12) is widely utilised in the study of the distribution of primes in number theory (Wolfram 1991) and here it arises in relativistic astrophysics. We have succeeded in fully describing the temporal behaviour of the model proposed by Kramer for a radiating star. The general solution of the Einstein field equations is given by (2)–(5), where $y = y(t)$ is governed by (12), in the Kramer model.

4. Physical Considerations

We now briefly consider some physical aspects of the general solution to the Kramer model. A graphical analysis of the energy density and pressure by Kramer showed that they are well-behaved in the interior of the star. In this paper we are concerned with the behaviour of $y = y(t)$ as this quantity governs the properties of the entire model. An inspection of (12) indicates that an analytic treatment of (12) is difficult because of the nonlinearity and the presence of the special function Li. The behaviour of $y(t)$ is complicated and difficult to interpret in general. Graphical plots of $y(t)$, with the assistance of the software package MATHEMATICA Version 2.0 (Wolfram 1991), indicate a monotonically decreasing function $y = y(t)$. This is consistent with Kramer's numerical integration of (11). It is important to observe that the second–order differential equation governing the behaviour of $y(t)$ essentially follows from the boundary condition (8), $p_\Sigma = (qB)_\Sigma$, which is equivalent to the radial flux of momentum across the hypersurface Σ (Bonnor *et al.* 1989). This should be a decreasing function for a radiating star. The condition $(dy/dt) < 0$ and the metric singularity at $y = -1$ restricts y to the range: $y_0 \geq y > -1$. A study of the ratio of pressure/energy density (p/μ) at the centre $(r = 0)$ yields an asymptotic $(y \to -1)$ equation of state $\mu + 3p = 0$, so that the pressure becomes negative.

To study the thermodynamics of the model we utilise the Gibb's fundamental form

$$T ds = du + pd\left(\frac{1}{\rho}\right) , \tag{13}$$

where $u = (\mu/\rho) - 1$ is the specific internal energy, s is the specific entropy and ρ is the rest mass density. The Gibb's relation (13) is applicable in the early stages of evolution when the fluid is close to thermal equilibrium. At the final stages of collapse this relation cannot be used as the model is far from equilibrium. In general relativity we have the relation

$$q^a = -Kh^{ab}(T_{,b} + Tu_{b;c}u^c) \tag{14}$$

linking the heat flow vector q^a to the temperature T. The quantity K represents the thermal conductivity and $h^{ab} = g^{ab} + u^a u^b$ is the projection tensor. The metric (2) and the heat flow (14) imply

$$q = -\left(\frac{K}{AB^2}\right)(TA)_{,r} . \tag{15}$$

The quantities μ and p have been found explicitly and, together with ρ, we can find the temperature T and the specific entropy s from (13). Then the thermal conductivity K results from (15) as the metric coefficients A and B have been found. When the collapsing star is far from equilibrium the Eckart formula (14) has to be modified. We need to incorporate second-order effects for a proper treatment of the thermodynamics in the latter stages of collapse. One possible approach is to utilise the extended irreversible thermodynamics of Israel and Stewart (1976, 1979). Here the relativistic formulation of the heat transport equation

$$\tau \frac{dq^a}{ds} + q^a = -Kh^{ab}(T_{,b} + Tu_{b;c}u^c) + \tau u^a q_b u^b_{;c} u^c \,, \tag{16}$$

where τ is the relaxation time, generalises (14). For the line element (2), equation (16) becomes

$$\tau(Bq)_{,t} + A(Bq) = -\frac{K}{B}(AT)_{,r} \,, \tag{17}$$

which relates the heat flow q to the temperature T. If $\tau = 0$ then (17) reduces to the Eckart formula (15). Clearly the second-order equations (16) and (17) are more difficult to analyse than the corresponding Eckart equations.

We now comment briefly on the physical viability of the model. The above calculation, for T and K in (15) and (17), is possible in principle. However, the complicated expressions for the gravitational and matter variables pose difficulties in practice. This is not a positive feature of the Kramer model as a detailed thermodynamical analysis cannot be completed. Also, there is no simple relationship between the energy density (3) and the pressure (5). For a realistic model we require a physical equation of state relating μ and p. An alternate approach is to begin by specifying an equation of state. The heat flow q in (4) is determined by the behaviour of y, and there is no freedom to control the rate of loss of energy. Perhaps in future work we should seek solutions that allow this freedom and generate models that more closely resemble the physical collapse of a radiating star.

The ansatz of Kramer (1992) is good in that it easily generates a solution from a static model if the condition of pressure isotropy is satisfied. However, this constrains the system *ab initio* and determines the behaviour of μ, p and q. The temperature T is also constrained by the initial static solution chosen. It is not possible to determine at the outset whether the ansatz will generate a physically reasonable model. A different approach would be to specify the behaviour of the heat flow q^a on physical grounds, and choose an equation of state. However, one is then faced with the problem of having to solve the field equations, which is a nontrivial exercise as the Kramer ansatz will not be applicable.

5. Conclusion

In conclusion we have completely specified the gravitational behaviour of the Kramer model and completely solved the Einstein field equations. Other choices of the interior metric are possible that could lead to physically acceptable models.

This is an area for future research. We have also pointed out that the initial static solution in the ansatz of Kramer constrains the behaviour of the system and may not necessarily lead to reasonable behaviour for the temperature. To overcome this problem would involve dropping the ansatz adopted here and to specify the behaviour of the heat flow. A detailed numerical analysis of the behaviour of K and T in (15) and (17), namely the thermodynamics, is the subject of a present investigation.

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