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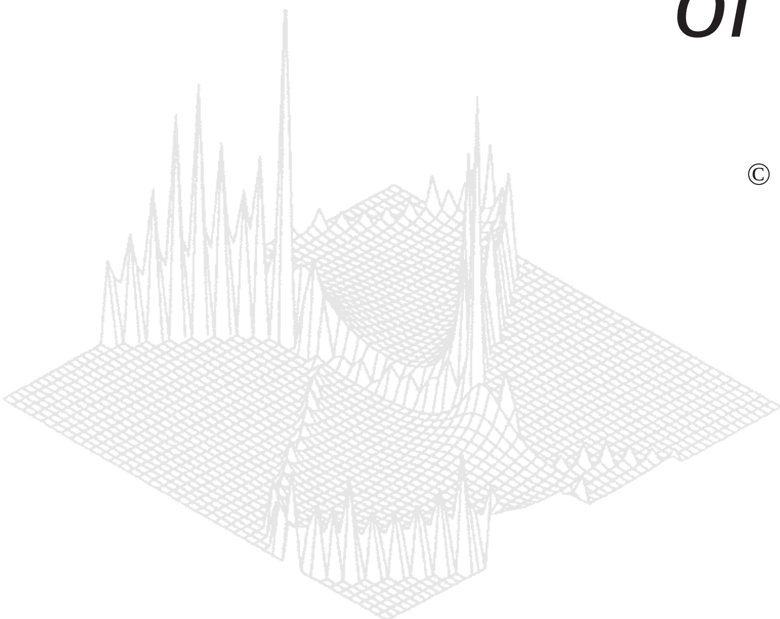

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Faddeev Approach to the Nucleon in the NJL model*

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Abstract

The nucleon and the delta are described as solutions of the relativistic Faddeev equation in the NJL model. We discuss the dependence of the baryon masses on the particular form of the four-Fermi interaction Lagrangians. Using the quark–diquark wave function, we calculate some bound state matrix elements such as the axial coupling constants, magnetic moments of the nucleon, the pion–nucleon sigma term and the proton–neutron mass difference. We also try to compare two pictures of describing the baryons in the NJL model, i.e. the mean-field approximation and the relativistic Faddeev approach. As a first step, we discuss how to improve the mean-field approximation by introducing an effective interaction. We also discuss the perturbative estimate of the deformation of the 'vacuum' in the Faddeev approach.

1. Introduction

The NJL model [1] is nowadays considered as an effective quark field theory for low energy hadron physics [2], and has been quite successful in the description of mesons as $q\bar{q}$ bound states in the Bethe–Salpeter (BS) framework. A direct extension of this approach to the description of baryons amounts to solving the relativistic Faddeev equation [3, 4]. In this work, we solve the relativistic Faddeev equation [5] in the ladder approximation to obtain the baryon masses. We take into account the qq interactions in the scalar $(0^+, T = 0)$ and the axial vector $(1^+, T = 1)$ diquark channels which are expected to dominate in the non-relativistic limit. We discuss the dependence of the baryon masses on the strength of the qq interaction. We calculate bound state matrix elements such as the axial coupling constants, the magnetic moments of the nucleon, the pion–nucleon sigma term and the proton–neutron mass difference using the nucleon wave function [6]. Finally, we try to understand the relation between the mean-field approximation and the Faddeev approach. As a first step, we discuss how to improve the former by introducing the effective interaction. We explain the problem of the latter by using an analogy to nuclear structure theory and discuss how to estimate the deformation of the 'core', which is missing in the Faddeev approach.

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2. Lagrangians and Effective Coupling Constants

If the four fermionic interaction Lagrangian $\mathcal{L}_I$ is explicitly specified, we can apply the Fierz identities to extract the interaction, in the pionic (0^-) $q\bar{q}$-channel, the scalar and axial vector qq-channels as

$$\mathcal{L}_{I,\pi} = -\frac{g_\pi}{2}\left(\bar{\psi}\gamma_5\boldsymbol{\tau}\psi\right)^2, \tag{2·1}$$

$$\mathcal{L}_{I,s} = g_s\sum_{c=1}^{3}\left(\bar{\psi}\left((\gamma_5 C)\tau_2\beta_c\right)\bar{\psi}^T\right)\cdot\left(\psi^T\left((C^{-1}\gamma_5)\tau_2\beta_c\right)^T\psi\right), \tag{2·2}$$

$$\mathcal{L}_{I,a} = g_a\sum_{c=1}^{3}\left(\bar{\psi}\left((\gamma_\mu C)(\boldsymbol{\tau}\tau_2)\beta_c\right)\bar{\psi}^T\right)\cdot\left(\psi^T\left((C^{-1}\gamma^\mu)(\tau_2\boldsymbol{\tau})\beta_c\right)^T\psi\right), \tag{2·3}$$

where in (2·1) we assume that the interaction Lagrangian is written in a Fierz symmetric form. In this way g_π represents both the direct and the exchange channel interactions. Here $C = i\gamma^2\gamma^0$ is the charge conjugation matrix and $(\beta_c)_{ij} = i\sqrt{\frac{3}{2}}\varepsilon_{cij}$ projects onto the colour $\bar{3}$ channel, while g_π, g_s, g_a are related to the coupling constants appearing in the original $\mathcal{L}_\mathrm{I}$. Due to chiral symmetry there is also an interaction term $\frac{1}{2}g_\pi(\bar{\psi}\psi)^2$ in the $0^+ q\bar{q}$ channel, which leads to the gap equation for the constituent quark mass M. From the interaction term (2·1) we calculate the $q\bar{q}$ t-matrix in the pionic channel in the ladder approximation and adjust the strength g_π to reproduce the experimental pion mass $m_\pi = 140$ MeV. Divergent integrals are regularized by introducing a sharp Euclidean cut-off Λ. For the calculation of the baryons we will treat g_s and g_a, or equivalently the ratios $r_\mathrm{s} = g_\mathrm{s}/g_\pi$ and $r_\mathrm{a} = g_\mathrm{a}/g_\pi$ as free parameters. These ratios reflect different possible forms of the interaction Lagrangian $\mathcal{L}_\mathrm{I}$, e.g. for the original NJL type interaction Lagrangian $\mathcal{L}_\mathrm{I} = g((\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\vec{\tau}\psi)^2)$ we have $r_\mathrm{s} = 2/13$, $r_\mathrm{a} = 1/13$, while for the colour current type $\mathcal{L}_\mathrm{I} = -g(\bar{\psi}\gamma_\mu\frac{\lambda_c}{2}\psi)^2$ we get $r_\mathrm{s} = 1/2$, $r_\mathrm{a} = 1/4$. From the interaction terms (2·2) and (2·3) we calculate the qq t-matrices in the scalar and axial vector channels by solving the BS equation in the ladder approximation. These t-matrices are then used as inputs in the relativistic Faddeev equation.

3. Relativistic Faddeev Equation

Generally the (exact) three-quark BS equation contains complicated two-body and three-body interactions. However, accepting the ladder approximation, only the separable two-body interactions are involved. In this case, if we apply the Faddeev [7] prescription, we are left with the following quark–diquark BS equation:

$$X = Z + ZGX\,. \tag{3·1}$$

Here Z describes the quark exchange, and has the form of a product of two-body vertex functions and the propagator of the exchanged quark. Further $G = tS$ is the three-body propagator with t the two-body (diquark) t-matrix in the ladder approximation and S the quark propagator. The quantities in (3·1) depend on the total momentum ($q = (E, \vec{0})$ in the c.m. frame) and on the relative momenta between the diquark and the quark, and moreover they are matrices in Dirac, flavour and colour spaces. The projections onto colour singlet and isospin $T = \frac{1}{2}$ or $\frac{3}{2}$ channels present no difficulties. We have used the helicity formalism of Jacob and Wick to carry out the projection onto the total spin $J = \frac{1}{2}$ or $\frac{3}{2}$. We diagonalize the kernels numerically to obtain the eigenvalues $\lambda(E)$ as a function of the total energy E and the right (left) eigenfunctions ψ_E ($\bar{\psi}_E$). The baryon mass $M_{\rm B}$ is then obtained from $\lambda(M_{\rm B}) = 1$, and the corresponding right (left) eigenfunctions are the quark–diquark wave (vertex) functions, respectively.

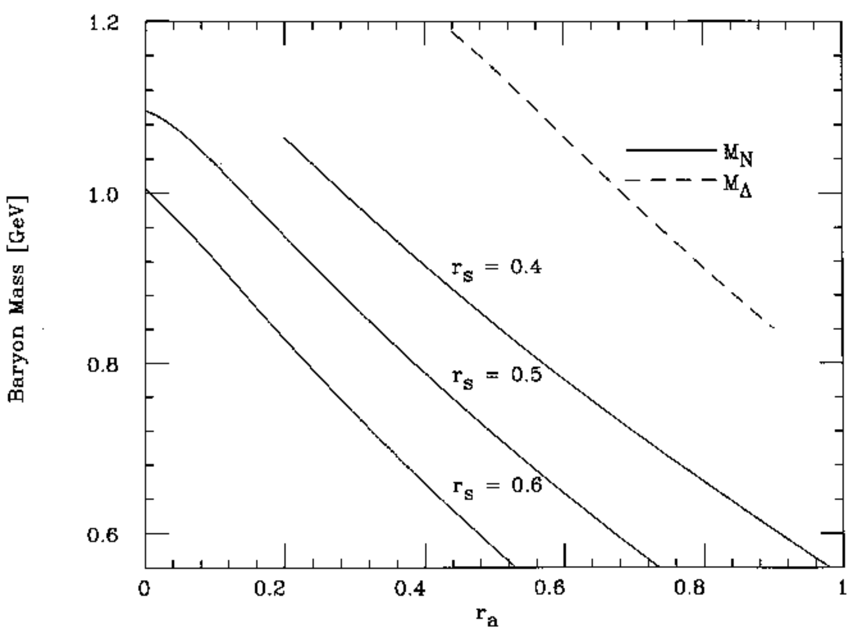

Fig. 1. Masses for the nucleon (solid line) and the delta (dashed line) plotted against the ratio $r_{\rm a} = g_{\rm a}/g_\pi$ for several values of $r_{\rm s} = g_{\rm s}/g_\pi$. (The scalar diquark channel is not involved in the calculation of the delta, i.e. the mass of the delta is independent of $r_{\rm s}$.)

Before showing the results, we explain how the parameters are determined. Since we truncate the qq interactions to the scalar and the axial-vector diquark channels, we have the following five parameters: the current quark mass m, the cutoff Λ, the pion channel effective coupling constant g_π and the qq effective coupling constants measured by g_π: $r_{\rm s} = g_{\rm s}/g_\pi$ and $r_{\rm a} = g_{\rm a}/g_\pi$ in the scalar and the axial-vector diquark channels. We impose on the first three parameters the following conditions: (1) the pion mass $m_\pi = 140$ MeV, (2) the pion decay constant $f_\pi = 93$ MeV and (3) the gap equation is satisfied with the constituent quark mass $M = 400$ MeV. The remaining two parameters $r_{\rm s}$ and $r_{\rm a}$ which reflect different possible forms of the interaction Lagrangian are treated as free parameters. We investigate how the baryon masses depend on them. The calculated masses of the nucleon and the delta are plotted in Fig. 1 against $r_{\rm a}$, with r_s being fixed to several values.* We see that both the scalar and the axial-vector diquark channels contribute to the nucleon state attractively, and that the axial-vector diquark channel contributes to the delta state attractively. If we take the colour current interaction Lagrangian ($r_\mathrm{s} = \frac{1}{2}$ and $r_\mathrm{a} = \frac{1}{4}$), the nucleon mass is 920 MeV, but the delta is not bound. If we take an interaction Lagrangian which corresponds to $r_\mathrm{s} = 0{\cdot}36$ and $r_\mathrm{a} = 0{\cdot}44$, then $M_\mathrm{N} = 920$ MeV and $M_\Delta = 1190$ MeV. We also find that the following linear relations are approximately valid:

$$M_\mathrm{N} \simeq -1{\cdot}1 r_\mathrm{s} - 0{\cdot}7 r_\mathrm{a} + 1{\cdot}7\ \mathrm{GeV}, \qquad M_\Delta \simeq -0{\cdot}76 r_\mathrm{a} + 1{\cdot}52\ \mathrm{GeV}\,. \quad (3{\cdot}2)$$

From the first relation, it follows that the qq interaction in the scalar diquark channel gives the dominant contribution. However, the interaction in the axial-vector diquark channel gives a rather large correction to it, and we should not neglect it for quantitative calculations. The above two relations give the nucleon-delta mass difference:

$$M_\Delta - M_\mathrm{N} = 1{\cdot}1 r_\mathrm{s} - 0{\cdot}06 r_\mathrm{a} - 0{\cdot}18\ \mathrm{GeV}\,, \qquad (3{\cdot}3)$$

and we conclude that the main mechanism for the nucleon–delta mass difference in the Faddeev approach is the qq interaction in the scalar diquark channel. Next we show the results of some bound state matrix elements in Table 1 for several values of r_s together with the corresponding nucleon mass (here we neglect the axial-vector diquark channel for simplicity). Here $G_\mathrm{A}^{(3)}$ and $G_\mathrm{A}^{(0)}$ are the iso-vector and iso-scalar axial coupling constants; $G_\mathrm{A}^{(3)}$ is close to the observed value, but $G_\mathrm{A}^{(0)}$ is rather large. The values of the magnetic moments μ_p and μ_n are too small in magnitude. In the quark–diquark approximation, it is reported that the axial-vector diquark channel gives an important correction [8]. As for the pion–nucleon sigma term $\Sigma_{\pi\mathrm{N}}$, we note that the vertex correction factor $\partial M/\partial m = 1{\cdot}35$ is already included in these values [8]. The result is rather close to the observed value. The proton–neutron mass difference is calculated to first order in $(m_\mathrm{d} - m_\mathrm{u})$ according to the following equation: $M_\mathrm{n} - M_\mathrm{p} = \langle \mathrm{n}|H|\mathrm{n}\rangle - \langle \mathrm{p}|H|\mathrm{p}\rangle \simeq (m_\mathrm{d} - m_\mathrm{u}) \int d^3x \langle \mathrm{N}|\bar{\psi}\tau_3\psi|\mathrm{N}\rangle$, where $|\mathrm{N}\rangle$ is the isospin symmetric nucleon state. We note that the results include the contribution from the electromagnetic interaction ($\delta M_\mathrm{elem.} = -0{\cdot}76$ MeV) [9]. We take $m_\mathrm{d} - m_\mathrm{u} = 5$ MeV, which follows from $m_\mathrm{d}/m_\mathrm{u} < 1{\cdot}76$ [9] and our value for $m = (m_\mathrm{d} + m_\mathrm{u})/2 = 8{\cdot}99$ MeV. The results for δM_np are about twice the observed value. Note, however, that the single quark vertex correction and the axial-vector diquark contribution are not yet included.

4. Mean-field Approximation and the Faddeev Approach

There are two methods to describe baryons in the NJL model: (1) the mean-field approximation [10] and (2) the relativistic Faddeev approach.† In the mean-field approximation, the results depend mainly on the $q\bar{q}$ interactions.

* The scalar diquark channel is not involved in the case of the delta due to its iso-scalar nature. Thus M_Δ is independent of r_s.

† The quark–diquark approximation is another method. However, this can be considered to be an approximate version of the relativistic Faddeev approach.

Table 1. Some bound state matrix elements together with the corresponding experimental values

$r_s = g_s/g_\pi$	$\frac{1}{2}$	$\frac{2}{3}$	$0{\cdot}8$	Experiment
M_N (MeV)	1096	934	764	938
$G_A^{(3)}$	1·17	1·20	1·20	1·25
$G_A^{(0)}$	0·92	0·83	0·81	0·31
μ_p	1·5	1·80	2·16	2·7
μ_n	−0·96	−1·38	−1·86	−1·9
$\Sigma_{\pi N}$ (MeV)	35·0	35·1	34·3	45 ± 7
δM_{np} (MeV)	2·74	2·74	4·84	1·3

Under the hedgehog ansatz, which takes account of the interactions in the pion and the isoscalar scalar meson channels, there seems to be no stable solution corresponding to a bound state, unless a three-body interaction Hamiltonian is included [10]. In fact, there exists a critical coupling constant g_c such that, if the coupling $g < g_c$, there is no localized single-particle solution, while if $g > g_c$, the baryon collapses.

On the other hand, in the Faddeev approach, the results depend on the qq interactions. The baryons are described as bound states of a quark and a diquark. It was found that there exist stable baryon states, if the qq interactions are strong enough. However, the Faddeev approach can be criticized since it does not properly take into account the effects of the 'deformed vacuum'. In terms of the nuclear structure theory, the 'vacuum' in the NJL model corresponds to the closed shell. The mean-field approximation of the baryon corresponds to the mean-field calculation of deformed nuclei. The relativistic Faddeev approach in the ladder approximation corresponds to the three-particle RPA on the spherical core. So the Faddeev approach takes into account the effect of the negative energy Dirac sea solely in terms of the RPA vacuum. However, if the deformation of the 'vacuum' due to the valence particles becomes very strong, it cannot be properly taken into account. As a first step, we propose two methods of how to improve both approaches. Numerical calculations following these two methods will be presented in future works.

(4a) Improvement of the Mean-field Approximation

As mentioned before, there exists a critical coupling constant g_c in the mean-field calculation of baryons in the NJL model. The Faddeev approach evaluates the attraction between quarks stronger than the mean-field approach. This is due to the fact that the qq interaction is iterated infinitely many times in terms of the diquark t-matrix. Thus it is natural to expect that we may obtain localized single particle solutions in the mean-field approach if we employ the diquark t-matrix $t(q^2)$ * as an effective interaction in the $q\bar{q}$ channel instead of the bare one, i.e. use $g_{\text{eff}}(q^2) = t(q^2)$ instead of g.

In Fig. 2 we plot the typical form of the effective interaction $g_{\text{eff}}(q^2)$ against q^2. The vertical dotted line is the diquark pole. We assume that the curve intersects with the horizontal line $g_{\text{eff}} = g_c$ below the diquark pole. We shall

* q is the total momentum of the qq system, or the momentum transfer in the $q\bar{q}$ system.

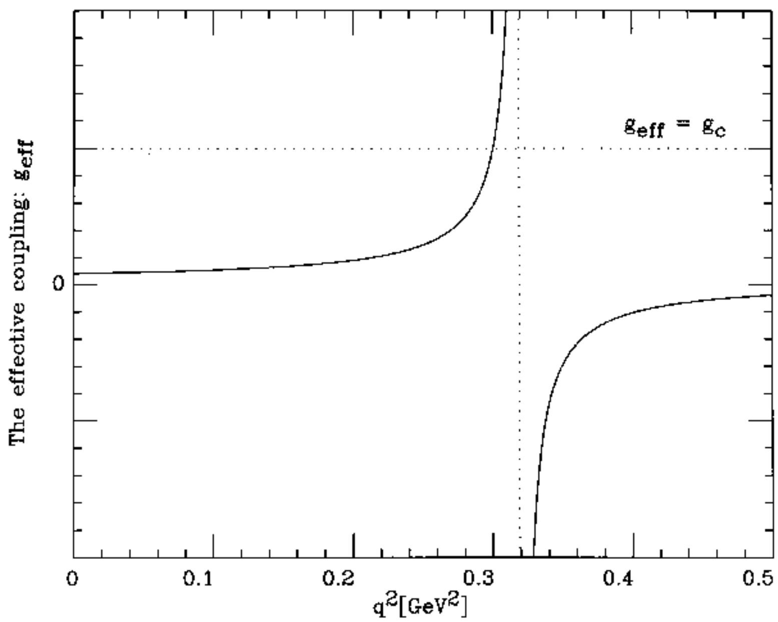

Fig. 2. Effective interaction introduced in Section 4*a* plotted against the squared total momentum of the two quark system under consideration. The vertical dotted line indicates the diquark pole. The line $g_{\text{eff}} = g_{\text{c}}$ corresponds to the critical coupling constant appearing in the mean-field approximation of the baryon.

refer to the horizontal coordinate of this intersection as the 'critical energy'. Let us imagine solving the mean-field equations self-consistently by iteration, starting from a trial state and a corresponding average qq energy. If this energy is below the 'critical energy', the effective coupling is weaker than g_{c}. Thus there will be no localized solution. Performing a cycle of the self-consistent procedure, *the effective coupling corresponding to the new state is stronger than the first one*, since, in the region $g_{\text{eff}} < g_{\text{c}}$, the wave functions tend to spread and the energy tends to get larger. On the other hand, if the trial energy is above the 'critical energy', the corresponding typical effective coupling is stronger than the critical coupling. So we will have a localized solution which would eventually collapse, if g_{eff} were a constant. After performing a cycle of the self-consistent procedure, *the effective coupling corresponding to the new state gets weaker than the first one*, since, in the region $g_{\text{eff}} > g_{\text{c}}$, the state tends to collapse and the energy becomes lower.* This will prevent the solution from collapsing. In this way, we conclude that a stable baryon solution may be obtained, if we use the effective coupling in the mean-field calculation instead of the bare coupling.

(4b) Effects of 'Meson Exchange' in the Faddeev Equation

In principle, there exists an exact version of the relativistic Faddeev equation with very complicated two-body and three-body interactions, which can perfectly

* If the baryon collapses, the spatial momenta get larger, and this makes the effective coupling smaller.

take into account the deformation of the 'vacuum'. However, such an equation is too hard to solve. Thus we propose estimating the deformation by perturbation theory, i.e. we treat the two-body contact interaction K_0 as the unperturbed one, and the non-separable two-body interaction δK induced by the meson exchange ($q\bar{q}$ exchange) diagram, which is considered to be the most important in the mean-field theory, as the perturbing one (the three-body interaction is neglected for simplicity). Then one can evaluate the deviation of the eigenvalue $\delta\lambda_{\mathrm{N}}$ of the Faddeev equation to first order in δK according to $\delta\lambda_{\mathrm{N}} = \langle \mathrm{N}|\delta K_{\mathrm{Fad}}|\mathrm{N}\rangle$, where $\langle \mathrm{N}|$ and $|\mathrm{N}\rangle$ refer to our unperturbed nucleon vertex (wave) function, and δK_{Fad} is the first order deviation of the Faddeev kernel induced by the perturbing two-body interaction δK. We will present numerical results in the near future. They will help to understand the relation between the mean-field and the Faddeev approach.

5. Conclusion

We solved numerically the relativistic Faddeev equation for the nucleon and the delta states in the NJL model in the ladder approximation, truncating the qq interaction to the scalar ($J^\pi = 0^+$) and the axial-vector ($J^\pi = 1^+$) diquark channels which are expected to dominate in the non-relativistic limit. We found that the qq interactions in both the scalar and the axial-vector diquark channel contribute to the nucleon state attractively for $r_{\mathrm{s}} > 0$ and $r_{\mathrm{a}} > 0$. We also found that the interaction in the axial-vector diquark channel contributes to the delta state attractively for $r_{\mathrm{a}} > 0$. The nucleon solution with a reasonable mass ($\simeq 920$ MeV) could be obtained with the colour current interaction Lagrangian, but a delta with a reasonable mass could not be obtained simultaneously from this Lagrangian. However, we found that there exists a certain class of the interaction Lagrangians which simultaneously reproduce reasonable values of the nucleon and delta masses. The delta–nucleon mass difference in the Faddeev approach was found to be mainly due to the qq interaction in the scalar diquark channel. As applications of the nucleon wave (vertex) function obtained through the diagonalization, we also calculated some bound state matrix elements such as the iso-scalar and the iso-vector axial coupling constants, the magnetic moments of the nucleon, the pion–nucleon sigma term and the proton–neutron mass difference. Finally, we tried to understand the relation between the mean-field approximation and the relativistic Faddeev approach. We proposed a method to improve the mean-field calculation by introducing an effective interaction. We also proposed estimating the effects of the 'deformed vacuum' in the Faddeev equation in the ladder approximation by treating the two-body interaction induced by the 'meson exchange' as a perturbing interaction. The numerical results will be shown in a future publication. We hope that these proposals will help us understand the relation between the two approaches.

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