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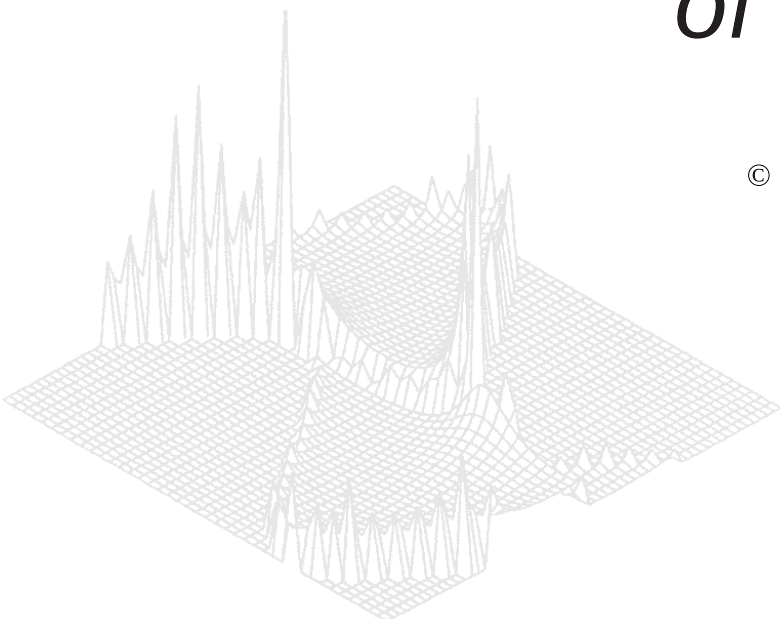

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Solving the Bethe–Salpeter Equation in Minkowski Space: Scalar Theories and Beyond*

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Abstract

The Bethe–Salpeter equation (BSE) for bound states in scalar theories is reformulated and solved in terms of a generalized spectral representation directly in Minkowski space. This differs from the conventional approach, where the BSE is solved in Euclidean space after a Wick rotation. For all but the lowest-order (i.e. ladder) approximation to the scattering kernel, the *naive* Wick rotation is invalid. Our approach generates the vertex function and Bethe–Salpeter amplitude for the entire allowed range of momenta, whereas these cannot be directly obtained from the Euclidean space solution. Our method is quite general and can be applied even in cases where the Wick rotation is not possible.

1. Introduction

The Bethe–Salpeter equation (BSE) [1] describes the two-body component of bound-state structure relativistically and in the language of quantum field theory (for an extensive review, see Ref. [2]; Ref. [3] is an exhaustive list of BSE literature prior to 1988). It has applications in, for example, calculation of electromagnetic form factors of two-body bound states and relativistic two-body bound state spectra and wavefunctions.

BSEs have been solved analytically for separable kernels and for scattering kernels in the ladder approximation. Solutions for BSEs have also been obtained for QCD-based models of meson structure in Euclidean space; these solutions must be analytically continued back to Minkowski space. It is important to note that analytical continuation back to Minkowski space from the Euclidean space solution is quite difficult even for the simple case of constituents interacting via simple particle exchange in the ladder approximation to the scattering kernel. In particular, BS amplitudes with time-like momenta cannot be unambiguously obtained from the Euclidean space solution without solving further singular integral equations. Furthermore, for any BSE with a non-ladder scattering kernel and/or with dressed propagators for the constituent particles, the proper implementation of this procedure (known as the Wick rotation [4]) itself is highly non-trivial. For these two reasons the direct solution of the Minkowski space BSE is preferable. Here we outline such a method for scalar theories, based on the perturbation theoretic integral representation (PTIR) of Nakanishi [5].

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The PTIR is a generalisation of the spectral representation for two-point Green functions to n-point functions. A general n-point function may be written as an integral over a weight distribution, which contains contributions from graphs at all orders in perturbation theory. As any graph with n fixed external legs can be written in PTIR form, this must also be true of any sum of such graphs. Hence the PTIR for a particular renormalised n-point function is an integral representation of the corresponding infinite sum of Feynman graphs with n fixed external legs.

The scalar–scalar BSE has been solved numerically in the ladder approximation after the Wick rotation by Linden and Mitter [6]. Here we present Minkowski space solutions to the ladder BSE, which will act as a check of our implementation of the approach to be used here. Our numerical solutions are obtained by using the PTIR to transform the equation for the proper bound-state vertex, which is an integral equation involving complex distributions, into a real integral equation. This equation may then be solved numerically for an arbitrary scattering kernel [11]. We will restrict our consideration of explicit numerical solutions to the ladder approximation, although the approach is a completely general one. Calculations for non-ladder kernels are under way and these results will be presented elsewhere [12]. As a specific example of a scalar theory to which our formalism may be applied, consider the $\phi^2\sigma$ model, which has a Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi\partial_\mu\phi - m^2\phi^2) + \tfrac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma - m_\sigma^2\sigma^2) - g\phi^2\sigma\,, \tag{1}$$

where g is the ϕ-σ coupling constant.

2. Formalism and PTIR

The Bethe–Salpeter equation in momentum space for a scalar–scalar bound state with scalar exchange is

$$\Phi(p,P) = -D(p_1)D(p_2)\int \frac{d^4q}{(2\pi)^4}\Phi(q,P)K(p,q;P)\,, \tag{2}$$

where Φ is the Bethe–Salpeter (BS) amplitude, and where K is the scattering kernel, which contains information about the interactions between the constituents of the bound state.

In Eq. (2), p_i is the four-momentum of the i^{th} constituent. We also define $p \equiv \eta_2 p_1 - \eta_1 p_2$, which is the relative four-momentum of the two constituents, and $P \equiv p_1 + p_2$ is the total four-momentum of the bound-state. The real positive numbers η_i are arbitrary, with the only constraint being that $\eta_1 + \eta_2 = 1$. In the case where the scalar constituents have equal mass, it is convenient to choose $\eta_1 = \frac{1}{2} = \eta_2$, and so henceforth these values of the η_i will be used. This notation is used, for example, by Itzykson and Zuber [7].

The quantities $D(p_i)$ are the propagators for the scalar constituents. We will use free propagators here for simplicity, although we could include arbitrary nonperturbative propagators by making use of their spectral representation [8]:

$$D(q^2) = -\frac{1}{m^2 - q^2 - i\epsilon} - \int_{(m+\mu)^2}^{\infty} d\alpha\ \frac{\rho_\phi(\alpha)}{\alpha - q^2 - i\epsilon}\,, \tag{3}$$

where $(m+\mu)$ is the invariant mass of the first threshold in D. It is relatively straight forward to generalise the discussion below to include $\rho_\phi(\alpha) \neq 0$. We now redefine the scattering kernel K such that $K(p,q;P) = iI(p,q;P)$, and rewrite the momenta of the constituents in terms of the relative momentum p and the bound-state momentum P. The BSE becomes

$$\Phi(p,P) = D(\tfrac{1}{2}P+p)D(\tfrac{1}{2}P-p) \int \frac{d^4q}{(2\pi)^4 i}\ \Phi(q,P)I(p,q;P)\,, \tag{4}$$

where I is the scattering kernel as referred to by Nakanishi in his review article [2].

In order to convert the BSE into a real integral equation, we will need to use the PTIR for the proper bound-state vertex and the scattering kernel [11]. The bound-state vertex may be represented as

$$\Gamma^{[l,m]}(q,P) = \mathcal{Y}_l^m(\Lambda^{-1}(P)q) \int_0^\infty d\alpha \int_{-1}^1 dz \frac{\rho_n^{[l]}(\alpha,z)}{[\alpha - (q^2 + zq\cdot P + \frac{1}{4}P^2) - i\epsilon]^n}\,. \tag{5}$$

The weight function $\rho_n^{[l]}(\alpha,z)$ of the vertex has support only for a finite region of the space spanned by the parameters α and z. A lower bound on the support is given by $\rho_n^{[l]}(\alpha,z) = 0$, unless

$$\begin{aligned} \alpha \geq \max \Bigg[&\frac{1-z}{2}(m+\mu)^2 + \frac{1+z}{2}\left(m+\mu-\sqrt{P^2}\right)^2, \\ &\frac{1-z}{2}(m+\mu-\sqrt{P^2})^2 + \frac{1+z}{2}(m+\mu)^2 \Bigg]. \end{aligned} \tag{6}$$

The PTIR for the vertex (three-point) function was originally derived by Nakanishi [5]; here we have also assumed that the vertex for bound states with non-zero angular momentum is given by the s-wave vertex multiplied by the appropriate solid harmonic [9]. The solid harmonic is an l^{th} order polynomial of its arguments, and can be written as

$$\mathcal{Y}_l^m(\vec{p}) = |\vec{p}|^l Y_l^m(\hat{p})\,, \tag{7}$$

with $Y_l^m(\hat{p})$ being the ordinary spherical harmonic for angular momentum quantum numbers l and m and where $\hat{p} \equiv \vec{p}/|\vec{p}|$ is a unit vector.

We have introduced a dummy parameter n, which will be of use in our numerical work since larger values of n produce smoother weight functions. The fact that n is arbitrary can be seen by integrating by parts with respect to α; in this way weight functions for different values of n may be connected [11].

We may use the PTIR for the bound-state vertex to derive the PTIR for the BS amplitude, since the two are related via

$$\Phi(p,P) = iD(\tfrac{1}{2}P+p)i\Gamma(p,P)iD(\tfrac{1}{2}P-p)\,. \tag{8}$$

We proceed by absorbing the two free scalar propagators into the expression for the vertex, Eq. (5), using the Feynman parametrization [10]. After some algebra we obtain

$$\Phi^{[l,m]}(p,P) = -i\mathcal{Y}_l^m(\Lambda^{-1}(P)p)\int_{-1}^{1} dz$$
$$+\int_{-\infty}^{\infty} d\alpha \frac{\varphi_n^{[l]}(\alpha,z)}{[m^2+\alpha-(p^2+zp\cdot P+\frac{1}{4}P^2)-i\epsilon]^{n+2}}\,, \qquad (9)$$

where the weight function for the BS amplitude, $\varphi_n^{[l]}(\alpha,z)$, vanishes when

$$\alpha < \min\ [0,(m+\mu+\sqrt{P^2})^2 - m^2 + \tfrac{1}{4}P]\,. \qquad (10)$$

To include the most general form of the scattering kernel in our derivation, we use the PTIR for the kernel:

$$I(p,q;P) = \sum_{\text{ch}} \int_0^{\infty} d\gamma \int_{\Omega} d\vec{\xi}$$
$$\times \frac{\rho_{\text{ch}}(\gamma,\vec{\xi})}{\gamma-(a_{\text{ch}}q^2+b_{\text{ch}}p\cdot q+c_{\text{ch}}p^2+d_{\text{ch}}P^2+e_{\text{ch}}q\cdot P+f_{\text{ch}}p\cdot P)-i\epsilon}\,, \qquad (11)$$

where the kernel parameters $\{a_{\text{ch}},\ldots,f_{\text{ch}}\}$ are linear combinations of the integration variables $\{\xi_1,\ldots,\xi_6\}$. Here we have defined, similar to before, $q\equiv(q_1-q_2)/2$. The expression for the kernel contains a sum over three different channels, labelled by st, tu and us.

The support properties of the kernel weight functions in each channel, ρ_{ch}, have been derived by Nakanishi [5]. They will not concern us here in our pure and generalized ladder treatments, since in both these cases the kernel weight functions are simply products of delta functions.

3. Derivation of Equations for Scalar Models

Armed with Eqs (5), (9) and (11), we can now derive real integral equations for the weight functions of the bound-state vertex and BS amplitude, both of which will be solved numerically by iteration.

If we consider Eq. (4), we note that it is necessary to combine four factors using Feynman parametrization. Having done this, we use the so-called 'self-reproducing' property of the solid harmonics [9, 11]:

$$\int d\vec{q}\ F(|\vec{q}|^2)\mathcal{Y}_l^m(\vec{q}+\vec{p}) = \mathcal{Y}_l^m(\vec{p})\int d\vec{q}\ F(|\vec{q}|^2)\,. \qquad (12)$$

This allows us to perform the integral over the loop momentum q. Note that in order for the loop-momentum integral to converge, we have the following restriction on the dummy parameter n:

$$l < 2n+2\,. \qquad (13)$$

This arises from a simple power-counting argument, and tells us the values of n that are valid for a particular partial wave. For example, for s-wave solutions we may choose arbitrary positive n.

After the loop momentum integration and some algebraic manipulation, we obtain the result

$$\varphi_n^{[l]}(\bar{\alpha},\bar{z}) = \int_0^\infty d\alpha \int_{-1}^1 dz\ \varphi_n^{[l]}(\alpha,z)\mathcal{K}_n^{[l]}(\bar{\alpha},\bar{z};\alpha,z)\,. \tag{14}$$

We will omit the explicit expression for the kernel function $\mathcal{K}$ for the sake of brevity.

We can derive an equivalent equation to Eq. (14), which has some advantages for numerical solutions of the BSE. Inserting the PTIR for the vertex, Eq. (5), into the vertex BSE gives

$$\Gamma(p,P) = \int \frac{d^4q}{(2\pi)^4} D(\tfrac{1}{2}P+q)D(\tfrac{1}{2}P-q)\Gamma(q,P)I(p,q;P)\,, \tag{15}$$

and once again using the self-reproducing property of the solid harmonics to perform the loop integral, we obtain the result

$$\rho_n^{[l]}(\bar{\alpha},\bar{z}) = \int_0^\infty d\alpha \int_{-1}^1 dz\ \rho_n^{[l]}(\alpha,z)\overline{\mathcal{K}}_n^{[l]}(\bar{\alpha},\bar{z};\alpha,z)\,. \tag{16}$$

Once again the explicit form of $\bar{\mathcal{K}}$ [12] will be omitted for brevity.

Once we solve for either $\varphi_n^{[l]}(\alpha,z)$ for the BS amplitude or $\rho_n^{[l]}(\alpha,z)$ for the vertex, we can evaluate both the BS amplitude and the vertex for any momenta p and P without solving any additional integral equation. It should be also mentioned that our approach is completely independent of the choice of inertial frame, so that no Lorentz boost is necessary to obtain the BS vertex for a moving bound state. These are very useful properties when applying the BS vertex to calculations of physical processes involving bound states.

4. Numerical Solutions and Results

For our numerical study, we have specialised to the case of s-wave ($l=0$) bound states interacting via a pure ladder kernel,

$$I(p,q;P) = \frac{g^2}{m_\sigma^2-(p-q)^2-i\epsilon}\,, \tag{17}$$

which corresponds to the following fixed set of kernel parameters: $a_{st}=1=c_{st}$, $b_{st}=-2$, $\gamma=m_\sigma^2$ for the st-channel, with the weight functions for the tu and us channels vanishing. We have also solved the BSE for a sum of the pure ladder kernel and a generalised ladder term, which has non-vanishing fixed values for the kernel parameters $d_{\rm ch}$, $e_{\rm ch}$ and $f_{\rm ch}$ [11].

For numerical solution of Eqs (14) and (16), it is convenient to define an 'eigenvalue' parameter λ, with λ being defined in terms of the coupling as $\lambda \equiv g^2/(4\pi)^2$, and writing the kernel functions $\mathcal{K}$ and $\overline{\mathcal{K}}$ as λK and $\lambda\overline{K}$, respectively. This having been done, we solve the BSE with the new kernel functions K and $\overline{K}$ as an eigenvalue equation, by iteration. Our approach enables us to obtain the eigenvalue as a function of the bound-state mass squared, as well as the weight functions for the BS amplitude and the vertex.

The BSE for the amplitude has been solved in a previous work [11], with an accuracy of the order of a few parts in 100. We have also solved here the BSE for the vertex, which has a simpler structure. These vertex solutions, obtained for $n = 2$ using an optimized grid and sophisticated integrator and interpolator, have an accuracy of approximately 1 part in 10^4. Our results for the eigenvalue for the case where $m_\sigma/m = 0{\cdot}5$, m_σ being the mass of the exchanged σ-particle, are shown in Table 1, and are plotted in Fig. 1. Some examples of vertex weight functions for various values of the bound-state mass are shown in Fig. 2. What is actually plotted for convenience is the rescaled weight function, $\rho \equiv \rho_n^{[\ell]}/\alpha^n$, where here $n = 2$ and $\ell = 0$. The parameter η represents the 'fraction of binding', $\eta \equiv M/2m$, with $M \equiv \sqrt{P^2}$ being the bound-state mass.

Table 1. Eigenvalues for the Wick-rotated (λ_{E}) and Minkowski-space (λ_{M}) solutions of the Bethe–Salpeter equation

The Wick-rotated values are from Linden and Mitter [6]

η	λ_E	λ_M	η	λ_E	λ_M
0	2·5658	2·56615	0·9	1·0349	1·03497
0·2	2·4984	2·49883	0·95	—	0·79528
0·4	2·2933	2·29371	0·99	0·5167	0·51684
0·6	1·9398	1·94020	0·999	0·3852	0·38530
0·8	1·4055	1·40560	1	0·3296	—

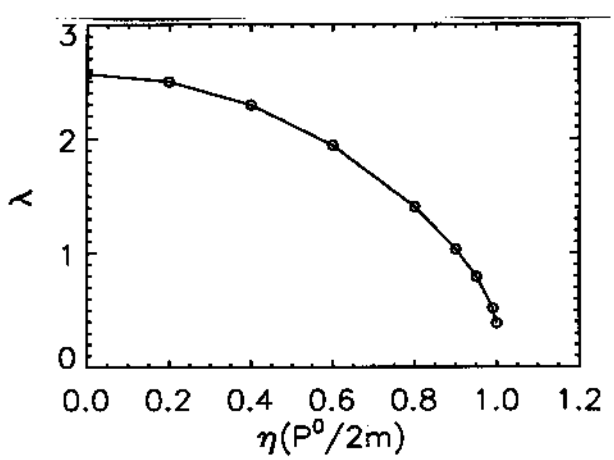

Fig. 1. Bound-state spectrum for equal constituent masses and an exchange mass of $m_\sigma = 0{\cdot}5m$.

5. Conclusions and Outlook

We have obtained numerical solutions of the Minkowski space BSE for scalar–scalar bound states in the pure ladder model. Our results agree exactly with those obtained in the Euclidean space treatment of Linden and Mitter [6]. Our

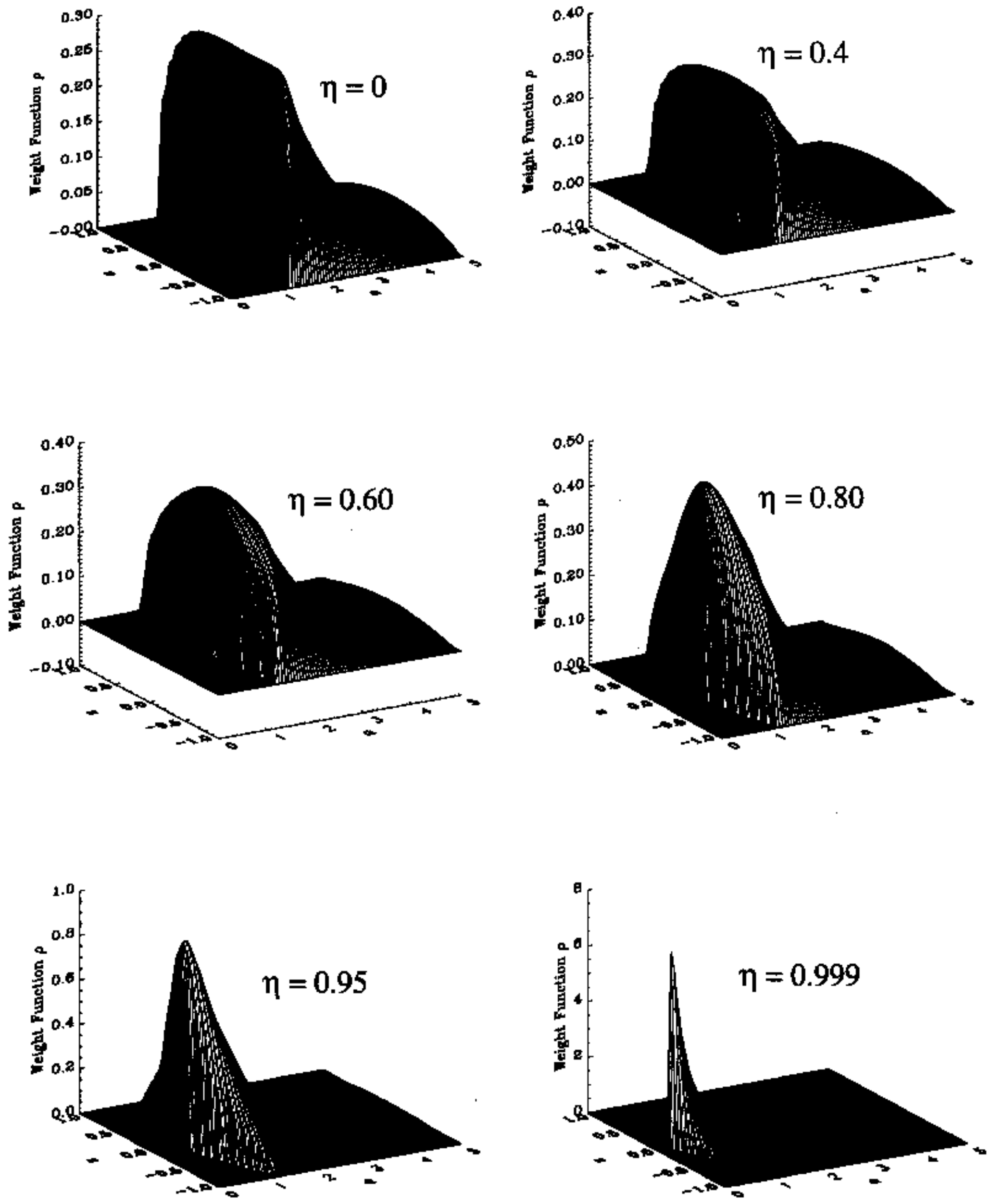

Fig. 2. Rescaled weight function $\rho \equiv \rho_2^{[0]}/\alpha^2$ of the bound-state vertex for various values of η in the pure ladder limit $\eta = 0$ corresponds to the case of a massless (i.e. Goldstone-like) bound state.

technique can be generalised to arbitrary scalar–scalar bound states, given that we know expressions for the kernel weight functions $\rho_{\rm ch}$ and the propagator spectral function $\rho(s)$. The key to our approach is to convert the BSE from an integral equation involving complex distributions into one involving functions, which is numerically soluble.

Our numerical solutions have been obtained for some simple choices of kernel weight function. It remains for us to carry out systematic studies for non-ladder kernels, orbital excitations, and comparison with other (approximate) methods such as solutions for separable kernels [12]. It will also be desirable to include spectral functions for the constituent particle propagators in a more sophisticated treatment e.g. the 'dressed ladder' kernel, which involves the simultaneous solution of self-energy Dyson–Schwinger equations and the Bethe–Salpeter equation.

To date the spinor–spinor and scalar–spinor BSEs have been solved only in the ladder approximation [13]. It will be important to attempt to formulate an approach for fermions similar to the one outlined here, so that we may solve the BSE involving spinors for more 'realistic' scattering kernels. We anticipate that the most challenging aspect of this will be generalising the PTIR to particles with non-zero spin. Since we would like to study the bound-state problem in QCD, for example the fermion–antifermion BSE for mesons, we also need to resolve the problem of incorporating confinement into the PTIR, and of including derivative couplings.

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