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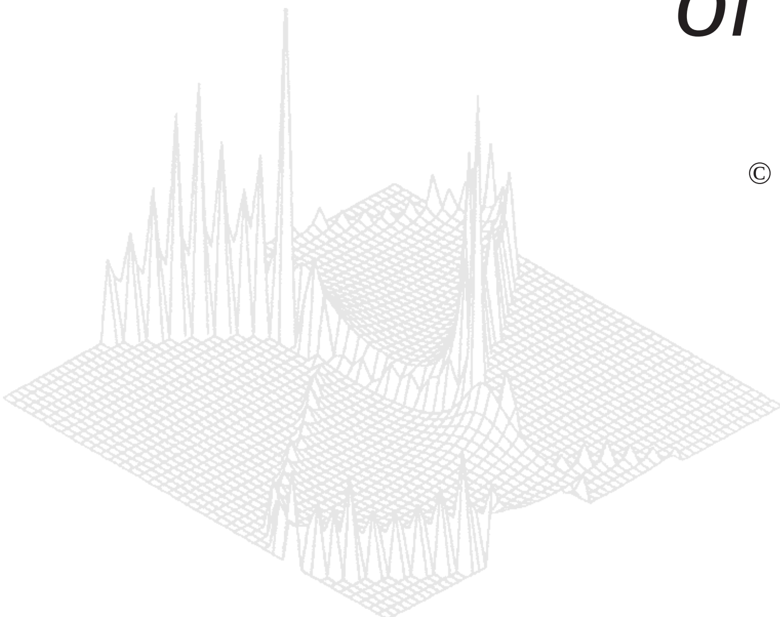

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A New Derivation for the Radiation Reaction Force

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Abstract

A new derivation for the radiation reaction on a point charge is presented. The field of the charge is written as a superposition of plane waves. The plane wave spectrum of the field consists of homogeneous plane waves which propagate away from the charge at the speed of light, and inhomogeneous plane waves which constitute the Coulomb field of the point charge. The radiation field is finite at the orbit of the point charge. The force acting on the charge due to this field is the well known Abraham–Lorentz radiation reaction.

1. Introduction

Maxwell's equations

$$\nabla \cdot \mathbf{E} = \varrho/\epsilon_0, \tag{1}$$

$$\nabla \times \mathbf{B} - \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}, \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0\,, \tag{4}$$

together with the Lorentz force density equation

$$\mathbf{f} = \varrho\mathbf{E} + \mathbf{J} \times \mathbf{B}\,, \tag{5}$$

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provide a complete and self-consistent theory for the classical electrodynamics of continuous distributions of charge. The field of any continuous distribution of charges and currents is completely determined by Maxwell's equations (with appropriate initial/boundary conditions) and the motion of the charges is determined by the Lorentz force density equation. The equations also lead to conservation theorems in terms of an energy–momentum tensor. These equations successfully describe a wide range of phenomena and provide the theoretical basis for many practical applications.

A continuous distribution of charge is an idealisation. Charges occur in nature as aggregates of charged fundamental particles which should be described, within the limits of validity of classical physics, as point charges. Maxwell's equations can be used to obtain the fields of point charges. However, it proved very difficult to generalise the force density equation to include the case of point charges. The obvious generalisation of equation (5) to an equation of motion for a point charge,

$$\frac{d\mathbf{p}}{dt} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})\,, \tag{6}$$

where $\mathbf{p}$ is the momentum of the point charge, e is its charge and $\mathbf{E}, \mathbf{B}$ are the fields of *other* charges, provides a very accurate description for the motion of point charges. The covariant version of equation (6)

$$\frac{dP^\mu}{d\tau} = eF^\mu_\nu U^\nu\,, \tag{7}$$

where P is the 4-momentum, τ is the proper time, F is the field tensor and U is the 4-velocity, accurately describes the motion at relativistic velocities. Equations (6) and (7) are not consistent with energy and momentum conservation because they do not include the effect of radiation. The effect of radiation can be included only by allowing some form of interaction between the charged particle and its *self*-field. This raises a fundamental difficulty because the self-field diverges at the position of the particle. It is recognised that this difficulty cannot be completely resolved by classical arguments, however it is desirable to develop a consistent classical theory for the electrodynamics of point charges. Such a theory must include a satisfactory treatment of radiation reaction.

Radiation reaction has been the subject of many investigations since the turn of the century. Significant contributions were made, but a completely satisfactory treatment is yet to be developed (Parrot 1987). The starting point for the early investigations (Lorentz 1904; Abraham 1903) was an extended model for the point charge (more readily available accounts of these works can by found in Jackson 1973 and Rohrlich 1959). The formula for the radiation reaction obtained from extended-particle models is in the form of a power series, with higher order terms dependent on some unverifiable form-factors. The structure-dependent terms can be eliminated by taking the limit of a point particle, but this leads to divergent self-energy and hence infinite mass. Dirac (1938) derived the radiation reaction force for a *point* charge by considering the energy–momentum flux through a world tube surrounding the world line of the charge. This very important contribution did not provide a satisfactory treatment of the diverging self-energy. More recent contributions were discussed by Rohrlich (1959) who showed the importance of incorporating an asymptotic boundary condition. Modern contributions include those of Barut (1990, 1992), Ford and O'Connell (1991, 1993), Herdgen (1992), Ianconescu and Horwitz (1992), Bosanac (1994) and Gaftoi *et al*. (1994).

The most objectionable aspect of extended-particle models is that the structure of fundamental particles is outside the domain of validity of classical physics. Also a classical extended charge is unstable because of the Coulomb force. The main disadvantage of point-charge models is the divergent self-energy. Also there is some evidence that the run-away solutions to the equation of motion can be traced back to the assumption of a point-charge (Burke 1970; Levine *et al*. 1977). The analysis presented in this paper does not depend on any structure-dependent features or any assumptions about the 'physical size' of the charge and the term *point charge* is used in this sense.

In this paper the radiation reaction force on a point charge is derived by expanding the field of the charge into a spectrum of plane waves. This expansion provides a unique and natural distinction between the radiation field and the Coulomb field. The radiation field is finite at the orbit of the charge and it propagates away from it at the speed of light. The Coulomb field does not detach from the charge and it diverges at the orbit. It is postulated that the charge does not interact with its Coulomb field. The radiation reaction is obtained in terms of the radiation field which does not diverge anywhere. The expression obtained for the radiation reaction is identical to that obtained by Lorentz, Abraham, Dirac and most recent workers. The difficulties associated with the equation of motion such as runaway solutions and preacceleration are not discussed here. The main advantage of the new derivation is that conservation of energy is not imposed as an extra condition in addition to the theory of Maxwell's equations. Energy conservation is automatically satisfied as one would expect in any theory which does not violate symmetry under time translation. Another advantage is that the divergent Coulomb field which does not contribute to the self-force is separated from the radiation field in a unique and physically plausible way.

The plane wave expansion of electromagnetic fields is described briefly in Section 2. The power and limitations of the technique are illustrated by applying it to the field of a Hertzian dipole in Section 3. The radiation reaction force on a point charge moving in a planar orbit is derived in Section 4. It is shown in Section 5 that the restriction to plane orbits is not necessary. The results are discussed in Section 6.

2. Plane Wave Spectrum Representation of Electromagnetic Fields

The theory and applications of the plane wave spectrum representation of electromagnetic fields are explained in an excellent monograph by Clemmow (1966). A brief summary is given below. The plane wave is the simplest solution to Maxwell's equations (1)–(4) in a source-free ($\varrho, \mathbf{J} = 0$) region. A general solution can be written as a linear superposition of plane waves. If there are some localised sources, it is not possible to have a plane wave representation which is valid everywhere in the source-free region; any given representation can be valid in at most a half-space. For the special case where the source is a surface current flowing in the $z = 0$ plane, the current density can be written as a Fourier integral

$$\mathbf{J}(x,y,z,t) = \delta(z) \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\ell \int_{-\infty}^{\infty} dm \times \exp\left[i\omega \left(\frac{\ell x + my}{c} + t \right) \right] (-2Q\,\mathbf{a}_x + 2P\,\mathbf{a}_y) \,, \tag{8}$$

where

$$\delta(z)Q(\ell,m,\omega) = \frac{\omega^2}{16\pi^3 c^2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \times \exp\left[-i\omega\left(\frac{\ell x + my}{c} + t \right) \right] (-\mathbf{J}\,.\,\mathbf{a}_x), \tag{9}$$

$$\delta(z)P(\ell,m,\omega) = \frac{\omega^2}{16\pi^3 c^2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \times \exp\left[-i\omega\left(\frac{\ell x + my}{c} + t \right) \right] (\mathbf{J}\,.\,\mathbf{a}_y), \tag{10}$$

c is the speed of light and $\mathbf{a}_x, \mathbf{a}_y$ are unit vectors in the x and y directions. The field of any of the Fourier components is a plane wave in the $z > 0$ half-space and another plane wave in the $z < 0$ half-space. These two waves are obtained by imposing the boundary conditions on the fields at $z = 0$. The tangential component of $\mathbf{E}$ is continuous at $z = 0$ and the discontinuity in $\mathbf{B}$ is given by

$$\{\mathbf{B}(z = 0^+) - \mathbf{B}(z = 0^-)\} \times \mathbf{a}_z = \mu_0 \mathbf{J}_s \,,$$

where $\mathbf{J}_s$ is the surface current density $\mathbf{J} = \delta(z)\mathbf{J}_s$. Also the Sommerfeld radiation condition (vanishing incoming radiation) is imposed. The boundary conditions for the normal components of the fields are automatically satisfied (Clemmow 1972, p. 23). The plane wave spectrum representation of the field is

$$\mathbf{B}(x,y,z,t) = \mu_0 \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\ell dm}{n} \exp\left[i\omega\left(\frac{\ell x + my \mp nz}{c} + t \right) \right] \times \big[\pm nP\,\mathbf{a}_x \pm nQ\,\mathbf{a}_y + (\ell P + mQ)\,\mathbf{a}_z \big], \tag{11}$$

$$\mathbf{E}(x,y,z,t) = \mu_0 c \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d\ell dm}{n} \exp\left[i\omega\left(\frac{\ell x + my \mp nz}{c} + t\right)\right]$$

$$\times \left[(\ell m P + (1-\ell^2)Q)\,\mathbf{a}_x - ((1-m^2)P + m\ell Q)\,\mathbf{a}_y\right.$$

$$\left.\mp\, n(mP - \ell Q)\,\mathbf{a}_z\right], \tag{12}$$

where $n = \sqrt{1-\ell^2-m^2}$ and the upper and lower signs apply for $z > 0$ and $z < 0$ respectively. It should be noted that the fields in either half-space can be written as in equations (11) and (12) even if the source is *not* a surface current. In this case, the above remark regarding the signs does not apply and the angular spectrum functions P and Q are not directly related to the source by equations (9) and (10). Instead, P and Q are obtained by matching the plane wave spectrum to a given field at $z = 0$.

3. Radiation Reaction on a Hertzian Dipole

The main objective of this work is to obtain the radiation reaction on a point charge. However, the analysis for a Hertzian dipole is much simpler and allows the physics of the problem to be examined without being obscured by tedious manipulations. The field of a Hertzian dipole (Jackson 1973, p. 395) consists of terms which are proportional to $1/r$ and other terms which are proportional to $1/r^2$ and $1/r^3$. The terms proportional to $1/r$ are usually identified as the radiation or far fields. The other terms are identified as near fields or Coulomb and induction fields. It is noted that neither the far field nor the near field satisfies Maxwell's equations; only the total field does. Also, both fields tend to infinity at $r = 0$. The plane wave spectrum representation of the field is more convenient for the purpose of separating the radiation fields from the Coulomb fields. Both the radiation and Coulomb fields defined in terms of the plane-wave spectrum satisfy Maxwell's equations. Also the radiation field is finite everywhere.

For the following analysis we choose the x axis parallel to the dipole so that we can use the results of the previous section. Equations (8), (9), (11) and (12) can be used to obtain the plane wave representation of the field (Clemmow 1966, p. 35). The expressions for $\mathbf{J}$ and Q are

$$\mathbf{J} = p\delta(x)\delta(y)\delta(z)\exp(i\omega_0 t)\,\mathbf{a}_x, \tag{13}$$

$$\delta(z)Q(\ell,m,\omega) = -\frac{p\omega^2}{8\pi^2c^2}\,\delta(\omega-\omega_0)\,, \tag{14}$$

where $\mathbf{p} = p\,\mathbf{a}_x$ is the dipole moment. The fields are obtained using equations (11), (12) and (14):

$$\mathbf{B}(x,y,z,t) = -\frac{\mu_0\omega_0^2 p}{8\pi^2 c^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\ell dm}{n}\exp\left[i\omega_0\left(\frac{\ell x + my \mp nz}{c} + t\right)\right]$$

$$\times\,[\pm n\,\mathbf{a}_y + m\,\mathbf{a}_z], \tag{15}$$

$$\mathbf{E}(x,y,z,t) = -\frac{\mu_0\omega_0^2 p}{8\pi^2 c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\ell dm}{n}\exp\left[i\omega_0\left(\frac{\ell x + my \mp nz}{c} + t\right)\right]$$

$$\times\,[(1-\ell^2)\,\mathbf{a}_x - m\ell\,\mathbf{a}_y \pm n\ell\,\mathbf{a}_z]. \tag{16}$$

Equations (15) and (16) give representations of the dipole field in the two half-spaces $z \rightleftharpoons 0$ as a superposition of plane waves of different ℓ, m. Waves with $\ell^2 + m^2 < 1$ have real n. These are homogeneous (surfaces of constant phase and constant magnitude are parallel) plane waves which travel away from the $z = 0$ plane. Waves with $\ell^2 + m^2 > 1$ have imaginary n. These are inhomogeneous (surfaces of constant phase and constant magnitude are not parallel) plane waves which do not travel away from the $z = 0$ plane. It is noted that travelling away from the $z = 0$ plane is not the same thing as travelling away from a point source such as a dipole, except in a small solid angle around the z axis. The *radiation* fields in a small solid angle around the z axis are given by the homogeneous part of the spectrum and are obtained from equations (15) and (16) with the integration in the $\ell - m$ plane limited to the disc $\ell^2 + m^2 < 1$. The *Coulomb* fields in this small solid angle are given by equations (15) and (16) with the integration in the $\ell - m$ plane limited to the region $\ell^2 + m^2 > 1$. The radiation fields are regular at the origin and their limits (as one approaches the dipole along the z axis) are

$$\mathbf{B}_{\mathrm{rad}}(0,t) = -\frac{\mu_0\omega_0^2 p}{8\pi^2 c^2}\exp(i\omega_0 t)\int_0^{2\pi} d\alpha\int_0^1\frac{\tau d\tau}{\sqrt{1-\tau^2}}$$

$$\times\,[\pm\sqrt{1-\tau^2}\,\mathbf{a}_y + \tau\sin\alpha\,\mathbf{a}_z]$$

$$= \mp\frac{\mu_0\omega_0^2 p}{4\pi c^2}\exp(i\omega_0 t)\int_0^1 \tau d\tau\mathbf{a}_y, \tag{17}$$

$$
\begin{aligned}
\mathbf{E}_{\text{rad}}(0,t) = & -\frac{\mu_0\omega_0^2 p}{8\pi^2 c}\exp(i\omega_0 t)\int_0^{2\pi} d\alpha \int_0^1 \frac{\tau d\tau}{\sqrt{1-\tau^2}} \\
& \times\left[(1-\tau^2\cos^2\alpha)\,\mathbf{a}_x - \tau^2\sin\alpha\cos\alpha\,\mathbf{a}_y \pm \tau\sqrt{1-\tau^2}\cos\alpha\,\mathbf{a}_z\right] \\
= & -\frac{\mu_0\omega_0^2 p}{4\pi c}\exp(i\omega_0 t)\int_0^1 d\tau \frac{\tau(1-\tau^2/2)}{\sqrt{1-\tau^2}}\,\mathbf{a}_x\,,
\end{aligned}
\tag{18}
$$

where $\ell = \tau\cos\alpha$ and $m = \tau\sin\alpha$. It follows that radiation fields at the origin are

$$
\mathbf{E}_{\text{rad}}(0,t) = \frac{-\mu_0\omega_0^2}{6\pi c}\,\mathbf{p}, \tag{19}
$$

$$
\mathbf{B}_{\text{rad}}(0,t) = \mp\frac{\mu_0\omega_0^2}{8\pi c^2}\,p\mathbf{a}_y. \tag{20}
$$

The $\mp$ sign in equation (20) corresponds to the limit of the magnetic field as the origin is approached from $z \rightleftharpoons 0$. It is noted that the time-average of the work done against the radiation field is given by

$$
W = -\text{Av}\left\{\int_{-\infty}^{\infty} dxdydz\mathbf{E}_{\text{rad}}\,.\,\mathbf{J}\right\} = \frac{\mu_0\omega_0^2}{12\pi c}\,p^2\,, \tag{21}
$$

where the time-average is taken over any complete number of cycles. The time-average of the work done against the radiation reaction is equal to the time-average of the radiated power (which is obtained by integrating the power flux over a closed surface). Thus conservation of energy is not imposed as an extra condition; it follows naturally from the interaction between the dipole and its radiation field.

The field in any small solid angle can represented as a suitable spectrum of plane waves and hence the radiation and Coulomb fields within this solid angle can be distinguished. It is not obvious that the radiation electric field at the dipole does not depend on the orientation of the solid angle. It is sufficient to show that the same value of the radiation field at the position of the dipole is obtained when the solid angle is oriented along the dipole axis as when it is oriented in a direction perpendicular to it. For this purpose, we consider the plane wave representation of a dipole parallel to the z axis. The current density of a Hertzian dipole parallel to the z axis and located at the origin is

$$
\mathbf{J} = \mathbf{p}\delta(x)\delta(y)\delta(z)\exp(i\omega_0 t)\,, \tag{22}
$$

where $\mathbf{p} = p\,\mathbf{a}_z$ is the dipole moment. The field of the dipole (Jackson 1973, p. 395) is given by

$$\mathbf{B} = \frac{\mu_0 p}{4\pi} \sin\theta \left(\frac{i\omega_0}{cr} + \frac{1}{r^2} \right) \exp\left[i\omega_0 \left(t - \frac{r}{c} \right) \mathbf{a}_\phi \right], \tag{23}$$

$$\mathbf{E} = \frac{\mu_0 cp}{4\pi} \left[\cos\theta \left(\frac{1}{r^2} - \frac{ic}{\omega_0 r^3} \right) \mathbf{a}_r + \sin\theta \left(\frac{i\omega_0}{cr} + \frac{1}{r^2} - \frac{ic}{\omega_0 r^3} \right) \mathbf{a}_\theta \right]$$
$$\times \exp\left[i\omega_0 \left(t - \frac{r}{c} \right) \right], \tag{24}$$

where r, θ and ϕ are the standard spherical coordinates. Obviously the current of this dipole does not lie in the $z = 0$ plane and hence equations (9) and (10) cannot be used to obtain the plane wave spectrum functions P and Q. These functions are obtained by comparing equations (11) and (23). The fields are given by

$$\mathbf{B}(x, y, z, t) = \mu_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\ell dm}{n} \exp\left[i\omega_0 \left(\frac{\ell x + my \mp nz}{c} + t \right) \right]$$
$$\times \left(nP\,\mathbf{a}_x + nQ\,\mathbf{a}_y \right), \tag{25}$$

$$\mathbf{E}(x, y, z, t) = \mu_0 c \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\ell dm}{n} \exp\left[i\omega_0 \left(\frac{\ell x + my \mp nz}{c} + t \right) \right]$$
$$\times \left[\pm(\ell mP + (1 - \ell^2)Q)\,\mathbf{a}_x \mp ((1 - m^2)P + m\ell Q)\,\mathbf{a}_y \right.$$
$$\left. - n(mP - \ell Q)\,\mathbf{a}_z \right], \tag{26}$$

where

$$\ell P + mQ = 0,$$

$$P(\ell, m, \omega_0) = \frac{\omega_0^2}{4\pi^2 c^2 \mu_0} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \exp\left[-i\omega_0 \left(\frac{\ell x + my}{c} + t \right) \right]$$
$$\times B_x(x, y, 0, t). \tag{27}$$

The expressions for the spectrum functions in the region $\ell^2 + m^2 < 1$ are obtained using equations (23) and (27):

$$P = \frac{p\omega_0^2}{8\pi^2 c^2}\,\frac{m}{n}\,, \qquad Q = \frac{-p\omega_0^2}{8\pi^2 c^2}\,\frac{\ell}{n}. \tag{28}$$

The radiation electric field at the origin is

$$\mathbf{E}_{\text{rad}}(0,t) = \frac{-\mu_0\omega_0^2 p}{4\pi c}\int_0^1 \frac{\tau^3 d\tau}{\sqrt{1-\tau^2}}\,\mathbf{a}_z = \frac{-\mu_0\omega^2}{6\pi c}\mathbf{p}. \tag{29}$$

Thus the limit of the radiation electric field at the dipole, when one approaches it along its axis, is the same as the limit when one approaches the dipole in a direction perpendicular to its axis [equation (19) above]. This result may seem strange because the radiation pattern of the dipole has a node along its axis. The node in the radiation pattern is evident from equations (23) and (24) since the terms proportional to $1/r$ have the factor $\sin\theta$ which vanishes along the z axis. This fact is also evident from the plane wave spectrum representation (equation 28) as the spectrum functions P and Q vanish for the wave propagating in the z direction ($n = 1, \ell = m = 0$). It is noted however that our definition for the radiation field in a small solid angle around the z axis is *not* the field of the wave with $n = 1$; the radiation field is defined as the field of all waves with real ℓ, m and n. This field does not vanish but it obviously decays along the z axis faster than $1/r$.

4. Radiation Reaction on a Point Charge moving in a Planar Orbit

Consider a point charge e moving in a planar orbit

$$\mathbf{r} = \xi(t)\,\mathbf{a}_x\, +\, \eta(t)\,\mathbf{a}_y. \tag{30}$$

The current density is

$$\mathbf{J} = e\mathbf{v}\delta(x-\xi(t))\delta(y-\eta(t))\delta(z)\,, \tag{31}$$

where

$$\mathbf{v} = \frac{d\xi}{dt}\,\mathbf{a}_x\, +\, \frac{d\eta}{dt}\,\mathbf{a}_y \tag{32}$$

is the velocity of the point charge. Since the orbit is planar, the fields in the two half-spaces $z > 0$ and $z < 0$ can be written as in equations (11) and (12) where the spectrum functions P and Q are related to the current density by equations (9) and (10). The electric field is

$$\begin{aligned}\mathbf{E}(x,y,z,t) = \frac{\mu_0 e}{16\pi^3 c} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d\ell dm}{n} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dt' \omega^2 \\ \times \left[\left(\ell m \frac{d\eta}{dt'} - (1-\ell^2)\frac{d\xi}{dt'} \right)\mathbf{a}_x + \left(\ell m \frac{d\xi}{dt'} - (1-m^2)\frac{d\eta}{dt'} \right)\mathbf{a}_y \right) \\ \mp n \left(m\frac{d\eta}{dt'} + \ell \frac{d\xi}{dt'} \right)\mathbf{a}_z \Bigg] \delta(x' - \xi(t'))\delta(y' - \eta(t')) \\ \times \exp\left[i\omega \left(t - t' + \frac{\ell(x-x') + m(y-y') - nz}{c} \right) \right]. \end{aligned} \tag{33}$$

The electric field at a point on the orbit of the point charge is

$$\begin{aligned}\mathbf{E}(\xi,\eta,0,t) = \frac{-\mu_0 e}{8\pi^2 c} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d\ell dm}{n} \int_{-\infty}^{\infty} dt' \left[\left(\ell m \frac{d\eta}{dt'} - (1-\ell^2)\frac{d\xi}{dt'} \right)\mathbf{a}_x \right. \\ \left. + \left(\ell m \frac{d\xi}{dt'} - (1-m^2)\frac{d\eta}{dt'} \right)\mathbf{a}_y \mp n \left(m\frac{d\eta}{dt'} + \ell\frac{d\xi}{dt'} \right)\mathbf{a}_z \right] \\ \times \frac{d^2}{dt'^2}\,\delta\left[t' - t - \frac{\ell(\xi(t') - \xi(t)) + m(\eta(t') - \eta(t))}{c} \right]. \end{aligned} \tag{34}$$

If we choose a frame of reference such that $\mathbf{v}(t) = 0$, the field at $(\xi(t), \eta(t), 0, t)$ is

$$\begin{aligned}\mathbf{E}(\xi,\eta,0,t) = \frac{-\mu_0 e}{8\pi^2 c} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d\ell dm}{n} \left[\left(\ell m \frac{d^3\eta}{dt^3} - (1-\ell^2)\frac{d^3\xi}{dt^3} \right)\mathbf{a}_x \right. \\ \left. + \left(\ell m \frac{d^3\xi}{dt^3} - (1-m^2)\frac{d^3\eta}{dt^3} \right)\mathbf{a}_y \mp n \left(m\frac{d^3\eta}{dt^3} + \ell\frac{d^3\xi}{dt^3} \right)\mathbf{a}_z \right]. \end{aligned} \tag{35}$$

The choice of this particular frame of reference makes it possible to identify the radiation field, in a small solid angle surrounding a line parallel to the z axis and passing through $\xi(t), \eta(t)$, with the homogeneous part of the plane wave spectrum (actually this identification is only valid near the orbit). The radiation electric field in this small solid angle is obtained from equation (35) by limiting the integrals in the $\ell - m$ plane to the disc $\ell^2 + m^2 < 1$,

$$\mathbf{E}_{\mathrm{rad}} = \frac{\mu_0 e}{6\pi c} \left(\frac{d^3\xi}{dt^3}\mathbf{a}_x + \frac{d^3\eta}{dt^3}\mathbf{a}_y \right) \tag{36}$$

and hence the radiation reaction force is

$$\mathbf{f}_{\mathrm{rad}} = \frac{\mu_0 e^2}{6\pi c}\frac{d^2\mathbf{v}}{dt^2} = \tfrac{2}{3}\frac{e^2}{4\pi\epsilon_0 c^3}\frac{d^2\mathbf{v}}{dt^2}. \tag{37}$$

This equation is valid only in the rest frame of the charge. The general equation is obtained by invoking Lorentz covariance and the requirement that the radiation reaction 4-vector must be perpendicular to the 4-velocity (see problem 17·4 in Jackson 1973). It follows that the covariant form of the radiation reaction force is

$$\mathbf{F}_{\mathrm{rad}} = \tfrac{2}{3}\frac{e^2}{4\pi\epsilon_0 c^3}\left[\frac{d^2\mathbf{U}}{d\tau^2} + \frac{1}{c^2}\left(\frac{d\mathbf{U}}{d\tau}\cdot\frac{d\mathbf{U}}{d\tau}\right)\mathbf{U}\right]. \tag{38}$$

5. Radiation Reaction on a Point Charge

It is shown in this section that the restriction to point charges moving in planar orbits is not necessary. Consider a point charge e moving in a general orbit

$$\mathbf{r} = \xi(t)\,\mathbf{a}_x + \eta(t)\,\mathbf{a}_y + \zeta(t)\,\mathbf{a}_z. \tag{39}$$

The current density is

$$\mathbf{J} = e\mathbf{v}\delta(x-\xi(t))\delta(y-\eta(t))\delta(z-\zeta(t)). \tag{40}$$

This current density can be considered as a superposition of elementary sources, each consisting of a short segment of the world line of the charge

$$\mathbf{J} = e\int dt'\mathbf{v}(t')\delta(x-\xi(t'))\delta(y-\eta(t'))\delta(z-\zeta(t'))\delta(t-t'). \tag{41}$$

The field of any of these elementary sources vanishes everywhere except on its future light cone. The world line of any point charge must be time-like and hence the self-field evaluated at any point on the orbit depends only on the local variables. This fact is evident from equation (36); the radiation field is determined by local variables not by an integral over the orbit. Variations to the rest of the orbit do not change the radiation field at a certain point on the orbit. The orbit of the charge is assumed planar in the previous section only for convenience in order to simplify the derivation.

6. Discussion and Conclusions

Representation of the field of a point charge as a spectrum of plane waves provides a means for separating the radiation from the Coulomb fields. The radiation field is finite at the orbit of the charge. If we postulate that the charge does not interact with its Coulomb field, the self-force is finite and is given by the force due to the radiation field. This makes it possible to derive the Abraham–Lorentz equation for the radiation reaction force without having to deal with divergent quantities. Also energy conservation does not have to be imposed as an extra condition.

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