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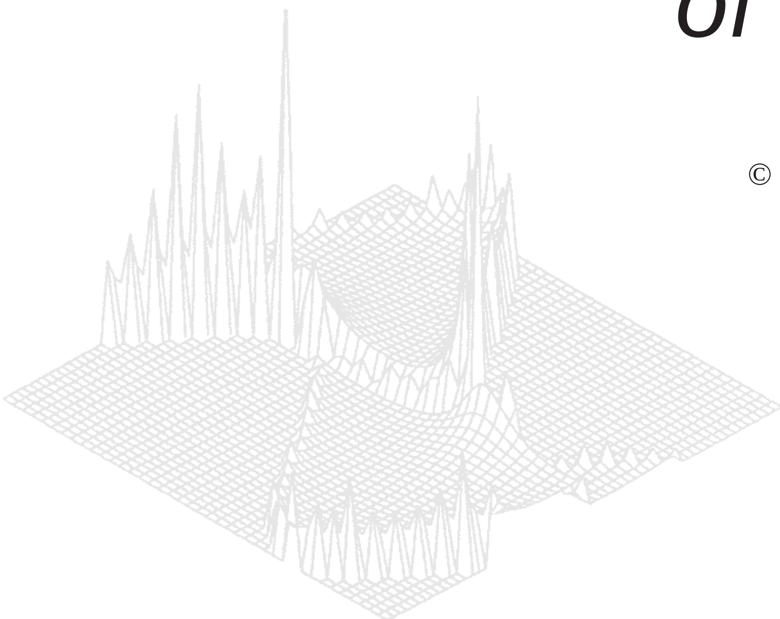

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General Coordinatisations of the Flat Space–Time of Constant Proper-acceleration

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Abstract

We examine space–times which are described by a metric of the form $ds^2 = V^2(X)dT^2 - U^2(X)dX^2 - dY^2 - dZ^2$ in which $V = V(X)$ and $U = U(X)$ are continuous functions of X only and which admit diffeomorphisms into Minkowski space–time. It is shown that such space–times are associated with rigidly accelerating frames of reference by appeal to the notion of a fundamental observer. The condition for the existence of the diffeomorphism is derived from first principles and some special cases of coordinates are discussed.

1. Introduction

The Rindler coordinates (η, ξ, Y, Z), defined by the transformation equations

$$x = \xi \cosh(\alpha_R \eta) \qquad \forall \alpha_R \neq 0\,, \tag{1}$$

$$t = \xi \sinh(\alpha_R \eta) \qquad \forall \alpha_R \neq 0, \tag{2}$$

$$y = Y,\ z = Z, \tag{3}$$

where (t, x, y, z) are the standard Minkowski coordinates, have proved to have considerable heuristic value. Although previously used by a number of authors (Einstein and Rosen 1935; Bergmann 1964; Møller 1972) these coordinates are so named because it was Wolfgang Rindler who discussed the existence of the analogy between the Kruskal (1960) diagram for 'extended' Schwarzschild space–time and the Minkowski diagram for a rigidly accelerating rod. This property of the Rindler coordinates has been employed in conjunction with Einstein's principle of equivalence to provide a flat space–time model which shares some of the properties of curved space–times, but is technically simpler. In particular the analogy between the static exterior region of a spherically symmetric black hole covered by Schwarzschild coordinates and the Rindler coordinates has been valuable in the study of some of the quantum theoretical properties of black holes (Davies 1975) and in other calculations dealing with the generalisation of quantum field theory to include cases involving accelerating particle detectors (Unruh 1974, 1976). The utility of the Rindler coordinates for the description of

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accelerated phenomena arises because the world-lines ξ = constant are hyperbolic in Minkowski space–time.

The Rindler coordinates cover only a region (wedge) of Minkowski space–time which is bounded by future and past event horizons. These event horizons play the part of the event horizon of the black hole in the analogy mentioned above. It is therefore sometimes useful to distinguish this sub-manifold from Minkowski space–time and to regard it as an independent space–time. In this work we look at a generalisation of such a space–time, which we will call 'the space–time of constant proper-acceleration', and find its general coordinatisations. We will be dealing, in particular, with static space–times described by a metric of the form

$$ds^2 = V^2(X)dT^2 - U^2(X)dX^2 - dY^2 - dZ^2 \,, \tag{4}$$

in which $V = V(X)$ and $U = U(X)$ are continuous positive functions of coordinate X only and which admit a diffeomorphism into Minkowski space. The requirement that there exist coordinate transformations into Minkowski space–time imposes a condition on the forms of the functions U and V and is sufficient to ensure that the resulting space–times have zero curvature. The admissable forms of the functions U and V are, therefore, equally well determined by the requirement that the space–time is flat.

The diffeomorphisms into Minkowski space–time which result from this procedure fall into two categories. One of these may be identified as generalised Poincaré transformations while the other consists of transformations into coordinate systems which cover the space–time of constant proper-acceleration. In the later category it will be shown that, as with the Rindler coordinates, the general inverse coordinate transformations always have a domain which is restricted to a quadrant, or wedge, in Minkowski space–time. On the other hand, the range of the inverse coordinate transformations is dependent upon the specific forms of the functions U and V.

Throughout this work quantities which are associated with the coordinate system in which the metric has the standard form of equation (4) will be denoted by capital letters and referred to as *coordinate quantities*, or sometimes as *accelerating coordinate quantities.* These lie in the image of the inverse coordinate transformation. On the other hand quantities associated with the domain of the inverse coordinate transformation will be referred to as *inertial quantities* and written in lower case letters.

2. Conditions for the Existence of Coordinate Transformations

Given a metric of the general form of equation (4) it will be assumed that there exists a diffeomorphism

$$t = t(T, X), \quad x = x(T, X), \quad y = Y, \quad z = Z \,, \tag{5}$$

whereby the coordinates of an event in the space–time described by equation (4) may be expressed in terms of the inertial coordinates (t, x, y, z) in Minkowski space–time. The metric expressed in equation (4) is assumed to be induced from the standard Minkowski metric

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \,,$$

by the diffeomorphism which is the inverse of equation (5).

In this section this property will be used to find the conditions which must be met by the functions U and V in order that the coordinate transformations exist.

Proposition 1: *The coordinates of an event* (T, X, Y, Z) *in the space–time described by a metric of the general form* $ds^2 = V^2(X)dT^2 - U^2(X)dX^2 - dY^2 - dZ^2$ *are related to the coordinates of the event in Minkowski space–time* (t, x, y, z) *by a coordinate transformation if and only if the derivative of the function* V *with respect to* X *is proportional to* U*.*

Proof: Under the assumed coordinate transformations the inertial coordinate differentials, dx^μ, are easily found. These may then be substituted into the Minkowski metric and the resulting equation compared to equation (4) to give the conditions

$$\left(\frac{\partial x}{\partial X}\right)^2 - \left(\frac{\partial t}{\partial X}\right)^2 = U^2\,, \tag{6}$$

$$\frac{\partial t}{\partial X}\frac{\partial t}{\partial T} = \frac{\partial x}{\partial X}\frac{\partial x}{\partial T}\,, \tag{7}$$

$$\left(\frac{\partial t}{\partial T}\right)^2 - \left(\frac{\partial x}{\partial T}\right)^2 = V^2\,. \tag{8}$$

Partial differentiation of equation (6) with respect to T and substitution of equation (7) gives

$$\left(\frac{\partial x}{\partial T}\right)^{-1}\frac{\partial}{\partial X}\frac{\partial x}{\partial T} - \left(\frac{\partial t}{\partial T}\right)^{-1}\frac{\partial}{\partial X}\frac{\partial t}{\partial T} = 0\,, \tag{9}$$

where it has been assumed that x and t are continuous functions of T and X with continuous partial derivatives so that the order of the mixed partials may be reversed.

Equation (9) may now be integrated to give the relationship between $\partial x/\partial T$ and $\partial t/\partial T$, and thus via equation (7) between $\partial x/\partial X$ and $\partial t/\partial X$, i.e.

$$\frac{\partial t}{\partial X} = \frac{\partial x}{\partial X}\,v_F(T)\,, \tag{10}$$

$$\frac{\partial x}{\partial T} = \frac{\partial t}{\partial T}\,v_F(T)\,, \tag{11}$$

where $v_F(T)$ is an arbitrary function of T which results from the integration. The notation v_F is used for consistency with later notation. Using equations (10), (11), (6) and (8), it can easily be shown that

$$\frac{\partial x}{\partial X} = \frac{U(X)}{\sqrt{1 - v_F^2(T)}}\,, \tag{12}$$

$$\frac{\partial t}{\partial X} = \frac{U(X) v_F(T)}{\sqrt{1 - v_F^2(T)}}\,, \tag{13}$$

$$\frac{\partial t}{\partial T} = \frac{V(X)}{\sqrt{1 - v_F^2(T)}}\,, \tag{14}$$

$$\frac{\partial x}{\partial T} = \frac{V(X) v_F(T)}{\sqrt{1 - v_F^2(T)}}\,. \tag{15}$$

Equation (12) may now be differentiated partially with respect to T to yield

$$\frac{\partial}{\partial T}\frac{\partial x}{\partial X} = U(X)\, v_F(T)\, [1 - v_F^2(T)]^{-3/2} \frac{\partial}{\partial T} v_F(T)\,, \tag{16}$$

while equation (15) is differentiated partially with respect to X to yield

$$\frac{\partial}{\partial X}\frac{\partial x}{\partial T} = v_F(T)\, [1 - v_F^2(T)]^{-1/2} \frac{\partial}{\partial X} V(X)\,. \tag{17}$$

Equations (16) and (17) are now identified after reversing one of the mixed partial derivatives to give that

$$[1 - v_F^2(T)]^{-1} \frac{\partial}{\partial T} v_F(T) = \frac{1}{U(X)} \frac{\partial}{\partial X} V(X)\,. \tag{18}$$

Since the left-hand side of equation (18) is a function of T only, while its right-hand side is a function of X only, both sides must be constant, say α_R, and hence using the right-hand side gives

$$\frac{\partial}{\partial X} V(X) = \alpha_R\, U(X) \tag{19}$$

as required.

Equation (19) is the only restriction on the forms of the functions U and V, other than the obvious conditions of differentiability and positivity defined above, so we are free to choose V, use equation (19) to find a suitable form for U, or vice versa, and then substitute into equation (4) to find the equation for the metric. Since the resulting metric describes a space–time for which a coordinate transformation into Minkowski space–time exists, the use of the coordinate transformation which induces this metric guarantees that the mapping is an isometry and therefore preserves curvature. The resulting space–times are, therefore, flat.

3. The Coordinate Transformations

An explicit form for the general transformation between the two space–times can now be found after evaluating the function v_F by equating the left-hand side of equation (18) to α_R and integrating with respect to T to give

$$v_F(T) = \tanh(\alpha_R T + \psi)\,, \tag{20}$$

where ψ is a constant of integration.

Using the result obtained in equation (20), equations (12), (13), (14) and (15) become

$$\frac{\partial x}{\partial X} = U(X)\cosh(\alpha_R T + \psi)\,, \tag{21}$$

$$\frac{\partial t}{\partial X} = U(X)\sinh(\alpha_R T + \psi)\,, \tag{22}$$

$$\frac{\partial x}{\partial T} = V(X)\sinh(\alpha_R T + \psi)\,, \tag{23}$$

$$\frac{\partial t}{\partial T} = V(X)\cosh(\alpha_R T + \psi)\,. \tag{24}$$

Integration of equations (23) and (24) gives respectively that

$$x = \frac{V(X)}{\alpha_R}\cosh(\alpha_R T + \psi) + M(X), \qquad \forall \alpha_R \neq 0\,, \tag{25}$$

$$t = \frac{V(X)}{\alpha_R}\sinh(\alpha_R T + \psi) + N(X), \qquad \forall \alpha_R \neq 0\,, \tag{26}$$

where M and N are arbitrary functions of X introduced via the integration. These functions may be evaluated by differentiating both equations partially with respect to X and then comparing the resulting equations to (21) and (22) respectively. From equation (25)

$$\begin{aligned}\frac{\partial x}{\partial X} &= \frac{1}{\alpha_R}\frac{\partial V(X)}{\partial X}\cosh(\alpha_R T + \psi) + \frac{\partial M(X)}{\partial X}\\ &= U(X)\cosh(\alpha_R T + \psi) + \frac{\partial M(X)}{\partial X}\,,\end{aligned} \tag{27}$$

while from equation (26)

$$\begin{aligned}\frac{\partial t}{\partial X} &= \frac{1}{\alpha_R}\frac{\partial V(X)}{\partial X}\sinh(\alpha_R T + \psi) + \frac{\partial N(X)}{\partial X}\\ &= U(X)\sinh(\alpha_R T + \psi) + \frac{\partial N(X)}{\partial X}\,,\end{aligned} \tag{28}$$

where in both cases equation (19) has been used. Comparing equations (27) and (28) to (21) and (22) respectively gives that $\partial M(X)/\partial X = \partial N(X)/\partial X = 0$ so that both M and N must be constants. Substitution into equations (25) and (26) gives the coordinate transformations in the cases for which $\alpha_R \neq 0$,

$$x = \frac{V(X)}{\alpha_R}\cosh(\alpha_R T + \psi) + M, \qquad \forall \alpha_R \neq 0\,, \tag{29}$$

$$t = \frac{V(X)}{\alpha_R}\sinh(\alpha_R T + \psi) + N, \qquad \forall \alpha_R \neq 0\,. \tag{30}$$

The metric independent constants M, N and ψ are determined by the particular choice of inertial coordinate system. In particular, the constants M and N depend upon the relative positions of the origins of the two coordinate systems. Without loss of generality let the inertial system be chosen so that

$$(T, X, Y, Z) = (0, 0, 0, 0) \Leftrightarrow (t, x, y, z) = (0, x_0, 0, 0)\,. \tag{31}$$

This choice implies that coordinate time is synchronised to the inertial standard time at $t = T = 0$, and that by changing the value of x_0 it is possible to effect a translation of the origin of the moving coordinate system along the inertial x-axis.

Substituting the conditions given by equation (31) into (29) and (30) gives the final forms for the coordinate transformations when $\alpha_R \neq 0$, i.e.

$$x = \frac{V(X)}{\alpha_R}\cosh(\alpha_R T + \psi) - \frac{V(0)}{\alpha_R}\cosh\psi + x_0, \qquad \forall \alpha_R \neq 0\,, \tag{32}$$

$$t = \frac{V(X)}{\alpha_R}\sinh(\alpha_R T + \psi) - \frac{V(0)}{\alpha_R}\sinh\psi, \qquad \forall \alpha_R \neq 0, \tag{33}$$

and $y = Y$, $z = Z$.

The $\alpha_R = 0$ Case

In the case $\alpha_R = 0$, equation (19) implies that $V(X)$ must be a constant. It will be seen that the function V in the metric determines, among other characteristics, the ratio of the rate of proper-time to accelerating-coordinate time at the position. If we now require that there should exist an accelerating-coordinate position, say $X = R$, at which this ratio is unity then $V(R) = 1$ and so if $V(X)$ is a constant it must also be unity. In this case equations (23) and (24) become

$$\frac{\partial x}{\partial T} = \sinh\psi\,, \tag{34}$$

$$\frac{\partial t}{\partial T} = \cosh\psi\,. \tag{35}$$

Integration of equations (34) and (35) gives that

$$x = [T + M(X)]\sinh\psi\,, \tag{36}$$

$$t = [T + N(X)]\cosh\psi \tag{37}$$

respectively, where once again M and N are arbitrary functions of X which have been introduced via the integration. Partial differentiation of these two equations with respect to X, comparison with (21) and (22) respectively, and then integrating gives that

$$M(X) = \tilde{U}(X)\coth\psi + p\,, \tag{38}$$

$$N(X) = \tilde{U}(X)\tanh\psi + q\,, \tag{39}$$

where p and q are constants of integration and $\tilde{U}(X)$ is defined so that $\tilde{U}(0) = 0$. Substituting equations (38) and (39) into (36) and (37) respectively, and introducing two new constants, P and Q, such that $P = p\sinh\psi$ and $Q = p\cosh\psi$, gives the required coordinate transformations when $\alpha_R = 0$:

$$x = \tilde{U}(X)\cosh\psi + T\sinh\psi + P\,, \tag{40}$$

$$t = T\cosh\psi + \tilde{U}(X)\sinh\psi + Q\,. \tag{41}$$

Once again the constants P and Q are determined by the relative positions of the origins of the coordinate systems. Using the choice of inertial frame given by (31), equations (40) and (41) give that

$$x = \tilde{U}(X)\cosh\psi + T\sinh\psi + x_0, \qquad \text{if} \quad \alpha_R = 0\,, \tag{42}$$

$$t = T\cosh\psi + \tilde{U}(X)\sinh\psi, \qquad \text{if} \quad \alpha_R = 0 \tag{43}$$

for the final forms of the coordinate transformations when $\alpha_R = 0$.

4. Inverse Coordinate Transformations

The following inverse coordinate transformations when $\alpha_R \neq 0$ are easily obtained:

$$V(X) = \alpha_R\left[\left(x - x_0 + \frac{V(0)}{\alpha_R}\cosh\psi\right)^2 - \left(t + \frac{V(0)}{\alpha_R}\sinh\psi\right)^2\right]^{\frac{1}{2}}, \tag{44}$$

$$T = \frac{1}{\alpha_R}\tanh^{-1}\left(\frac{t + [V(0)/\alpha_R]\sinh\psi}{x - x_0 + [V(0)/\alpha_R]\cosh\psi}\right) - \frac{\psi}{\alpha_R}\,, \tag{45}$$

and $y = Y$, $z = Z$.

For $\alpha_R = 0$ the inverse transformations are

$$\tilde{U}(X) = (x - x_0)\cosh\psi - t\sinh\psi\,, \tag{46}$$

$$T = t\cosh\psi - (x - x_0)\sinh\psi\,, \tag{47}$$

and $Y = y$, $Z = z$, where the final forms for equations (44) and (46) can be obtained once $V(X)$ and $U(X)$ are known.

It is interesting to note at this stage that from equation (45)

$$\tanh(\alpha_R T + \psi) = \frac{t + [V(0)/\alpha_R]\sinh\psi}{x - x_0 + [V(0)/\alpha_R]\cosh\psi}\,,$$

and therefore

$$\left| t + \frac{V(0)}{\alpha_R}\sinh\psi \right| < \left| x - x_0 + \frac{V(0)}{\alpha_R}\cosh\psi \right|. \tag{48}$$

In the $\alpha_R \neq 0$ case, therefore, the inverse coordinate transformations have a domain which is only that region of Minkowski space–time described by equation (48). This region is bounded by the 'light cones' represented by the straight lines with slopes ± 1 passing through the event,

$$(t, x) = \left(-\frac{V(0)}{\alpha_R}\sinh\psi, \quad x_0 - \frac{V(0)}{\alpha_R}\cosh\psi \right),$$

which, therefore, dissect the Minkowski space–time into the four distinct wedges

$$\begin{aligned}
\text{Region (I)} \quad & x - x_0 + \frac{V(0)}{\alpha_R}\cosh\psi > \left| t + \frac{V(0)}{\alpha_R}\sinh\psi \right|, \\
\text{Region (II)} \quad & t + \frac{V(0)}{\alpha_R}\sinh\psi > \left| x - x_0 + \frac{V(0)}{\alpha_R}\cosh\psi \right|, \\
\text{Region (III)} \quad & x - x_0 + \frac{V(0)}{\alpha_R}\cosh\psi < -\left| t + \frac{V(0)}{\alpha_R}\sinh\psi \right|, \\
\text{Region (IV)} \quad & t + \frac{V(0)}{\alpha_R}\sinh\psi < -\left| x - x_0 + \frac{V(0)}{\alpha_R}\cosh\psi \right|.
\end{aligned}$$

Of these, only regions (I) and (III) satisfy equation (48). By convention we shall adopt the right-hand wedge, region I, as the region of Minkowski space–time to which the coordinate transformations map events from the (T, X, Y, Z) coordinate system.

An observer in region (I) can send light signals to observers in region (II) but not to regions (III) and (IV). Similarly observers in region (I) can detect signals sent from region (IV) but not from regions (II) and (III). It follows from this that no communication is possible between observers in regions (I) and (III), and further that the light cones represent event horizons for observers in these regions. In terms of the (T, X, Y, Z) coordinate system the equation for the event horizon becomes simply $X = E$, where E is a constant satisfying

$$V(E) = 0\,. \tag{49}$$

It is possible to choose V so that equation (49) is satisfied only at $-\infty$, for example, so it does not follow from the existence of the event horizons in Minkowski space–time that there exists an event horizon at a finite position in a flat space–time described by equation (4). This depends, instead, on the form of the function V.

5. Properties of the Space–time

In this section some of the properties of the flat space–time of constant proper-acceleration will be examined by looking at the properties of the fundamental observers of this space–time. By the term 'fundamental observers of a space–time' we have in mind the usual idea employed in cosmology. In cosmology a space–time can be regarded as a model universe by specifying a family of observers which follow world-lines which are characteristic of the average motion of matter in the universe being modeled (Martin 1988; Bondi 1952; Weinberg 1976). Such world-lines are then referred to as *fundamental world-lines* and observers following these world-lines are *fundamental observers* of the model universe.

We can regard the space–time of constant proper-acceleration as a model universe and take the fundamental observers of this space–time to correspond to the observers composing a frame of reference. Such a space–time can then be *said to be naturally adapted to the description of this frame of reference.* By a *frame of reference* we shall mean **an ensemble of physical observers** with well determined physical inter-relationships which are equipped with an ideal standard clock, and an ideal standard measuring rod. We shall use the coordinates derived above to define the fundamental observers of the space–times under consideration by taking the equation for the world-line of a fundamental observer to be the straight line

$$(X, Y, Z) = (F, F_Y, F_Z), \qquad \forall T \,,$$

where (F, F_Y, F_Z) is a constant. It also proves convenient to distinguish one of these fundamental observers located at $X = R$ where $V(R) = 1$, and call this the *reference observer* of the space–time.

It will be seen that the characteristics of fundamental observers in the space–times under consideration fall into two categories determined by whether or not α_R vanishes. It is therefore convenient to consider the cases $\alpha_R \neq 0$ and $\alpha_R = 0$ separately. In what follows we use the fact that the Y and Z coordinates are identifiable with the corresponding Minkowski coordinates, y and z, and simplify the notation by suppressing these coordinates in the calculations. The results are true on an arbitrary (T, X) hypersurface.

(5a) The $\alpha_R \neq 0$ Cases

Let x_F denote the spatial coordinate in Minkowski coordinates of the fundamental observer located at $X = F$. Substituting into the coordinate transformations and solving simultaneously gives the following equation for the world-line of fundamental observers:

$$\left(x_F - x_0 + \frac{V(0)}{\alpha_R}\cosh\psi\right)^2 - \left(t + \frac{V(0)}{\alpha_R}\sinh\psi\right)^2 = \frac{V^2(F)}{\alpha_R^2} \,. \tag{50}$$

Since x_0, α_R, $V(0)$ and $V(F)$ are all constants, this is the equation of a hyperbola. The fundamental observers of the space–time, therefore, follow hyperbolic world-lines with respect to the inertial coordinate system and therefore have constant proper-acceleration (Rindler 1977) regardless of the form of the function V. Any two such space–times can, therefore, only be distinguished by the values of proper-acceleration assigned to different fundamental observers.

The term 'flat space–time of constant proper-acceleration' is not meant to imply that all of the fundamental observers have the same value of proper-acceleration. In fact, it will be shown that it is not possible to find a flat space–time described by a metric of the form of equation (4) in which all fundamental observers have one and the same value of proper-acceleration, unless that proper-acceleration is zero. It will also be shown that the zero proper-acceleration case corresponds to the $\alpha_R = 0$ case to be discussed later in this section.

Solving equation (50) for x_F, differentiating with respect to t and substituting from (47) gives that

$$v_F = \tanh(\alpha_R T + \psi) = v_F(T)\,. \tag{51}$$

The function $v_F(T)$, therefore, which was introduced in equations (10) and (11) as an arbitrary function resulting from an integration is precisely the velocity of the accelerating fundamental observer with respect to the inertial coordinate system. Note also that at time $T = 0$ the velocity becomes simply $v_F = \tanh\psi$, so the constant ψ determines the velocity of the fundamental observers at this time. It is interesting that v_F is independent of the accelerating coordinate position F of the fundamental observer in accelerating coordinates.

The acceleration a_F of the fundamental observers can also be found relative to the inertial coordinate system by differentiating equation (51) once again with respect to t. Thus

$$a_F = \alpha_R\ \mathrm{sech}^2(\alpha_R T + \psi)\frac{dT}{dt}\,. \tag{52}$$

The derivative dT/dt is found by differentiating equation (33) with respect to T after letting $X = F$ and inverting to give

$$\frac{dT}{dt} = \frac{1}{V(F)}\ \mathrm{sech}(\alpha_R T + \psi)\,. \tag{53}$$

Substituting this equation into (52) gives the acceleration of the fundamental observers relative to the inertial coordinate system, i.e.

$$a_F = \frac{\alpha_R}{V(F)}\ \mathrm{sech}^3 \alpha_R T + \psi\,. \tag{54}$$

Note that the acceleration of the fundamental observers reaches a maximum at $T = -\psi/\alpha_R$ and then approaches zero as T tends to infinity.

(5b) Proper-time for the Fundamental Observers ($\alpha_R \neq 0$)

In general the standard proper-time τ_F as determined by a fundamental observer at F is not the same as coordinate time. The relationship between the accelerating-coordinate time and proper-time for a fundamental observer located at $X = F$ will be found after first finding the relationship between the proper-time at F and the inertial standard time. This is done by starting with the standard special relativistic result

$$dt = \gamma d\tau \,, \tag{55}$$

where $\gamma = (1 - u^2)^{-1/2}$ and u is the velocity determined by the inertial observer at rest, of the inertial frame of reference with respect to which the moving observer remains at rest. It follows from the standard clock hypothesis of special relativity that the velocity of the momentarily co-moving inertial frame (MCIF) of the accelerating standard clock in equation (51) can be substituted for u, so

$$\gamma_F(T) = \cosh(\alpha_R T + \psi) \,, \tag{56}$$

and substitution into (55) gives that

$$dt = d\tau_F \cosh(\alpha_R T + \psi) \,.$$

The inertial time differential may be eliminated from this equation by use of the inverse of equation (53) so that $V(F) = d\tau_F/dT$ or upon integration

$$\tau_F = V(F)T \,, \tag{57}$$

where we have used the added condition that accelerating-coordinate time is synchronised to standard proper-time at $\tau_F = T = 0$.

This important and interesting result provides a physical interpretation for the function V. The function V evaluated at F is the ratio of the rate of proper-time to accelerating-coordinate time at the position $X = F$.

(5c) Proper-distance for the Fundamental Observers ($\alpha_R \neq 0$).

In addition to a standard clock each fundamental observer is equipped with a standard measuring rod by means of which the fundamental observer located at $X = F$ can determine the proper-distance to events in its immediate neighbourhood. The total proper rod-distance between two points in the accelerated space–time can then be found by disposing along the X-axis fundamental observers who lay down standard measuring rods end to end. In general, the proper-distance from a fundamental observer to a neighbouring point will not be the same as the accelerating-coordinate distance. By analogy to the method used earlier for proper-time the relationship between the proper-distance differential, $d\sigma_F$, at $X = F$ and the inertial coordinate differential, dx, will be found first, by using the standard special relativistic result,

$$dx = d\sigma/\gamma. \tag{58}$$

Once again, it follows from the definition of a standard measuring rod, that u may be identified with the velocity of the MCIF of the fundamental observer. Thus from equation (56), equation (58) becomes

$$dx = d\sigma_F \ \mathrm{sech}(\alpha_R T_m + \psi) \,, \tag{59}$$

where T_m denotes the accelerating coordinate time when the measurement is carried out.

The relationship between the accelerating-coordinate differential dX and the inertial-coordinate differential can be found after first noting that any measurement by an inertial observer of the length of a moving interval requires a determination of the position of the ends of the interval at the same inertial time, t_m. Substituting $t = t_m$ in the inverse coordinate transformation, equation (44), and differentiating implicitly with respect to x gives that

$$dx = dX\, U(X)\mathrm{sech}(\alpha_R T_m + \psi)\,, \tag{60}$$

where equation (19) and the coordinate transformation equation (29) have been used. Finally, comparing equations (59) and (60) gives that $d\sigma_F/dX = U(X)$ or upon integration that

$$\sigma_F = \frac{V(X)}{\alpha_R} - \frac{V(F)}{\alpha_R}\,, \tag{61}$$

where σ_F denotes the proper-distance interval from the fundamental observer at F to the point X.

Equation (61) provides a physical interpretation for the function $U(X)$ in the metric. It is the ratio of the proper-distance differential to accelerating-coordinate differential at position X.

(5d) Proper-acceleration for the Fundamental Observers ($\alpha_R \neq 0$)

The proper-acceleration α_F of the fundamental observer at $X = F$ is by definition the acceleration determined with respect to the MCIF at F. It is related to the acceleration determined with respect to the inertial coordinate system by the usual result of special relativity, $\alpha_F = a_F \gamma_F^3$, which from equation (56) becomes

$$\alpha_F = a_F \cosh^3(\alpha_R T + \psi)\,.$$

Using equation (54) gives finally

$$\alpha_F = \frac{\alpha_R}{V(F)}\,. \tag{62}$$

As expected, the proper-acceleration is dependent on the position of the fundamental observer but is independent of time. Note also that in order for the proper-acceleration to take the same value for all fundamental observers, i.e. to be independent of F, the function V must be a constant, which from equation (19) above implies that $\alpha_R = 0$, and thus from (62) that the proper-acceleration vanishes. Thus, as mentioned earlier, with the exception of the trivial case in which α_F vanishes, it is not possible to find a flat space–time described by a metric of the form of equation (4) in which all of the fundamental observers have one and the same value of proper-acceleration.

Note also that equation (62) may also be evaluated at the position of the reference observer which shows that the constant of proportionality α_R introduced in (19) may be identified as the proper-acceleration of the reference observer.

This result may be used to provide a second interpretation for the function V in the metric. From equation (62) $V(F) = \alpha_R/\alpha_F$ so the value of the function V at F is the ratio of the proper-acceleration of the reference observer to the proper-acceleration of a fundamental observer located at $X = F$.

(5e) The $\alpha_R = 0$ Cases

Once again let x_F denote the inertial-coordinate position of the fundamental observer located at $X = F$. Using the coordinate transformations for the $\alpha_R = 0$ cases and solving we obtain that the equation of the world-lines for the fundamental observers is

$$x_F = t \tanh\psi + \tilde{U}(F)\mathrm{sech}\psi + x_0 \,. \tag{63}$$

This is the equation of a straight line with slope $\tanh\psi$. In the $\alpha_R = 0$ case, therefore, all of the fundamental observers of the space–time described by the metric in equation (4) travel at the same constant velocity

$$v_F = \tanh\psi \tag{64}$$

with respect to the (t, x, y, z) coordinate system.

(5f) Proper-time for the Fundamental Observers $(\alpha_R = 0)$

Once again the relationship between the time shown by moving coordinate clocks, keeping coordinate time T, and moving standard clocks, keeping proper-time, can be found after first finding the relationship between the proper-time and the inertial standard time. Substituting equation (64) into (55) gives that

$$dt = d\tau_F \cosh\psi \,. \tag{65}$$

Differentiation of equation (45) with respect to T, however, shows that $dt = dT \cosh\psi$ so (65) becomes simply $dT = d\tau_F$. In this special case, therefore, the coordinate clocks and the moving standard clocks keep the same time. The relationship between the proper-time of the fundamental observers and the inertial standard time may be easily derived from equations (64) and (65), and becomes

$$t = \tau_F \cosh\psi = \frac{\tau_F}{\sqrt{1 - v_F^2}} \,, \tag{66}$$

which is the standard time-dilation of special relativity.

In the $\alpha_R = 0$ case, therefore, all of the fundamental observers move with the same velocity along the x-axis and the (T, X, Y, Z) coordinate system is inertial. The only effect of the function $\tilde{U}(F)$ is to define a scale for the X-axis. This axis may be chosen to have a linear scale simply by letting $U(X) = 1$ so that $\tilde{U}(X) = X$, in which case the coordinate transformations reduce to the standard Poincaré transformations of special relativity.

6. Special Cases

It has been shown that space–times described by a metric of the general form of equation (4) are characterised by fundamental observers all of whom have constant proper-acceleration and therefore follow hyperbolic world-lines. Since all fundamental world-lines are hyperbolic any two space–times can be distinguished only by the values of proper-acceleration which are assigned to particular fundamental observers by the functions $V(X)$ and $U(X)$ in the metric. It has also been shown that many of the important properties of these space–times are generic since they are determined only by the general form of the metric. Other properties, however, will be determined by the particular choices for $V(X)$ and $U(X)$. Special cases of the general coordinate systems derived in this paper have appeared in the literature in other contexts. In this section we consider two examples and point out a few of the special properties of these coordinates.

(6a) Rindler Coordinates

The Rindler coordinates arise from the work carried out here simply by requiring that the accelerating coordinate displacement of an object from the reference observer be precisely equivalent to the proper rod-distance to the object from the reference observer, i.e. $\sigma_R = X - R$ where X denotes the coordinate position of the object. With this requirement equation (61) yields that

$$V(X) = \alpha_R X - \alpha_R R + 1\,. \tag{67}$$

It follows from equations (67) and (19) that $U(X) = 1$ and the resulting metric for the space–time is

$$ds^2 = (\alpha_R X - \alpha_R R + 1)^2 (dT)^2 - (dX)^2 - (dY)^2 - (dZ)^2\,.$$

The standard form for the Rindler (1966) metric is obtained by locating the reference observer at $R = 1/\alpha_R$, in which case the metric becomes

$$ds^2 = \alpha_R^2 X^2 (dT)^2 - (dX)^2 - (dY)^2 - (dZ)^2\,,$$

and the standard Rindler transformations

$$x = \xi \cosh(\alpha_R \eta) \qquad \forall \alpha_R \neq 0\,, \tag{68}$$

$$ct = \xi \sinh(\alpha_R \eta) \qquad \forall \alpha_R \neq 0\,, \tag{69}$$

and $y = Y$, $z = Z$, result from the general coordinate transformations (32) and (33). We have also chosen to let the origins of the accelerating coordinate system and inertial coordinate system coincide, the velocity of the accelerating reference observer relative to the inertial frame to be zero at time $t = 0$, i.e.

$$\left.\begin{array}{r} x_0 = 0 \\ \psi = 0 \end{array}\right\} \text{ at } t = T = 0\,, \tag{70}$$

and we have chosen to write $X = \xi$ and $T = \eta$. With this choice and using equation (49), the event horizon for the Rindler space–time is located at the origin of the accelerating coordinate system and the fundamental observers have hyperbolic world-lines.

A commonly employed alternative realisation of the Rindler coordinates results from locating the reference observer at the accelerating coordinate origin, $R = 0$, in which case the metric takes the form

$$ds^2 = (\alpha_R X + 1)^2 (dT)^2 - (dX)^2 - (dY)^2 - (dZ)^2 \,, \tag{71}$$

and the coordinate transformations become

$$x = \left(X + \frac{1}{\alpha_R}\right) \cosh\,(\alpha_R T) - \frac{1}{\alpha_R} \qquad \forall \alpha_R \neq 0 \,,$$
$$t = \left(X + \frac{1}{\alpha_R}\right) \sinh\,(\alpha_R T) \qquad \forall \alpha_R \neq 0 \,,$$

with $y = Y$, $z = Z$, where the conditions given by equation (70) have been employed. In this case the event horizon is located at $-1/\alpha_R$.

It is interesting to note that for comparison with the results for other space–times to be discussed below, the coordinate velocity of light in Rindler coordinates is

$$\frac{dX_\gamma}{dT} = \pm(\alpha_R X + 1) \,, \tag{72}$$

where the alternative coordinate system described by the metric form in equation (71) has been adopted. This result is obtained by starting with the equation of the light cone in inertial coordinates, transforming into accelerating coordinates and differentiating with respect to coordinate time. The general result takes the form

$$\frac{dX_\gamma}{dT} = \pm \frac{V(X_\gamma)}{U(X_\gamma)} \,, \tag{73}$$

from which (72) is obtained by the specific choice of the metric, equation (71). Note that, in general, for the class of space–times we are considering the speed of light is not constant and is equal to its Minkowski value only at the position of the reference observer.

(6b) Lass's Coordinates and the Uniform Gravitational Field

Lass (1963) introduced a coordinate transformation between an accelerating coordinate system and an inertial coordinate system which is based upon a relatively simple and compelling set of hypotheses, and which leads to the metric

$$ds^2 = e^{2\alpha_R X} dT^2 - e^{2\alpha_R X} dX^2 - dY^2 - dZ^2 \,.$$

An alternative derivation of Lass's transformations was carried out by Marsh (1965), while Romain (1964) has analysed the hypotheses employed in Lass's derivation. Lass's metric satisfies equation (19) with $V(X) = U(X) = e^{\alpha_R X}$ and therefore describes a flat space–time of the type discussed in this paper. The coordinate transformations (which agree with those found by Lass) can be found from equations (32) and (33) and become

$$x = \frac{1}{\alpha_R} e^{\alpha_R X} \cosh(\alpha_R T) - \frac{1}{\alpha_R}\,,$$
$$t = \frac{1}{\alpha_R} e^{\alpha_R X} \sinh(\alpha_R T)\,,$$

once the conditions given in equation (70) are employed. Some of the properties of these coordinates are discussed in Lass's paper but it is possible here to call attention to two particularly interesting properties. Firstly, from equation (73) the coordinate-velocity of light takes the Minkowski value at *all* points in this accelerating space–time. Among the coordinatisations of accelerating frames of reference in a flat space–time this is a unique property of Lass's coordinates and could equally well have been used as the basis for the derivation of the metric. The requirement that $|dX\gamma/dT| = 1$ gives, from equation (73), that $V(X) = U(X)$ which from (19) implies the exponential form for V and U.

The second particularly interesting property of Lass's coordinates results from the coordinate-acceleration of a particle released from rest by one of the fundamental observers. The world-line of such a freely-falling particle is found by writing the equation for the particle in inertial coordinates, transforming to accelerating coordinates and differentiating twice with respect to T to yield

$$\frac{d^2 X_{ff}(T)}{dT^2} = -\alpha_R \left\{ \frac{V(X_{ff}(T))}{U(X_{ff}(T))}\ \mathrm{sech}^2(\alpha_R T - \alpha_R T_r) \right.$$
$$\left. + \left[\frac{V^2(X_{ff}(T))\ U'(X_{ff}(T))}{\alpha_R U^3(X_{ff}(T))} - \frac{V(X_{ff}(T))}{U(X_{ff}(T))} \right] \tanh^2(\alpha_R T - \alpha_R T_r) \right\}, \quad (74)$$

where

$$U'(X_{ff}(T)) = \frac{dU(X_{ff}(T))}{d(X_{ff}(T))}\,,$$

and $X_{ff}(T)$ denotes the position of the freely-falling particle in general accelerating coordinates.

Substituting for V and U in equation (74), using Lass's metric and evaluating the resulting equation at the time of release, $T = T_r$, gives that

$$\frac{d^2 X_{ff}(T)}{dT^2} = -\alpha_R\,.$$

In Lass's coordinates, therefore, the initial coordinate-acceleration of the freely-falling object has magnitude equal to the proper-acceleration of the reference observer of the space–time and is therefore independent of position.

It has been shown (Desloge 1989; Takagi 1989) that a suitable form for the metric of the two-dimensional space–time describing a rigid frame of reference at rest in a uniform gravitational field takes the form

$$ds^2 = e^{2\alpha X} dT^2 - dX^2 \,.$$

The essential characteristic of this space–time is that, α, the initial acceleration of a particle released from rest by an observer at rest in the gravitational field, is independent of position. This metric does not satisfy equation (19) and the resulting space–time is not flat, i.e. the space–time of a uniform gravitational field is still a curved space–time. The space–time described by Lass's metric, however, does retain this essential property of a uniform gravitational field while still being a flat space–time.

7. Conclusion

It has been shown that it is possible to find a general class of rigid accelerating frames of reference which may be covered by coordinate systems which possess non-local coordinate transformations to Minkowski space–time. Observers composing these frames of reference have constant proper-acceleration and are restricted to wedges in Minkowski space–time. We take the resulting sub-manifolds of Minkowski space–time to be the 'space–times of constant proper-acceleration' and the observers composing the frames of reference to be the fundamental observers of the space–times. It has been shown, further, that it is not possible to find a flat space–time with a metric of the form of equation (4) in which all of the fundamental observers have one and the same value of proper-acceleration. The coordinate transformations relating the accelerating-coordinate systems to an inertial coordinate system have been found for the general case and then employed to find some of the general properties of these space–times.

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