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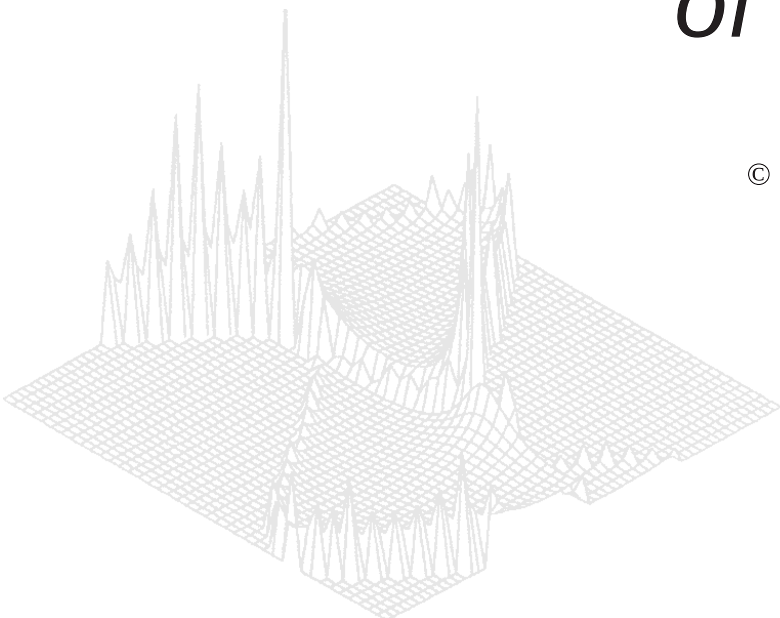

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A New Class of Brans–Dicke Cosmological Models with Causal Viscous Fluid

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Abstract

The behaviour of solutions to the Brans–Dicke equations of Friedmann–Robertson–Walker models with bulk viscous fluid source described by full (i.e. non-truncated) causal nonequilibrium thermodynamics is investigated. A new class of cosmological solution is obtained for all spatial sections ($k = 0, \pm 1$).

1. Introduction

One of the most important modifications to Einstein's theory of general relativity is Brans–Dicke theory (BDT) (Brans and Dicke 1961), which introduced a scalar field ϕ into the field equations to make things less reliant on the absolute properties of space. Great emphasis in recent years has been put on BDT, because cosmological models constructed from BDT first provided a way to terminate the inflationary era (so-called 'old', 'new' and 'chaotic' inflation) without fine tuning (Mathiazhagan and Johri 1984; La and Steinhardt 1989; La *et al.* 1989; Linde 1990; Barrow and Maeda 1990; Holman *et al*. 1991). Moreover BDT can be considered like the usual induced one if there is no scalar field potential $V(\phi)$. More detailed surveys on BDT can be found in Singh and Rai (1983) and Singh and Singh (1987).

Most treatments of cosmology regard the fluid as being a perfect fluid. However, bulk viscosity is expected to play an important role in cosmology at certain stages in the evolution of the Universe (Ellis 1971; Misner 1968; Hu 1983). It can be of interest to study Brans–Dicke cosmological models with bulk viscosity. This has been studied in the non-causal theory in papers by Johri and Sudharsan (1989), Pimentel (1994) and Beesham (1996). Recently, Banerjee and Beesham (1996) have considered Brans–Dicke cosmology with a causal viscous fluid in the full theory of nonequilibrium thermodynamics. They have found exact solutions for a spatially flat ($k = 0$) model by making the assumption $\phi \sim R^{\alpha}$ (Johri and Sudharsan 1989).

In this paper, we investigate a new class of Brans–Dicke cosmological models with a causal viscous fluid in the full theory of nonequilibrium thermodynamics.

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Exact solutions are obtained for all spatial sections ($k = 0, \pm 1$). It is found that the solutions (for $k = 0$) are consistent with Banerjee and Beesham (1996).

The necessary physical conditions forced on the models by observation for our universe are $R \geq 0$, $\dot{R} > 0$, $\phi > 0$, $\rho > 0$ and $\Pi < 0$. There is possibly a physical constraint on ω. But present day experiment and observation does not give sufficient information about this. So in this paper we will consider its value to be free except for the above constraints (see Xing and You-lin 1993).

The paper is organised as follows. In Section 2 we introduce the basic equations of BDT and the causal evolution equation (full theory). Section 3 is devoted to finding the exact solutions of the models. The paper ends with our conclusions in Section 4.

2. Field Equations

The gravitational field equations with usual notation for BDT may be written as

$$G_{ab} + \frac{\omega}{\phi^2}\left[\phi_{,a}\phi_{,b} - \tfrac{1}{2} - g_{ab}\phi^{,c}\phi, c\right] + \frac{1}{\phi}\left[\phi_{,a;b} - \Box\phi g_{ab}\right] = \frac{1}{\phi}T_{ab}\,, \tag{1}$$

$$\Box\phi = \frac{1}{(2\omega+3)}T\,, \tag{2}$$

where ϕ is the scalar field, the constant ω is the Brans–Dicke parameter, g_{ab} is the metric tensor, $\Box$ is the D'Alembertian wave operator and T is the trace of the energy moment tensor T_{ab}. We are using units in which both $8\pi G$ and c are equal to unity.

Let us consider a homogeneous and isotropic universe represented by the FRW metric

$$ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi\right], \tag{3}$$

where $R(t)$ is the scale factor. The constant $k = 0, \pm 1$ defines the curvature of the spatial section.

The Brans–Dicke field equations (1)–(2) with the above metric take the form

$$3\left(\frac{\dot{R}}{R}\right)^2 + 3\frac{\dot{R}\dot{\phi}}{R\phi} + \frac{3k}{R^2} - \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 = \frac{\rho}{\phi}\,, \tag{4}$$

$$2\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 + \frac{\ddot{\phi}}{\phi} + \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + 2\frac{\dot{R}\dot{\phi}}{R\phi} + \frac{k}{R^2} = \frac{-P}{\phi}\,, \tag{5}$$

$$\frac{\ddot{\phi}}{\phi} + 3\frac{\dot{R}\dot{\phi}}{R\phi} = \frac{1}{(2\omega+3)\phi}(\rho - 3P)\,. \tag{6}$$

An overdot denotes the time derivative, ρ and P being the density and pressure of the fluid respectively. To include the effect of viscosity we have to replace the pressure P by an effective pressure P_{eff}. The latter is given by

$$P_{\text{eff}} = P + \Pi \,, \tag{7}$$

where P and Π stands for the equilibrium hydrostatic and scalar viscous pressure (bulk viscous stress) respectively. The equilibrium pressure is assumed to obey the usual equation of state

$$P = \gamma\rho \,, \tag{8}$$

where the parameter γ is a constant $(0 \leq \gamma \leq 1)$.

The causal evolution equation for the bulk viscosity Π is given by

$$\Pi + \tau\dot{\Pi} = -3\xi H - \frac{\epsilon\tau H}{2}\left[3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T}\right] . \tag{9}$$

Here $T \geq 0$ is the absolute temperature, $H = \dot{R}/R$ is the Hubble parameter, ξ is the bulk viscosity coefficient, which cannot become negative otherwise the principle of entropy increase would be violated, and the coefficient τ denotes the relaxation time for the transient bulk viscous effects. Causality demands that $\tau > 0$. In equation (9), $\epsilon = 0$ gives the truncated theory, i.e. it corresponds to the case where the term in square brackets in equation (9) is negligible in comparision with the other terms, while the full theory has $\epsilon = 1$ and the non-causal (Eckart) theory has $\tau = 0$. The causal theory and non-causal theory have been surveyed in Maartens (1995), Zimdahl (1996) and Grøn (1990). Maartens (1995) discussed the drawbacks of Eckart theory and the truncated theory of nonequilibrium thermodynamics. Further, he argued that the best currently available theory for analysing the dissipative process is the full (i.e. non-truncated) causal thermodynamics of Israel and Stewart (1976).

To proceed further we must specify the dependence of ξ and τ on the energy density. It is usual to assume (see Pavoń *et al.* 1991; Maartens 1995; Zimdahl 1996) the following *ad hoc* laws:

$$\xi = \xi_0\rho^q \quad \text{and} \quad \tau = \frac{\xi}{\rho} \,, \tag{10}$$

and the equation (8) of state for P, where $\xi_0 \geq 0$ and $q \geq 0$ are constants. If $q = 1$, then equation (10) may correspond to a radiative fluid, whereas $q = \frac{3}{2}$ may represent a string dominated universe (see Murphy 1973; Barrow 1988). However, more realistic models are based on values of q lying in the range $0 \leq q \leq \frac{1}{2}$ (Santos *et al.* 1985).

On rearranging and integrating equation (9), we get

$$T = T_0 \exp\left[\frac{2f(t)}{\epsilon}\right] \exp\left[\frac{2g(t)}{\epsilon}\right] \frac{\Pi^{(2/\epsilon)}R^3\tau}{\xi} \,. \tag{11}$$

Here T_0 is an integration constant, while the functions $f(t)$ and $g(t)$ are, respectively, the antiderivatives of $(1/\tau)$ and $(3\rho H/\Pi)$. Equation (11) gives the temperature as a function of time. We shall evaluate $f(t)$ and $g(t)$ for $k = 0$, -1, 1 in the following section.

Equations (4)–(6) in combination with (7) and the equation of state now take the form

$$3\left(\frac{\dot{R}}{R}\right)^2 + 3\frac{\dot{R}\dot{\phi}}{R\phi} + \frac{3k}{R^2} - \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 = \frac{\rho}{\phi}\,, \tag{12}$$

$$2\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 + \frac{\ddot{\phi}}{\phi} + \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + 2\frac{\dot{R}\dot{\phi}}{R\phi} + \frac{k}{R^2} = -\frac{(\gamma\rho + \Pi)}{\phi}\,, \tag{13}$$

$$\frac{\ddot{\phi}}{\phi} + 3\frac{\dot{R}\dot{\phi}}{R\phi} = \frac{1}{(2\omega + 3)\phi}[\rho - 3\gamma\rho - 3\Pi]\,. \tag{14}$$

Equations (12)–(14) lead to the continuity equation,

$$\dot{\rho} + 3H(\rho + \rho\gamma + \Pi) = 0\,. \tag{15}$$

We have in equations (12)–(14) three independent equations in four unknown variables, viz. $R(t)$, $\phi(t)$, $\rho(t)$ and $\Pi(t)$, and as such the system does not have a unique solution.

3. Models

By a combination of equations (12)–(14), we have

$$\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{\omega}{3}\left[\frac{\ddot{\phi}}{\phi} - \tfrac{1}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + 3\frac{\dot{R}\dot{\phi}}{R\phi}\right]. \tag{16}$$

In most of the investigations (see e.g. Johri and Sudharsan 1989) a relationship between the scale factor $R(t)$ and scalar field $\phi(t)$ of the form $\phi \sim R^\alpha$ is assumed. In the present work, however, no such assumption is made *a priori*. We shall rather assume a scale factor $R(t)$ such that

$$\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = 0\,, \tag{17}$$

which on integration yields

$$R(t) = (-kt^2 + At + B)^{\frac{1}{2}}\,, \tag{18}$$

where A and B are integration constants. Using equation (17) in (16), and integrating, we get

$$\phi^{\frac{1}{2}} = \int \frac{C_1}{R^3} dt \,. \tag{19}$$

For the sake of simplicity the second integration constant is taken to be zero. To proceed for exact solutions, it is convenient to consider the three cases $k = 0$, -1 and $+1$ separately.

(3a) Case $k = 0$

Equation (18) reduces to

$$R(t) = (At + B)^{\frac{1}{2}} \,. \tag{20}$$

Using equation (20) in (19) leads to

$$\phi(t) = \frac{C_1^2}{A^2(At + B)} \,. \tag{21}$$

Then from equations (12), (13), (20) and (21) we have

$$\rho = \rho_0 (At + B)^{-3} \,, \tag{22}$$

$$\Pi = \Pi_0 (At + B)^{-3} \,, \tag{23}$$

where ρ_0 and Π_0 are constants given by

$$\rho_0 = C_1^2 \left(\tfrac{3}{4} + \frac{\omega}{2} \right) , \qquad \Pi_0 = C_1^2 \left[\tfrac{3}{4} + \frac{\omega}{2} \right] - \gamma \rho_0 \,.$$

The solution is completely specified in terms of the coupling parameter ω. Using equations (10), (20), (23) and (24), one can find $f(t)$ and $g(t)$ as

$$f(t) = \frac{(At + B)^{3q-2}}{A \xi_0 \rho_0^{q-1} (3q - 2)} \,, \tag{24}$$

$$g(t) = \ln \left[(At + B)^{(3\rho_0 / 2\Pi_0)} \right] \,. \tag{25}$$

One can easily identify that the results (with $B = 0$) obtained in this case are consistent with Banerjee and Beesham (1996) (for $\alpha = -2$ and $\beta = 2$).

(3b) Case $k = -1$

In this case equation (18) reads

$$R(t) = (t^2 + At + B)^{\frac{1}{2}} . \tag{26}$$

Substituting equation (26) into (19) leads to

$$\phi(t) = \frac{C_1^2 (A + 2t)^2}{L_1^2 (t^2 + At + B)}$$

and the density $\rho(t)$ and bulk viscosity $\Pi(t)$ have the form

$$\rho(t) = C_1^2 (t^2 + At + B)^{-3} \left[\frac{3(A + 2t)^2}{4L_1} - 2\omega \right] , \tag{28}$$

$$\Pi(t) = C_1^2 (t^2 + At + B)^{-3} \left[\frac{-3\gamma(A + 2t)^2}{4L_1} + \frac{(t^2 + At + B)}{L_1} + 2\omega(\gamma - 1) - 3 \right] , \tag{29}$$

where $L_1 = 4B - A^2$ is a constant. It may be difficult to find the general value of $f(t)$. However we have calculated $f(t)$ for some particular values of q. To conserve space we present $f(t)$ for $q = 0$, 1, which is given by

$$f(t) = \frac{(-A^3 d_1 + 4ABd_1 - 2A^2 d_1 t + 8Bd_1 t - 6AC_1^2\omega - 12C_1^2\omega t)}{\xi_0 L_1^2 (t^2 + At + B)} + \frac{(A^3 d_1 - 4ABd_1 + 2A^2 d_1 t - 8Bd_1 t - 2AC_1^2\omega - 4C_1^2\omega t)}{2\xi_0 L_1 (t^2 + At + B)^2} + \frac{[4(-(A^2 d_1) + 4Bd_1 - 6C_1^2\omega)]}{\xi_0 L_1^{\frac{5}{2}}} \arctan\left(\frac{A + 2t}{\sqrt{L_1}}\right) \quad \text{for } q = 0 , \tag{30}$$

$$f(t) = \frac{t}{\xi_0} \qquad \text{for } q = 1 , \tag{31}$$

Here $d_1 = 3C_1^2/4L_1$ is a constant. The function $g(t)$ takes the form

$$g(t) = \frac{3}{(-24\gamma+8)}\left[\frac{2M_1 - AM_2}{\sqrt{L_1}}\arctan\left(\frac{A+2t}{\sqrt{L_1}}\right)\right]$$

$$- \frac{3}{(-24\gamma+8)}\left[\frac{2M_3 - AM_4}{\sqrt{L_2}}\operatorname{arctanh}\left(\frac{A+2t}{\sqrt{L_2}}\right)\right]$$

$$+ \frac{3}{(-24\gamma+8)}\ln\left[(t^2 + At + B)^{M_2/2}(t^2 + At + k_1)^{M_4/2}\right], \tag{32}$$

where $L_2 = A^2 - 4K_1$ is a constant, K_1 is a constant given by

$$K_1 = \left[\frac{(4B - 3\gamma A^2) + 4[2\omega(\gamma-1) - 3]L_1}{-12\gamma + 4}\right] \tag{33}$$

and M_1, M_2, M_3 and M_4 are constants given by

$$M_1 = \frac{-12AB + 3A^3 - 8\omega L_1 A}{K_1 - B}, \quad M_2 = \frac{-24B + 6A^2 - 16\omega L_1}{K_1 - B},$$

$$M_3 = \frac{12AK_1 - 3A^3 + 8\omega L_1 A}{K_1 - B}, \quad M_4 = \frac{24K_1 - 6A^2 - 16\omega L_1}{K_1 - B}.$$

For the solutions to be real, we put the conditions that $L_1 > 0$ and $L_2 > 0$. The model contracts from $R_0 = \sqrt{B}$ at $t_0 = 0$ to a minimum of $R_{\min} = \sqrt{L_1}$ and thereafter expands forever.

(3c) Case $k = +1$

In this case the scale factor $R(t)$, scalar field $\phi(t)$, density $\rho(t)$, and bulk viscosity $\Pi(t)$ are given by

$$R(t) = (-t^2 + At + B)^{\frac{1}{2}}, \tag{34}$$

$$\phi(t) = \frac{C_1^2(A-2t)^2}{L_3^2(-t^2 + At + B)}, \tag{35}$$

$$\rho(t) = C_1^2(-t^2 + At + B)^{-3}\left[-\frac{3(A-2t)^2}{4L_3} - 2\omega\right], \tag{36}$$

$$\Pi(t) = C_1^2(-t^2+At+B)^{-3}\left[\frac{3\gamma(A-2t)^2}{4L_3} + \frac{(-t^2+At+B)}{L_3} + 2\omega(\gamma-1)-3\right], \tag{37}$$

where $L_3 = 4B + A^2$ is a constant. The function $f(t)$ takes the form:

$$f(t) = \frac{(A^3d_2 + 4ABd_2 - -2A^2d_2t - 8Bd_2t + 6AC_1^2\omega - 12C_1^2\omega t)}{\xi_0 L_3^2(-t^2+At+B)} + \frac{(-A^3d_2 - 4ABd_2 + 2A^2d_2t + 8Bd_2t + 2AC_1^2\omega - 4C_1^2\omega t)}{2\xi_0 L_3(-t^2+At+B)^2} + \frac{[4(A^2d_2 + 4Bd_2 + 6C_1^2\omega)]}{\xi_0 L_3^{\frac{5}{2}}}\arctan\left(\frac{A+2t}{\sqrt{L_3}}\right) \quad \text{for } q = 0\,, \tag{38}$$

where $d_2 = -3C_1^2/4L_3$ is a constant and $f(t)$ has the same form as in the previous case for $q = 1$.

The function $g(t)$ is given by

$$g(t) = \frac{-3}{(-24\gamma+8)}\left[\frac{2M_5 - AM_6}{\sqrt{L_3}}\operatorname{arctanh}\left(\frac{A-2t}{\sqrt{L_3}}\right)\right] + \frac{3}{(-24\gamma+8)}\left[\frac{2M_7 - AM_8}{\sqrt{L_4}}\operatorname{arctanh}\left(\frac{A-2t}{\sqrt{L_4}}\right)\right] + \frac{3}{(-24\gamma+8)}\ln\left[(-t^2+At+B)^{M_6/2}(-t^2+At+k_2)^{M_8/2}\right], \tag{39}$$

where $L_4 = A^2 + 4K_2$ is a constant, K_2 is a constant given by

$$K_2 = \left[\frac{(4B + 3\gamma A^2) + 2[2\omega(\gamma-1)-3]L_3}{-12\gamma+4}\right], \tag{40}$$

and M_5, M_6, M_7 and M_8 are constants given by

$$M_5 = \frac{12AB + 3A^3 + 8\omega L_2 A}{K_2 - B}\,, \quad M_6 = \frac{-24B - 6A^2 - 16\omega L_2}{K_2 - B}\,,$$

$$M_7 = \frac{-12AK_2 - 3A^3 - 8\omega L_2 A}{K_2 - B}\,, \quad M_8 = \frac{24K_2 + 6A^2 + 16\omega L_2}{K_2 - B}\,.$$

Again conditions $L_3 > 0$ and $L_4 > 0$ ensure that the solutions are real. The model expands from $R = 0$ at time $t_0 = (A - \sqrt{L_3})/2$, reaches a maximum $R_{\max} = \sqrt{L_3}/4$ at $t_{\max} = A/2$ and finally contracts to $R = 0$ at $t_f = (A + \sqrt{L_3})/2$.

4. Conclusions

In this paper we have discussed Brans–Dicke cosmological models with a bulk viscous fluid source described by full causal thermodynamics. We have found a new class of exact solutions for all spatial sections ($k = 0, \pm 1$). For the case $k = 0$ our solutions are found to be consistent with the available results of Banerjee and Beesham (1996). For $k = -1$, the model first contracts to a minimum and then expands forever, whereas the model first expands to a maximum and then contracts for $k = +1$. The temporal behaviour of the absolute temperature T is also determined in all three cases.

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