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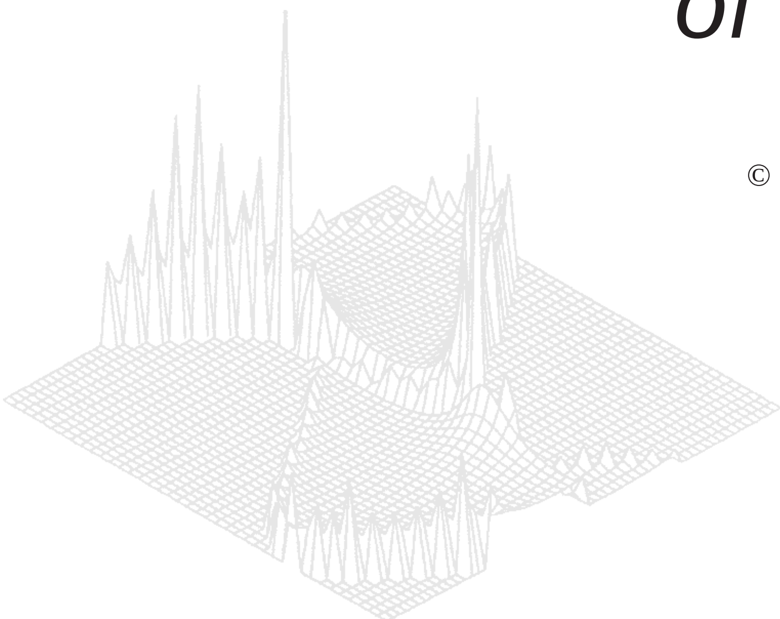

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Massive Gravitons in General Relativity

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Abstract

In the framework of a generalisation of linear gravitation to the case when the gravitons have nonzero rest mass, we obtain a result analogous to that obtained by Regge and Wheeler, that is, the energy of the gravitational waves is trapped in the 'material' (interior) metric of the curved space–time. We show that the concept of a nonzero rest mass graviton may be defined in two ways: (i) phenomenologically, by introducing of a mass term in the linear Lagrangian density, as in Proca electrodynamics, and (ii) self-consistently, by solving Einstein's equations in the conformally flat case. We find that the rest mass of the graviton may be given in terms of the three fundamental constants (gravitational, Planck, and light velocity constants) and it is a function of the density of cosmic matter.

1. Introduction

Some years ago, Droz-Vincent (1959, 1966) proposed a slight additional term in Einstein's equations in order to take into account a possible nonzero mass of the graviton. In the present work we follow an inverse simple way. We start from the unique generalisation of the linear gravitational equations that can admit nonzero mass gravitons, and only *a posteriori* we look for a solution of the Einstein equations.

Thus, a straightforward generalisation of linear gravitoelectromagnetics is parametrised by a constant $\Lambda_g = (\hbar/m_g c)$ which can be interpreted as a measure of the Compton wavelength of a massive vectorial spin-one graviton (*gravinon*) associated to some components of the gravitational field in linear gravitation. The interaction potential is of the Yukawa type and finite range, with a range $\sim\Lambda_g$. The conclusion is that the conventional linear massless gravitoelectromagnetics would be inapplicable over distances exceeding Λ_g.

We first consider space as flat, and we introduce the linear gravitational field that interacts with matter or other fields in an appropriate way, so that the analogy with the electromagnetic field is maintained as complete as possible. This 'field picture' of gravitation seems worth investigating, because it might lead to the idea of new gravitational experiments based on well-known electromagnetic phenomena.

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Then, by considering the coupling of the gravitational waves with a curved space–time, we find that the energy of the gravitational waves is trapped in the metric of the curved space–time in the presence of matter. This result is similar to that obtained by Regge and Wheeler (1957) who, investigating the behaviour of a Schwarzschild singularity subjected to a small nonspherical perturbation, found that ‘geons derive all their mass and energy from gravitational waves trapped in the metric’.

Finally, a screened gravitational potential of the Proca–Yukawa type is directly obtained from Einstein’s equations, as a conformally flat solution.

2. Maxwell-type Equations of the Gravitational Field

When four-dimensional space–time is split into space plus time, the electromagnetic field tensor $F^{\alpha\beta}$ breaks into two parts, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$. Analogously, the general relativistic gravitational potential field, the space–time metric tensor $g^{\alpha\beta}$ breaks into three parts: an electric-like part g^{00}, whose gradient is the Newtonian acceleration $\mathbf{g}$ or gravitoelectric field $\mathbf{E}_g$, a magnetic-like part g^{0i}, whose curl is the gravitomagnetic field $\mathbf{B}_g$, and a spatial metric g^{ij}, whose curvature tensor is the curvature of space.

Although there are theoretical arguments and experimental evidence for the conventional view that the gravitational acceleration of antimatter must be as that of matter (Goldman *et al*. 1986; Ericson and Richter 1990), the concept of a gravitational field (or at least some components of the gravitational field) analogous to the electromagnetic field is supported, for example, by the following reasons (Ciubotariu 1991; Ciubotariu *et al*. 1993):

(A) *For the case of a weak field (linear approximation):*

(i) Newton’s law of gravity is analogous to Coulomb’s law.

(ii) The linearised Einstein equations have the same form as the Maxwell equations.

(iii) The geodesic equations have the same form as the Lorentz equation of motion.

(iv) Shapiro *et al*. (1988) claimed to have confirmed the existence of gravitomagnetic geodetic precession of the orbit of the moon to within $2\cdot 0\%$ using laser-ranging data [see (vii) below].

(B) *For the case of a strong field (nonlinear exact solutions):*

(i) There exists a class of solutions of the linearised field equations which is also a solution of the exact nonlinear field equations (Gürses and Gürsey 1975; Baekler and Gürses 1987).

(ii) When two gravitational waves collide they tend to focus each other as is the case in the nonlinear optics of strong laser radiations.

(iii) The gravitational field of axially symmetric and reflection-symmetric systems, in the near-field approximation, has a structure very similar to the electric-type solutions of electromagnetic theory. In particular, the first effect of a variable gravitational field is to induce a magnetic-like field.

(iv) There exists a gravitational analogue of the electromagnetic Faraday rotation.

(v) Some theories of quasars and galactic nuclei rely on the gravitomagnetic field of a supermassive black hole for energy storage, power generation, jet formation, and jet alignement (Thorne 1983).

(vi) The axes of inertial frames, placed near a massive spinning source, precess with respect to points far away from the source ('Lense–Thirring effect', see Mashoon *et al*. 1984).

(vii) The spin axis of a gyroscope, freely falling in the gravitational field of a spinning body, is dragged due to the angular momentum of the central source, since the central body is thought of as dragging the metric with it to some extent (Schiff 1960). This effect has also been interpreted as a spin–spin interaction in analogy to the phenomenon in atomic physics.

So far nobody has observed gravitational waves and thus these waves, as a linearised effect, are important in the same measure in which the Maxwell equations are important for linear optics. Should we detect any first-order gravitational waves, interest in the nonlinear characteristics of propagation and the exact solutions will certainly grow.

A thorough investigation on the possible linear theories of a tensor field by Weyl (1944) showed that the linear theory is incapable of explaining the correct advance of the perihelion of Mercury. But, considerable support was provided for the flat space interpretation when Papapetrou (1948) discovered that the exact equations of general relativity can be brought into a form in which the linear part is supplemented by the energy–momentum tensor of the gravitational field as a source in the field equations. Adopting the De Donder condition, Papapetrou succeeded in putting Einstein's equations in the simpler form

$$\frac{\partial^2 \gamma^{\alpha\beta}}{\partial x^\mu \partial x_\mu} \equiv \Box \gamma^{\alpha\beta} = \kappa^2 \Theta^{\alpha\beta}, \qquad \gamma^{\alpha\beta} = \sqrt{-\det(g_{\mu\nu})}\gamma^{\alpha\beta}\,, \tag{1}$$

where $\Theta^{\alpha\beta}$ is the energy–momentum tensor of the whole system including the gravitational field itself. Taking the field equations for a free gravitational field as

$$\frac{\partial^2 \gamma^{\alpha\beta}}{\partial x^\mu \partial x_\mu} = 0\,, \tag{2}$$

we can regard the entire interaction as caused by $\Theta^{\alpha\beta}$.

Peng (1983, 1990) discussed a set of Maxwell-like linear equations that arise in the slow-motion $v/c \ll 1$ weak-field limit of Einstein's field equations. In this case, the equations of general relativity can be written in terms of separate space–time coordinates, and one can introduce the gravitoelectric ($\mathbf{E}_g$) and gravitomagnetic ($\mathbf{B}_g$) fields. The set of governing equations may be called gravitoelectromagnetics and they are (with $G = c = 1$)

$$\nabla \times \mathbf{B}_g = -4\pi \mathbf{j}_m + \frac{\partial \mathbf{E}_g}{\partial t}\,, \tag{3}$$

$$\nabla \times \mathbf{E}_g = -\frac{\partial \mathbf{B}_g}{\partial t}\,, \tag{4}$$

$$\nabla \cdot \mathbf{B}_g = -4\pi\rho_m\,, \tag{5}$$

$$\nabla \cdot \mathbf{B}_g = 0, \qquad \mathbf{B}_g = \nabla \times \mathbf{A}_g\,, \tag{6}$$

where $\mathbf{j} = \rho_m \mathbf{v}$ is the mass current density, $\rho_m = nm$ is the proper mass density, $\mathbf{A}_g = A^i_g \equiv \left(\frac{1}{4}\bar{h}^{01}, \frac{1}{4}\bar{h}^{02}, \frac{1}{4}\bar{h}^{03}\right)$ (see Ciubotariu 1991 for notation) is the gravitomagnetic vector potential, and

$$E^i_g = G^{00i} \equiv \tfrac{1}{4}\left(\bar{h}^{00,i} - \bar{h}^{0i,0}\right) = \tfrac{1}{4}\left(-\bar{h}^{00}_{,i} - \bar{h}^{0i}_{,0}\right). \tag{7}$$

If we choose $\frac{1}{4}\bar{h}^{00}_{,i} = \phi_g$ so that

$$\mathbf{E}_g = -\nabla\phi_g - \frac{\partial A_g}{\partial t}\,, \tag{8}$$

then the gravitational 4-vector potential can be introduced by

$$A^\alpha_g \equiv (\phi_g, \mathbf{A}_g)\,, \tag{9}$$

$$F^{\alpha\beta}_g = \frac{\partial A^\beta_g}{\partial x_\alpha} - \frac{\partial A^\alpha_g}{\partial x_\beta}, \quad F_{g\alpha\beta} = \frac{\partial A_{g\beta}}{\partial x^\alpha} - \frac{\partial A_{g\alpha}}{\partial x^\beta} \equiv A_{g\beta,\alpha} - \partial A_{g\alpha,\beta} \tag{10}$$

with $F^{\alpha\beta}_g$ describing the gravitoelectromagnetic field tensor.

For instance, the gravitomagnetic field induced by the rotation of the Earth, at a geographical latitude of 45°, is given by [see e.g. Ljubicic and Logan 1992, eq. (1), where $G \neq c \neq 1$]

$$|\mathbf{B}_g| = \left|\frac{GI}{2cR^3}\left[\boldsymbol{\omega} - \frac{3(\boldsymbol{\omega}\cdot\mathbf{R})\mathbf{R}}{R^2}\right]\right| \simeq 20{\cdot}4 \times 10^{-6}\ \mathrm{m\,s^{-2}}$$

$$\simeq 10^{-14}\ \mathrm{s^{-1}} \quad (G = c = 1 \text{ units})\,, \tag{11}$$

where G is the gravitational constant, R is the radius of the Earth, I is the moment of inertia of the Earth about an axis through its centre, and ω is the angular velocity associated with the Earth's rotation.

Quantitatively, the linearising approximation is extremely well justified on the surface of any plane in our solar system. Even on the surface of a white dwarf, the deviation of g_{00} from unity does not exceed a value of the order of 10^{-5}. Therefore, we can postulate, *ab initio*, the Maxwell-type gravitational equations for *linear gravitation*.

3. Gravitoelectromagnetics in a Curved Space–Time

We consider a space–time manifold described by the 'material' (interior) metric tensor $g_{\alpha\beta}$. In addition, we allow a gravitational wave ('free' gravitational field) characterised by the gravitational 4−vector potential ${A_g}^{\alpha} \equiv (\phi_g,\, \mathbf{A}_g)$ to travel through and, also, interact with that tensor field. If we assume that the gravitational wave does not disturb the metric, the Lagrangian density for this system can be written as

$$\begin{aligned} L = & -\frac{1}{16\pi}\sqrt{-g}{F_g}^{\alpha\beta}F_{g\alpha\beta} + \frac{1}{k}\sqrt{-g}g^{\alpha\beta}R_{\alpha\beta} \\ & + \chi\frac{1}{k}\sqrt{-g}g^{\alpha\beta}R_{\alpha\beta}{A_g}^{\mu}A_{g\mu}\,, \end{aligned} \tag{12}$$

where the first term refers to the free gravitoelectromagnetic field, described by the gravitoelectromagnetic field tensor ${F_g}^{\alpha\beta}$, the second term is the Lagrangian of the curved space–time with $R_{\alpha\beta}$ being the Ricci tensor, and the last term specifies the coupling, with χ being the proportionality constant. If the metric tensor $g_{\alpha\beta}$ is varied in the Lagrangian density, we obtain the equations of motion (field equations) in the form

$$\begin{aligned} &\left(1 + \chi{A_g}^{\mu}A_{g\mu}\right)G_{\alpha\beta} + \chi\left[RA_{g\alpha}A_{g\beta} - g_{\alpha\beta}\Box\left({A_g}^{\mu}A_{g\mu}\right) + \left({A_g}^{\mu}A_{g\mu}\right)_{;\alpha;\beta}\right] \\ &\qquad = -kT_{g\alpha\beta} \equiv -\frac{8\pi G}{c^4}T_{g\alpha\beta}\,, \end{aligned} \tag{13}$$

where $G_{\alpha\beta}$ is the Einstein tensor,

$$T_{g\alpha\beta} = \frac{1}{16\pi}{F_g}^{\mu\nu}F_{g\mu\nu}\eta_{\alpha\beta} - F_{g\mu\alpha}F_{g\nu\beta}\eta^{\mu\nu} \tag{14}$$

is the stress-energy tensor of the gravitoelectromagnetic field, and the semicolon denotes the covariant derivative. Taking the trace of this equation we find

$$R = -3\chi\Box\left({A_g}^{\mu}A_{g\mu}\right)\,. \tag{15}$$

On the other hand, performing the variation of the Lagrangian (12) with respect to the gravitational 4−vector potential ${A_g}^{\alpha}$ we obtain the equations of gravitomagnetics in the curved space–time described by the metric tensor $g_{\alpha\beta}$:

$${F_g}^{\alpha\beta}_{\ ;\beta} + \frac{\chi}{k}R{A_g}^{\alpha} = 0 \tag{16}$$

or

$${F_g}^{\alpha\mu}_{\ ;\mu} - \frac{3}{k}\chi^2{A_g}^{\alpha}\Box\left({A_g}^{\mu}A_{g\mu}\right) = 0\,, \tag{17}$$

and the resulting nonlinearity due to the coupling of the gravitational radiation with curvature of space–time is clearly exhibited. In other words, we split the gravitational field into linear and nonlinear parts. We treat the linear part as a free gravitational field, while the nonlinear part is regarded as the direct interaction between the gravitons and matter. Thus, interacting gravitons become massive gravitons. We note that whilst the gauge invariance holds for Minkowskian (flat) gravitoelectromagnetics (equations 3–6) in the case of the coupling with the curvature tensor (equation 16) this invariance is excluded.

4. Massive Gravitoelectromagnetics

Discussions about gravitoelectric and gravitomagnetic fields on astrophysical or atomic distance scales involve the tacit assumption that it is valid to extrapolate Maxwell-like equations for gravitoelectromagnetics to arbitrarily large or small distance scales. Of course, it is important to investigate whether other versions of non-Maxwellian gravitoelectromagnetics have implications for gravitational phenomena. In this work, a novel approximation in the linear gravitation based on gravitoelectromagnetics with massive vector is presented.

We define a model of the universe (a gravitating system) as a collection of dust particles and nonzero rest mass gravitons which exhibits collective mode behaviour. The required field equations are of the Proca type (see e.g. Byrne 1977, and references therein) which we adapt to our gravitational case with $c \neq 1$:

$$\nabla \times \mathbf{B}_g = -\frac{4\pi}{c}\mathbf{j}_m + \frac{1}{c}\frac{\partial \mathbf{E}_g}{\partial t} - \frac{1}{\Lambda_g^2}\mathbf{A}_g\,, \tag{18}$$

$$\nabla \times \mathbf{E}_g = -\frac{1}{c}\frac{\partial \mathbf{B}_g}{\partial t}\,, \tag{19}$$

$$\nabla \cdot \mathbf{E}_g = -4\pi\rho_m - \frac{1}{\Lambda_g^2}\phi_g\,, \tag{20}$$

$$\nabla \cdot \mathbf{B}_g = 0\,, \tag{21}$$

where $\mathbf{E}_g$ and $\mathbf{B}_g$ are related to ϕ_g and $\mathbf{A}_g$ just as in Maxwell-type equations of the gravitational field, and

$$\Lambda_g = \frac{\hbar}{m_g c} \tag{22}$$

is the reduced Compton wavelength of the graviton. In contrast to the Maxwell-type theory of linear gravitation, the potentials ϕ_g and $\mathbf{A}_g$ are directly measurable quantities so that gauge invariance is not possible, and the Lorentz gauge condition

$$\nabla \cdot \mathbf{A}_g + \frac{1}{c}\frac{\partial \phi_g}{\partial t} = 0 \tag{23}$$

is required in order to conserve mass. Since $m_g \neq 0$ is not consistent with gauge invariance, the Proca generalisation of gravitoelectromagnetics could be aesthetically defective in the eyes of many theoretical physicists. However, the only certain statements about the value of m_g that can be made must be based on experiment. We note that in the full nonlinear theory (Deser 1971) the presence of massive gravitons breaks coordinate invariance just as it breaks gauge invariance in our massive linear gravitoelectromagnetics.

Another distinctive contrast between Maxwell's gravitoelectromagnetics and that of Proca is that the latter admits nonzero mass and current densities even in the absence of gravitational fields; for example

$$\rho_m = -\frac{1}{4\pi\Lambda_g^2}\phi_g \neq 0 \quad \text{and} \quad \mathbf{j}_m = -\frac{c}{4\pi\Lambda_g^2}\mathbf{A}_g \neq 0,$$

with $\mathbf{E}_g = -\nabla\phi_g - \partial\mathbf{A}_g/\partial t = 0$ and $\mathbf{B}_g = \nabla \times \mathbf{A}_g = 0$ are solutions of equations (18)–(21). Hence, there are large regions of the universe containing currents or mass densities that contain no gravitational fields. Furthermore, the perfect cosmological principle that the universe should be homogeneous and locally isotropic throughout the course of time becomes now a natural property of the massive gravitoelectromagnetics.

As the Coulomb force is due to the exchange of a virtual quantum, or photon, in our theory the gravitational force between particles is likewise due to a massive vectorial graviton, a quasiparticle which we call *gravinon*, of effective mass m_g and maximum speed c. Massive gravitoelectromagnetics becomes conventional Maxwell's gravitoelectromagnetics in the limit $m_g \to 0$. For $m_g \neq 0$ equations (18)–(21) predict fields which are different from those predicted by the Maxwell gravitoelectromagnetics. For example, in the case of a static point mass placed at the origin, the gravitostatic potential is given by a Yukawa–Proca potential

$$A_g^0 \equiv \phi_g = \frac{\text{const}}{r}\exp\left(-\frac{1}{\Lambda_g}r\right) \tag{24}$$

in spherical coordinates. Equation (24) implies a finite range (Λ_g) type interaction for this type of gravitoelectromagnetics, and masses interact only with other masses within this range. Hence, if the homogeneous mass is spread over a large distance $D \gg \Lambda_g$, isotropy of masses guarantees that the gravitational field, and thus the gravitostatic force, vanishes everywhere.

We note that by solving the Schrödinger equation with the screened potential (24) (Hook and Hall 1991), we find that the bound states exist only if the screening length Λ_g is greater than the 'gravitational Bohr radius' a_{g0}. If we consider that the gravinon mass is equal to the best available upper limit on the photon rest mass ($m_{\text{photon}} = 3 \times 10^{-53}$ g, Byrne 1977), we find that Λ_g is of the order of the diameter of Pluto's orbit ($\Lambda_g \approx 10^{15}$ cm). This result is in agreement with the idea of possible evidence for gravitational Bohr orbits as a cosmic version of ordinary quantum mechanics (DerSarkissian 1985).

The wave corresponding to the massive spin-one field is dispersive in a vacuum, that is

$$\omega^2 = \left(k^2 + \frac{1}{\Lambda_g^2}\right) c^2 \,. \tag{25}$$

Then, from its definition, the group speed is

$$v_{\text{group}} = c\left(1 - \frac{c^2}{\Lambda_g^2 \omega^2}\right)^{\frac{1}{2}} \tag{26}$$

and thus, the velocity of energy propagation is frequency dependent.

At this point we ask: How could we solve Einstein's field equations so that their solution in the weak-field approximation coincides with the solution of equations (18)–(21) of massive Proca gravitoelectromagnetics where the massive gravitons are introduced as a supplimentary hypothesis (see Section 6).

5. Absorption of Gravitational Waves

Since the graviton is assumed to have a non-vanishing mass, a mass term has to be added to the Lagrangian density describing the massless graviton. Thus, the Lagrangian density in the absence of matter becomes

$$L = -\frac{1}{16\pi} F_g{}^{\alpha\beta} F_{g\alpha\beta} + \frac{1}{8\pi\Lambda_g^2} A_g{}^{\alpha} A_{g\alpha} \,. \tag{27}$$

The field equations then follow from the Euler–Lagrange equations of motion

$$F_g{}^{\alpha\beta}{}_{,\beta} + \frac{1}{\Lambda_g^2} A_g^{\alpha} = 0 \,. \tag{28}$$

It should be noted that these are the only possible linear generalisations of linear gravitation. As already indicated, as a result of the inclusion of the 'coupling constant' $(1/\Lambda_g)$, the potentials ϕ_g, $\mathbf{A}_g$ now acquire real physical characteristic, i.e. they now become observable. Another conclusion is that there would be three states of polarisation for massive gravitons. Apart from two transverse polarisations, there also exists a longitudinal one, and the faster the longitudinal graviton moves, the weaker is the associated gravitoelectric field $\mathbf{E}_g$. As was argued above, the gravitoelectric field diminishes exponentially with distance and the corresponding flux lines fade away, even in vacuo. Furthermore, the gravitomagnetic field $\mathbf{B}_g$ is also affected and the associated lines are compressed around the equator (see Novak 1989).

If we compare equation (16) of gravitoelectromagnetics in curved space–time to equation (28) of massive gravitoelectromagnetics defined in flat space–time, we see that the similarity between these is rather striking. Thus, it is reasonable to write the relation

$$m_g^2 \frac{c^2}{\hbar^2} \equiv \frac{1}{\Lambda_g^2} = \frac{\chi}{k} R \, . \tag{29}$$

It follows that the mass of a gravinon vanishes if $R = 0$, that is, if the curvature constant (scalar) of the four-dimensional curved space–time is zero. Hence, in a non-empty curved space–time, but not only with pure electromagnetic field (when $T_\alpha{}^\alpha = R = 0$), a graviton acquires a mass as a result of its interaction with surroundings. In other words, the mass of the graviton is directly related to the curvature scalar of space–time. As the range of the massive graviton is finite (see Section 4), we obtain a result analogous to that obtained by Regge and Wheeler (1957), that is, the energy of the gravitational waves is trapped in the metric of curved space–time.

6. Massive Gravitons as a Direct Consequence of Einstein Equations

One of the authors (Ciubotariu 1991) has shown that the gravitational waves may be absorbed by a background cosmic fluid (a 'false vacuum') with negative pressure, and described by an energy–momentum density tensor of the form

$$T_{\alpha\beta} = \left(\rho + \frac{p}{c^2}\right) \delta_\alpha^0 \delta_\beta^0 - \frac{p}{c^2} g_{\alpha\beta} \equiv -\frac{p}{c^2} g_{\alpha\beta}, \qquad p = -\rho c^2 \, . \tag{30}$$

Now, we want to solve the Einstein field equations

$$R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R = -\frac{8\pi G}{c^4} T_{\alpha\beta} \equiv -\frac{8\pi G}{c^4} \rho g_{\alpha\beta} \tag{31}$$

in the weak-field approximation. Since we intend to obtain a solution 'with mass' of the Proca–Yukawa type, and the conformal relativity is a theory of mass (see, for example, Ingraham 1978*a*, 1978*b*, 1978*c*, 1978*d*, 1978*e*), we consider a space–time which is conformally flat and has the metric tensor

$$g_{\alpha\beta} = e^{\psi} \eta_{\alpha\beta} \, , \tag{32}$$

where ψ is some function of the coordinates. By conformally flat we mean that the Weyl conformal curvature tensor vanishes, i.e.

$$C_{\alpha\beta\gamma\delta} \equiv R_{\alpha\beta\gamma\delta} + 2 g_{[\alpha[\gamma} \left(R_{\delta]\beta]} - \tfrac{1}{6} g_{\delta]\beta]} R \right) = 0 \, , \tag{33}$$

where the brackets enclosing a pair of indices act as antisymmetrisers. Here $R_{\alpha\beta}$ is the Ricci tensor, computed to be

$$R_{\alpha\beta} = \frac{\partial^2\psi}{\partial x^\alpha \partial x^\beta} - \tfrac{1}{2}\frac{\partial\psi}{\partial x^\alpha}\frac{\partial\psi}{\partial x^\beta} + \tfrac{1}{2}\eta_{\alpha\beta}\eta^{\mu\nu}\left(\frac{\partial^2\psi}{\partial x^\mu \partial x^\nu} + \frac{\partial\psi}{\partial x^\mu}\frac{\partial\psi}{\partial x^\nu}\right)$$

$$\simeq \frac{\partial^2\psi}{\partial x^\alpha \partial x^\beta} + \tfrac{1}{2}\eta_{\alpha\beta}\eta^{\mu\nu}\frac{\partial^2\psi}{\partial x^\mu \partial x^\nu}\,, \tag{34}$$

where

$$R_\alpha{}^\beta = \frac{\partial^2\psi}{\partial x^\alpha \partial x_\beta} + \tfrac{1}{2}\delta_\alpha^\beta\frac{\partial^2\psi}{\partial x^\mu \partial x_\mu}, \qquad R = 3\Box\psi, \qquad \Box\psi \equiv \frac{\partial^2\psi}{\partial x^\mu \partial x_\mu}\,. \tag{35}$$

Now, Einstein's equations (31) become

$$\frac{\partial^2\psi}{\partial x^\alpha \partial x^\beta} + \tfrac{1}{2}\eta_{\alpha\beta}\eta^{\mu\nu}\frac{\partial^2\psi}{\partial x^\mu \partial x^\nu} - \tfrac{3}{2}g_{\alpha\beta}\Box\psi = -\frac{8\pi G}{c^4}\rho g_{\alpha\beta} \tag{36}$$

or

$$\frac{\partial^2\psi}{\partial x^\alpha \partial x_\beta} + \tfrac{1}{2}\delta_\beta^\alpha\frac{\partial^2\psi}{\partial x^\mu \partial x_\mu} - \tfrac{3}{2}\delta_\beta^\alpha\Box\psi = -\frac{8\pi G}{c^4}\rho(1+2\psi)\delta_\beta^\alpha\,. \tag{37}$$

It follows, therefore, that

$$\Box\psi - \frac{64\pi G}{3c^4}\rho\psi = \frac{32\pi G}{3c^4}\rho\,, \tag{38}$$

that is, in the static case

$$\nabla^2\psi - \frac{64\pi G}{3c^4}\rho\psi = \frac{32\pi G}{3c^4}\rho\,. \tag{39}$$

If we compare this equation with the Maxwell's equation (20) we may write

$$\nabla^2\psi - \frac{1}{\Lambda_g^2}\psi = \frac{1}{2\Lambda_g^2}, \qquad \frac{1}{\Lambda_g^2} \equiv m_g^2\frac{c^2}{\hbar^2} = \frac{64\pi G}{3c^4}\rho\,, \tag{40}$$

and thus we have obtained the Proca–Yukawa scalar field equation.

We notice immediately that the graviton can have a non-zero rest mass ($m_g \neq 0$) only as a result of the interaction with matter ($\rho \neq 0$). We have for the first time an expression for the rest mass of the graviton in terms of the gravitational constant G and the mean density of matter in the universe:

$$m_g = \frac{8\hbar}{c^3}\left(\frac{\pi G}{3}\rho\right)^{\frac{1}{2}}\,. \tag{41}$$

Since the curvature scalar $R = g^{\alpha\beta} R_{\alpha\beta}$ is expressible in terms of the density ρ [see equations (30) and (31)],

$$R = \frac{32\pi G}{c^4} \rho \,, \tag{42}$$

we readily obtain

$$m_g^2 \frac{c^2}{\hbar^2} = \tfrac{2}{3} R \,, \tag{43}$$

which is in complete agreement with the relation (29) obtained by another approach. This proves that our theory is consistent.

7. Concluding Remarks and Comment

The main point of this work has been to show how the inclusion of the Compton wavelength of the graviton in linear gravitoelectromagnetics leads to nonlinear gravitons, whose mass is directly related to the curvature scalar which means the presence of matter. A gravitational wave propagating through a medium is influenced by the latter and this effect is manifested via the generation of massive gravitons. In other words, the medium (matter) generates the scalar of curvature which is proportional to the square of the mass of graviton. Thus, we have a geometric interpretation of the mass of a graviton.

On the other hand, we deal with two metric tensors: a flat-space *a priori* given metric corresponding to linear gravitational waves, and the ordinary metric tensor corresponding to an interior solution (with $R \neq 0$) of Einstein equations. The physical situation which describes the interaction between the two metrics is the following: Freely running gravitational waves generate massive gravitons by the interaction with an interior gravitational field of the matter that they encounter.

In the final part of the work we obtained a conformally flat solution of Einstein's equations which, in a linear approximation, describes a Proca–Yukawa potential.

Of course, the rest mass of the graviton is very small, but study of the interaction of gravitons with cosmic matter and other gravitational fields in the first approximation is motivated in part by the fact that the graviton reaches us from the most distant galaxies after a time $\sim 10^{17}$ s. During this long interval, even a very weak interaction might lead to observable effects over such an extremely long time.

Quantisation of the exact general theory relativity may lead to some important consequences, but this is a very difficult program. While it is being carried out, in this work we have attempted to extract new physics out of an approximate treatment using existing mathematical procedures.

As we have seen above, in the linear approximation of Einstein's gravitational field equations, the metric tensor satisfies the same type of equations as those of electromagnetic potentials. Accordingly, in the quantisation procedure, there will be many analogies between the gravitational and electromagnetic fields. In the classical theory of the electromagnetic field, the field quantities are composed of two parts: the transverse part for which energy and momentum form a four-vector and behave like the energy and momentum of particles as regards its transformation properties, and the part for which this is not the case. In the quantum electromagnetic field theory, only the first part is subjected to a quantisation, giving rise to photons which behave like particles, while the second remains unquantised. The latter part composed of longitudinal and scalar parts, in the presence of matter, gives rise to the Coulomb interaction between charged particles both in the classical and in the quantum electromagnetic theories. Similarly, in the gravitational field it is reasonable to quantise in such a way that the unquantised part gives the gravitational potential corresponding to the Newtonian potential or to the Proca–Yukawa potential in the case of massive gravitons. We note that the unquantised part has the same form for both macroscopic and microscopic phenomena.

Further developments of this work will refer to the use of the formalism for the gauge theories. It may be shown how a graviton can acquire a nonvanishing mass using the effect of spontaneous symmetry breaking in analogy to the Higgs model, and the mass generation of gravitons as a result of dynamical symmetry breaking. In addition, the introduction of a gravitational superconductivity concept where the graviton acquires a mass, thus manifesting a Meissner gravitational effect (Ciubotariu 1996), may also be considered.

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