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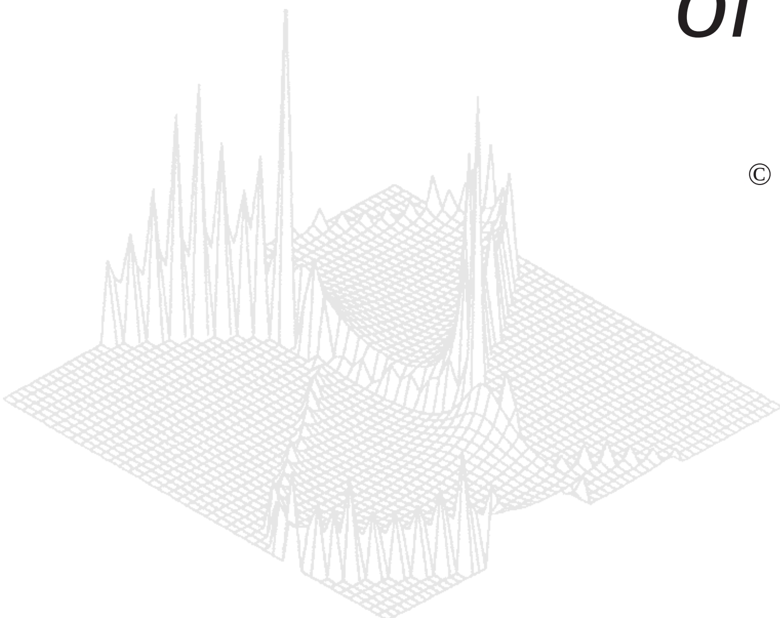

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Wave Equations of Cooper Pairs as Quasiparticles in Superconducting Fermion Systems*

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Abstract

The superconducting pairing of fermions is studied in the framework of the functional intergral approach. The bi-local composite scalar and vector fields are introduced to describe the singlet and triplet pairings. The static (time-independent) fields are the superconducting order parameters. From the variational principle for the effective action of the composite fields we derive the generalised Ginzburg–Landau equations. They are also the extensions of the BCS gap equations.

1. Introduction

Before the foundation of the microscopic theory of superconductivity by Bardeen, Cooper and Schrieffer (1957) a phenomenological theory of the superconducting many-fermion systems based on the Landau (1937) theory of phase transitions had been proposed by Ginzburg and Landau (1950). In the Ginzburg–Landau (GL) theory the physical quantity characterising the superconductivity of the system is some superconducting order parameter $\Delta(\mathbf{R})$. With their physical intuition Ginzburg and Landau considered $\Delta(\mathbf{R})$ as some single quasiparticle quantum-mechanical wave function and derived its equation of motion—the GL equation. In the presence of a static magnetic field with the vector potential $\mathbf{A}(\mathbf{R})$ it is the nonlinear Schrödinger equation

$$\frac{1}{2m^*}\big(-i\nabla - e^*\mathbf{A}\big)^2\Delta(\mathbf{R}) + \alpha\Delta(\mathbf{R}) + \beta\ |\Delta(\mathbf{R})|^2\Delta(\mathbf{R}) = 0\,, \tag{1}$$

with some effective mass m^* and effective charge e^*.

In the BCS microscopic theory the physical origin of superconductivity in the many-electron system is the electron pairing—the formation of Cooper (1956) pairs due to the attractive interaction of electrons in solids caused by phonon exchange (Fröhlich 1950). The superconducting order parameter $\Delta(\mathbf{R})$ is the expectation value of the product of two destruction operators for the electrons with the opposite spin projections $\psi_\uparrow(\mathbf{R})$ and $\psi_\downarrow(\mathbf{R})$:

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$$\Delta(\mathbf{R}) \ = \langle\psi_\uparrow(\mathbf{R})\psi_\downarrow(\mathbf{R})\rangle \,. \tag{2}$$

Here $\langle ...\rangle$ denotes the mean value in the equilibrium grand canonical ensemble at the given temperature T:

$$\langle ...\rangle = \frac{Tr[...e^{-\beta(H-\mu N)}\ ]}{Tr[e^{-\beta(H-\mu N)}\ ]}\,, \tag{3}$$

where H is the Hamiltonian of the system, N is the total particle number operator, μ is the chemical potential,

$$\beta \ = \frac{1}{kT}\,,$$

and k is the Boltzmann constant. By means of the Green function technique, starting from the quantum mechanical equation of motion for the electron destruction and creation operators $\psi_\alpha(\mathbf{R})$ and $\psi_\alpha^+(\mathbf{R})$, $\alpha \ = \ \uparrow$ and $\downarrow$, in the microscopic BCS theory, Gorkov (1959) has derived the GL equation (1) for the order parameter (2) and shown that

$$e^* = \ 2e\,,$$

where e is the electron charge. The GL equation (1) is indeed the quantum-mechanical equation for the wave function of a Cooper pair of two electrons.

The relation (2) is the definition of the superconducting order parameter in the special case when two coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ of the electrons coincide

$$\mathbf{r}_1 \ = \ \mathbf{r}_2 \ = \ \mathbf{R}\,,$$

and their spin projections α and β are opposite $\alpha = \ \uparrow,\ \beta = \ \downarrow$. In general the electron pairing is characterised by the following superconducting order parameters:

$$\Delta_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2) \ = \ \big\langle\psi_\alpha(\mathbf{r}_1)\psi_\beta(\mathbf{r}_2)\big\rangle\,. \tag{4}$$

As the elements of a 2×2 matrix they can be written in the form

$$\Delta_{\alpha\beta}(\mathbf{r}_1,\ \mathbf{r}_2) = (\sigma_2)_{\alpha\beta}\Delta(\mathbf{r}_1,\mathbf{r}_2) + (\sigma_2\boldsymbol{\sigma})_{\alpha\beta}\boldsymbol{\Delta}(\mathbf{r}_1,\mathbf{r}_2)\,, \tag{5}$$

where $\boldsymbol{\sigma}$ is a vector with the components $\sigma_i, i = 1,\ 2,\ 3$, being the Pauli matrices. The scalar and vector fuctions $\Delta(\mathbf{r}_1,\mathbf{r}_2)$ and $\boldsymbol{\Delta}(\mathbf{r}_1,\mathbf{r}_2)$ are the superconducting order parameters for electron pairing in the singlet and triplet states respectively. In general they are functions of the c.m. coordinate and the relative one of the electrons in a Cooper pair

$$\Delta(\mathbf{r}_1,\mathbf{r}_2) = \Delta(\mathbf{R};\ \boldsymbol{\rho}), \qquad \boldsymbol{\Delta}(\mathbf{r}_1,\mathbf{r}_2) = \boldsymbol{\Delta}(\mathbf{R};\ \boldsymbol{\rho})\,,$$

$$\mathbf{R} = \frac{\mathbf{r}_1+\mathbf{r}_2}{2},\ \ \boldsymbol{\rho} = \mathbf{r}_1 - \mathbf{r}_2\,.$$

The quantity (2) is the special case of the scalar function $-i\Delta(\mathbf{R};\ \boldsymbol{\rho})$:

$$\Delta(\mathbf{R}) = -i\Delta(\mathbf{R};\ 0)\,.$$

Consider the special case of the singlet pairing

$$\Delta_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2) = (\sigma_2)_{\alpha\beta}\ \Delta(\mathbf{r}_1,\ \mathbf{r}_2)$$

due to the attractive interaction between the electrons with the two-body potential $V(\mathbf{r}_1 - \mathbf{r}_2)$. If the external field is absent, then due to the translational invariance we can assume that the superconducting order parameter $\Delta(\mathbf{r}_1, \mathbf{r}_2)$ depends only on the relative coordinate of two electrons:

$$\Delta(\mathbf{r}_1, \mathbf{r}_2) = \Delta(\mathbf{r}_1 - \mathbf{r}_2) = \Delta(\boldsymbol{\rho})\,.$$

Its Fourier transform $\tilde{\Delta}(\mathbf{p})$,

$$\Delta(\boldsymbol{\rho}) = \frac{1}{(2\pi)^3}\int e^{i\mathbf{p}\boldsymbol{\rho}}\tilde{\Delta}(\mathbf{p})d^3p\,, \tag{6}$$

satisfies the nonlinear integral equation

$$\tilde{\Delta}(\mathbf{p}) = \frac{1}{(2\pi)^3}\int \tilde{V}(\mathbf{p}-\mathbf{q})\frac{1}{2E(\mathbf{q})}th\frac{\beta E(\mathbf{q})}{2}\tilde{\Delta}(\mathbf{q})d^3q\,, \tag{7}$$

$$E^2(\mathbf{q}) = E_0^2(\mathbf{q}) + |\tilde{\Delta}(\mathbf{q})|^2\,, \tag{8}$$

derived from the BCS theory in the mean field approximation (Schrieffer 1964). In the formulae (7) and (8) $\tilde{V}(\mathbf{p})$ is the Fourier transform of the potential $V(\boldsymbol{\rho})$, $E_0(\mathbf{q})$ is the difference of the kinetic energy of the electron with the momentum $\mathbf{q}$ and the chemical potential μ. For a free electron with mass m

$$E_0(\mathbf{q}) = \frac{\mathbf{q}^2}{2m} - \mu\,.$$

The relation (8) means that $|\tilde{\Delta}(\mathbf{q})|$ is the gap in the energy spectrum of electrons, and the formula (7) is called the BCS gap equation.

For the Fourier transforms of the $\mathbf{R}$-independent superconducting order parameters of the electron system with both singlet and triplet pairings there exists a system of two nonlinear integral equations which are the extensions of the BCS gap equation (7). These equations have been established not only for the system with the phonon exchange mechanism, but also in the models of high-T_c superconductivity with other microscopic mechanisms (Chen *et al.* 1994; Chernyshev *et al.* 1994; Das and Mahanty 1994; Feng *et al.* 1994; Hieu 1994*a*; Inoue *et al.* 1994; Kostyrko *et al.* 1994; Matsukawa and Fukuyama 1992; Rice and Wang 1993; Tanamoto *et al.* 1992, 1993; Unger and Fulde 1993; Zhang *et al.* 1994).

The Ginzburg–Landau equation (1) is an equation for the values of the superconducting order parameter $\Delta(\mathbf{R};\boldsymbol{\rho})$ at $\boldsymbol{\rho}=0$. By means of the Green function technique several authors (Joynt 1990; Hieu 1994*b*; Soininen *et al.* 1994; Xu *et al.* 1995) have derived the systems of extended Ginzburg–Landau equations for the Fourier transforms of the superconducting order parameters with respect to the relative coordinate $\boldsymbol{\rho}$. It is natural to pose two equations: Are there further extensions of the BCS gap equations for these $\mathbf{R}$-dependent functions $\tilde{\Delta}(\mathbf{R};\mathbf{p})$ and $\tilde{\boldsymbol{\Delta}}(\mathbf{R};\mathbf{p})$? What is the connection between them and the extended Ginzburg–Landau equations? The answers to these questions will be given later as a consequence of the establishment of equations of motion for the bi-local fields describing the Cooper pairs as some composite quasiparticles in the superconducting many-fermion systems.

In order to derive the dynamical equations for the superconducting order parameters we can apply different methods: the operator method based on the construction of the state vectors of the many-fermion system in terms of the fermion destruction and creation operators with or without the use of the Bogolubov transformation (Bardeen *et al.* 1957; Bogolubov 1958; Bogolubov *et al.* 1959; de Gennes 1966; Schrieffer 1964), the Green function method with or without the use of the dispersion relations (Abrikosov *et al.* 1963; Doniach and Sondheimer 1974; Eliashberg 1960; Gorkov 1959; Hieu 1994*a*, 1994*b*; Nambu 1960; Xu *et al.* 1995) and the functional integral technique (Alexandrov and Rubin 1993; Hieu 1994*c*; Noh *et al.* 1995; Sakita 1985; Stoof 1993; van Weert 1991; Weinberg 1994).

In this paper we present some results of the application of the functional integral method to the study of the superconducting many-fermion systems. By means of the Hubbard–Stratonovich transformation (Hubbard 1959; Stratonovich 1958) we introduce the bi-local quantum fields describing the Cooper pairs as some composite quasiparticles—the cooperons. We establish the effective action of the systems in terms of these bi-local quantum fields and derive their equations of motion. We shall see that to some approximation these field equations are extensions of the BCS gap equations as well as of the GL equations.

In Section 2 we start from the simple case of singlet pairing in the system with some effective two-body attractive potential $-V(\mathbf{r}_1-\mathbf{r}_2)$. In Section 3 we show that in this system there must exist also triplet pairing and derive the corresponding dynamical equations. The general case with the simultaneous presence of both singlet and triplet pairings is studied in Section 4. In Section 5 we consider the t-J model of high-T_c superconductivity and derive also systems of nonlinear integral equations for the singlet and triplet superconducting order parameters.

2. Singlet Pairing in the System with Two-body Attractive Potential $V(\mathbf{r}_1\text{-}\mathbf{r}_2)$

We work in the Matsubara imaginary time formalism and start from the total Hamiltonian with some two-body attractive potential $V(\mathbf{r}_1-\mathbf{r}_2)$:

$$H(\tau)=H_0(\tau)\;+\;H_{\rm int}(\tau)\,, \tag{9}$$

$$H_0(\tau)=\int\overline{\psi}_\alpha(\mathbf{r},\tau)\bigg(-\frac{\nabla^2}{2m}\bigg)\psi_\alpha(\mathbf{r},\;\tau)d\mathbf{r}\,, \tag{10}$$

$$H_{\rm int}(\tau) = -\tfrac{1}{2}\int V(\mathbf{r}_1 - \mathbf{r}_2)\overline{\psi}_\alpha(\mathbf{r}_1,\ \tau)\psi_\alpha(\mathbf{r}_1,\ \tau)\overline{\psi}_\beta(\mathbf{r}_2,\ \tau)\psi_\beta(\mathbf{r}_2,\ \tau)d\mathbf{r}_1 d\mathbf{r}_2\,. \tag{11}$$

Here τ denotes the imaginary time. Using the Fierz rearrangement we can rewrite the interaction Hamiltonian in the new form convenient for the introduction of their composite field variables:

$$\begin{aligned} H_{\rm int}(\tau) = {} & -\tfrac{1}{4}\int V(\mathbf{r}_1 - \mathbf{r}_2)\ [\overline{\psi}_\alpha(\mathbf{r}_1,\tau)(\sigma_2)_{\alpha\beta}\overline{\psi}_\beta(\mathbf{r}_2,\ \tau)\psi_\gamma(\mathbf{r}_2,\ \tau)(\sigma_2)_{\gamma\Delta}\psi_\Delta(\mathbf{r}_1,\ \tau) \\ & +\overline{\psi}_\alpha(\mathbf{r}_1,\tau)(\boldsymbol{\sigma}\sigma_2)_{\alpha\beta}\overline{\psi}_\beta(\mathbf{r}_2,\ \tau)\psi_\gamma(\mathbf{r}_2,\ \tau)(\sigma_2\boldsymbol{\sigma})_{\gamma\Delta}\psi_\Delta(\mathbf{r}_1,\ \tau)]\ d\mathbf{r}_1 d\mathbf{r}_2\,. \end{aligned} \tag{12}$$

The first term on the right side of (12) is the physical origin of the pairing of two fermions in their singlet spin state, the singlet pairing, while the second one is responsible for the triplet pairing.

For simplicity we begin our study by considering only the pairing interaction in the singlet spin state—the singlet pairing interaction expressed by the first term on the right side of (12):

$$H^{(0)}_{\rm int}(\tau) = -\tfrac{1}{4}V(\mathbf{r}_1 - \mathbf{r}_2)\overline{\psi}_\alpha(\mathbf{r}_1,\ \tau)(\sigma_2)_{\alpha\beta}\overline{\psi}_\beta(\mathbf{r}_2,\ \tau)\psi_\gamma(\mathbf{r}_2,\ \tau)(\sigma_2)_{\gamma\Delta}\psi_\Delta(\mathbf{r}_1,\ \tau)\ d\mathbf{r}_1\ d\mathbf{r}_2\,. \tag{13}$$

This is the physical origin of the formation of Cooper pairs in the singlet spin state—the singlet cooperons described by some scalar composite field $\Phi(\mathbf{r}_1, \mathbf{r}_2, \tau)$. For the fermion system with interaction Hamiltonian (13) the partition function equals

$$\begin{aligned} Z^{(0)} = {} & \int [D\psi][D\overline{\psi}]\ e^{-\int_0^\beta d\tau\int \overline{\psi}_\alpha(\mathbf{r},\ \tau)[\frac{\partial}{\partial\tau}-\frac{\nabla^2}{2m}-\mu]\ \psi_\alpha(\mathbf{r},\ \tau)d\mathbf{r}} \\ & \times e^{\frac{1}{4}\int_0^\beta d\tau\int V(\mathbf{r}_1-\mathbf{r}_2)\overline{\psi}_\alpha(\mathbf{r}_1,\ \tau)(\sigma_2)_{\alpha\beta}\overline{\psi}_\beta(\mathbf{r}_2,\ \tau)\psi_\gamma(\mathbf{r}_2,\ \tau)(\sigma_2)_{\gamma\Delta}\psi_\Delta(\mathbf{r}_1,\ \tau)d\mathbf{r}_1 d\mathbf{r}_2}\,. \end{aligned} \tag{14}$$

Introducing the bi-local scalar fields $\Phi(\mathbf{r}_1, \mathbf{r}_2,\ \tau)$ and $\overline{\Phi}(\mathbf{r}_1, \mathbf{r}_2)$, the functional integral over these fields

$$C^{(0)} = \int [D\Phi][D\overline{\Phi}]\ e^{-\int_0^\beta d\tau\int V(\mathbf{r}_1-\mathbf{r}_2)\overline{\Phi}(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)\Phi(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)d\mathbf{r}_1 d\mathbf{r}_2}\,, \tag{15}$$

and shifting the integration variables

$$\begin{aligned} \Phi(\mathbf{r}_1, \mathbf{r}_2, \tau) &\to \Phi(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau) + \tfrac{1}{2}\psi_\gamma(\mathbf{r}_2,\ \tau)(\sigma_2)_{\gamma\Delta}\psi_\Delta(\mathbf{r}_1,\ \tau)\,, \\ \overline{\Phi}(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau) &\to \overline{\Phi}(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau) + \tfrac{1}{2}\overline{\psi}_\gamma(\mathbf{r}_1,\ \tau)(\sigma_2)_{\gamma\Delta}\overline{\psi}_\Delta(\mathbf{r}_2,\ \tau)\,, \end{aligned} \tag{16}$$

we derive the Hubbard–Stratonovich transformation

$$\begin{aligned} & e^{\frac{1}{4}\int_0^\beta d\tau\int V(\mathbf{r}_1-\mathbf{r}_2)\overline{\psi}_\alpha(\mathbf{r}_1,\ \tau)(\sigma_2)_{\alpha\beta}\overline{\psi}_\beta(\mathbf{r}_2,\ \tau)\psi_\gamma(\mathbf{r}_2,\ \tau)(\sigma_2)_{\gamma\Delta}\psi_\Delta(\mathbf{r}_1,\ \tau)d\mathbf{r}_1 d\mathbf{r}_2} \\ & \quad = \frac{1}{C^{(0)}}\int [D\Phi][D\overline{\Phi}]\ e^{-\int_0^\beta d\tau\int V(\mathbf{r}_1-\mathbf{r}_2)\overline{\Phi}(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)\Phi(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)d\mathbf{r}_1 d\mathbf{r}_2}\,. \\ & \quad \times\ e^{-\frac{1}{2}\int_0^\beta d\tau\int V(\mathbf{r}_1-\mathbf{r}_2)[\overline{\Phi}(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)\psi_\beta(\mathbf{r}_2,\ \tau)(\sigma_2)_{\beta\alpha}\psi_\alpha(\mathbf{r}_1,\ \tau)+h.c.]\ d\mathbf{r}_1 d\mathbf{r}_2}\,. \end{aligned} \tag{17}$$

Using this transformation on the right side of (14) and performing the functional integration over the fermionic variables $\psi(\mathbf{r},\ \tau)$ and $\overline{\psi}(\mathbf{r},\ \tau)$, we rewrite the partition function $Z^{(0)}$ in the form of some functional integral over the bi-local scalar composite fields $\Phi(\mathbf{r}_1,\ \mathbf{r}_2,\tau)$ and $\overline{\Phi}(\mathbf{r}_1,\ \mathbf{r}_2,\tau)$:

$$Z^{(0)} = \ \text{const} \ \int [D\Phi][D\overline{\Phi}]e^{-S_{\text{eff}}[\Phi,\ \overline{\Phi}]}\,, \tag{18}$$

$$S_{\text{eff}}[\Phi,\ \overline{\Phi}] = \int_0^\beta d\tau \int V(\mathbf{r}_1 - \mathbf{r}_2)\overline{\Phi}(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)\Phi(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)d\mathbf{r}_1 d\mathbf{r}_2 - W[\Phi,\overline{\Phi}]\,, \tag{19}$$

$$e^{W[\Phi,\ \overline{\Phi}]} = \frac{1}{Z_0}\int [D\psi][D\overline{\psi}]e^{-\int_0^\beta \int \overline{\psi}_\alpha(\mathbf{r},\ \tau)[\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu]\psi_\alpha(\mathbf{r},\ \tau)d\mathbf{r}}\,.$$

$$\times\, e^{-\frac{1}{2}\int_0^\beta d\tau \int V(\mathbf{r}_1 - \mathbf{r}_2)[\overline{\Phi}(\mathbf{r}_1,\mathbf{r}_2,\ \tau)\psi_\beta(\mathbf{r}_2,\ \tau)(\sigma_2)_{\beta\alpha}\psi_\alpha(\mathbf{r}_1,\ \tau) + h.c.]d\mathbf{r}_1 d\mathbf{r}_2}\,, \tag{20}$$

$$Z_0 = \int [D\Phi][D\overline{\Phi}]e^{-\int_0^\beta \int \overline{\psi}_\alpha(\mathbf{r},\tau)[\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu]\psi_\alpha(\mathbf{r},\ \tau)d\mathbf{r}}\,. \tag{21}$$

Here $W[\Phi,\ \overline{\Phi}]$ is a functional power series in the fields $\Phi(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)$ and $\overline{\Phi}(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)$. It has the form

$$W[\Phi,\ \overline{\Phi}] = 1 + \sum_{n=1}^\infty W^{(2n)}[\Phi,\ \overline{\Phi}]\,, \tag{22}$$

where $W^{(2n)}[\Phi,\ \overline{\Phi}]$ is some functional of order n with respect to the fields of each type $\Phi(\mathbf{r}_1,\ \mathbf{r}_2,\tau)$ and $\overline{\Phi}(\mathbf{r}_1,\mathbf{r}_2,\ \tau)$. From the variational principle we derive the field equation

$$\frac{\Delta S_{\text{eff}}[\Phi,\ \overline{\Phi}]}{\Delta\overline{\Phi}(\mathbf{r},\ \mathbf{r}',\ \tau)} = V(\mathbf{r} - \mathbf{r}')\Phi(\mathbf{r},\ \mathbf{r}',\ \tau) - \frac{\Delta W[\Phi,\ \overline{\Phi}]}{\Delta\overline{\Phi}(\mathbf{r},\ \mathbf{r}',\ \tau)} = 0\,. \tag{23}$$

Consider the equation of motion (23) for the special class of τ-independent bi-local fields

$$\Phi(\mathbf{r},\ \mathbf{r}',\ \tau) = \Phi(\mathbf{r},\ \mathbf{r}') = \Phi(\mathbf{R};\ \boldsymbol{\rho})\,, \tag{24}$$

$$\mathbf{R} = \frac{\mathbf{r} + \mathbf{r}'}{2}, \quad \boldsymbol{\rho} = \mathbf{r} - \mathbf{r}'\,. \tag{25}$$

For convenience in writing the equations we set

$$\Delta(\mathbf{R};\ \boldsymbol{\rho}) = V(\boldsymbol{\rho})\Phi(\mathbf{R};\ \boldsymbol{\rho}) = V(\mathbf{r} - \mathbf{r}')\Phi(\mathbf{r},\ \mathbf{r}')\,, \tag{26}$$

denote $\tilde{\Delta}(\mathbf{R};\mathbf{p})$ and $\tilde{V}(\mathbf{p})$ as the Fourier transforms of $\Delta(\mathbf{R};\ \boldsymbol{\rho})$ and $V(\boldsymbol{\rho})$ with respect to the relative coordinate $\boldsymbol{\rho}$

$$\Delta(\mathbf{R};\ \boldsymbol{\rho}) = \frac{1}{(2\pi)^3}\int e^{i\mathbf{p}\boldsymbol{\rho}}\tilde{\Delta}(\mathbf{R};\ \mathbf{p})d\mathbf{p}\,, \tag{27}$$

$$V(\boldsymbol{\rho}) = \frac{1}{(2\pi)^3}\int e^{i\mathbf{p}\boldsymbol{\rho}}\tilde{V}(\mathbf{p})d\mathbf{p}\,, \tag{28}$$

and introduce the operator

$$K(-i\nabla;\ \mathbf{p}) = \frac{E_0(\mathbf{p}-i\frac{\nabla}{2}) + E_0(\mathbf{p}+i\frac{\nabla}{2})}{th\frac{\beta E_0(\mathbf{p}-i\frac{\nabla}{2})}{2} + th\frac{\beta E_0(\mathbf{p}+i\frac{\nabla}{2})}{2}}\,\frac{th\frac{(\beta E_0\mathbf{p})}{2}}{E_0(\mathbf{p})}\,, \tag{29}$$

$$\nabla = \frac{\partial}{\partial\mathbf{R}}, \qquad E_0(\mathbf{p}) = \frac{\mathbf{p}^2}{2m} - \mu\,.$$

We suppose that the $\mathbf{R}$-dependence of $\tilde{\Delta}(\mathbf{R};\ \mathbf{q})$ is very weak and neglect the derivative $\nabla\tilde{\Delta}(\mathbf{R};\ \mathbf{q})$ in the terms nonlinear with respect to $\tilde{\Delta}(\mathbf{R};\ \mathbf{q})$ and its derivatives. Then the equation of motion (23) has the approximate form

$$\begin{aligned}\tilde{\Delta}(\mathbf{R};\ \mathbf{p}) = \frac{1}{(2\pi)^3}\int \tilde{V}(\mathbf{p}-\mathbf{q})\Big\{&\frac{1}{2E_0(\mathbf{q})}th\frac{\beta E_0(\mathbf{q})}{2}\left[\frac{1}{K(-i\nabla;\ \mathbf{q})} - 1\right]\\ &+\frac{1}{2E(\mathbf{R};\ \mathbf{q})}th\frac{\beta E(\mathbf{R};\ \mathbf{q})}{2}\Big\}\tilde{\Delta}(\mathbf{R};\ \mathbf{q})d\mathbf{q}\,,\end{aligned} \tag{30}$$

where $E(\mathbf{R};\mathbf{q})$ is determined by the formula

$$E^2(\mathbf{R};\ \mathbf{q}) = E_0^2(\mathbf{q}) + |\tilde{\Delta}(\mathbf{R};\ \mathbf{q})|^2\,. \tag{31}$$

This is the extension of the BCS gap equation to the case of the weakly $\mathbf{R}$-dependent superconducting order parameter $\Delta(\mathbf{R},\ \boldsymbol{\rho})$. If we neglect the derivatives of order higher than 2 then the relation (30) reads

$$\begin{aligned}\tilde{\Delta}(\mathbf{R};\mathbf{p}) = \frac{1}{(2\pi)^3}\int \tilde{V}(\mathbf{p}-\mathbf{q})\Big\{&\frac{1}{2E_0(\mathbf{q})}th\frac{\beta E_0(\mathbf{q})}{2}\Gamma_{ij}(\mathbf{q})\frac{\partial^2}{\partial R_i\partial R_j}\\ &+\frac{1}{2E(\mathbf{R};\ \mathbf{q})}th\frac{\beta E(\mathbf{R};\ \mathbf{q})}{2}\Big\}\tilde{\Delta}(\mathbf{R};\ \mathbf{q})d\mathbf{q}\,,\end{aligned} \tag{32}$$

$$\Gamma_{ij}(\mathbf{q}) = \frac{1}{8E_0(\mathbf{q})}\left[1 - \frac{\beta E_0(\mathbf{q})}{sh\beta E_0(\mathbf{q})}\right]\frac{\partial^2 E_0(\mathbf{q})}{\partial q_i\partial q_j} + \frac{\beta^2}{16ch^2\frac{\beta E_0(\mathbf{q})}{2}}\frac{\partial E_0(\mathbf{q})}{\partial q_i}\frac{\partial E_0(\mathbf{q})}{\partial q_j}\,. \tag{33}$$

If on the right side of (32) we expand the function

$$\frac{1}{E(\mathbf{R};\ \mathbf{p})} th \frac{\beta E(\mathbf{R};\ \mathbf{p})}{2}$$

into a power series of $|\tilde{\Delta}(\mathbf{R};\ \mathbf{q})|$ and neglect terms of order higher than 2, then we have the Ginzburg–Landau equation

$$\tilde{\Delta}(\mathbf{R};\ \mathbf{p}) = \frac{1}{(2\pi)^3}\int \tilde{V}(\mathbf{p}-\mathbf{q})\frac{1}{2E_0(\mathbf{q})} th \frac{\beta E_0(\mathbf{q})}{2}\Bigg\{1+\gamma_{ij}(\mathbf{q})\frac{\partial^2}{\partial R_i \partial R_j} - \frac{1}{2E_0^2(\mathbf{q})}\left[1-\frac{\beta E_0(\mathbf{q})}{sh\beta E_0(\mathbf{q})}\right]|\tilde{\Delta}(\mathbf{R};\ \mathbf{q})|^2\Bigg\}\tilde{\Delta}(\mathbf{R};\ \mathbf{q})d\mathbf{q}\,. \tag{34}$$

Therefore equation (30) can also be considered as the generalised GL equation for the weakly **R**-dependent wave functions of the Cooper pairs. In the presence of some static electromagnetic field with the vector potential $\mathbf{A}(\mathbf{R})$ we replace

$$\frac{\partial}{\partial \mathbf{r}_i} \to \frac{\partial}{\partial \mathbf{r}_i} + i\frac{e}{c}\mathbf{A}(\mathbf{r}_i)$$

and therefore

$$\nabla \to \nabla + i\frac{e}{c}\Big[\mathbf{A}(\mathbf{r}_1)+\mathbf{A}(\mathbf{r}_2)\Big] \approx \nabla + i\frac{2e}{c}\ \mathbf{A}(\mathbf{R})\,.$$

3. Triplet Pairing in the System with Two-body Attractive Potential $V(\mathbf{r_1}-\mathbf{r_2})$

Now we study the role of the second term in the expession (12) of the interaction Hamiltonian:

$$H_{int}^{(1)}(\tau) = -\tfrac{1}{4}\int V(\mathbf{r}_1-\mathbf{r}_2)\overline{\psi}_\alpha(\mathbf{r}_1,\ \tau)(\sigma_i\sigma_2)_{\alpha\beta}\overline{\psi}_\beta(\mathbf{r}_2,\ \tau)\psi_\gamma(\mathbf{r}_2,\ \tau)(\sigma_2\sigma_i)_{\gamma\Delta}\psi_\Delta(\mathbf{r}_1,\ \tau)d\mathbf{r}_1 d\mathbf{r}_2\,. \tag{35}$$

This is the physical origin of the formation of Cooper pairs in the triplet spin state—the triplet cooperons described by some vector composite field with the components $\Phi^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)$. The partition function of the fermion system with the pairing interaction Hamiltonian (35) equals

$$Z^{(1)} = \int [D\psi][D\overline{\psi}]e^{-\int_0^\beta d\tau\int\overline{\psi}_\alpha(\mathbf{r},\ \tau)\left[\frac{\partial}{\partial\tau}-\frac{\nabla^2}{2m}-\mu\right]\psi_\alpha(\mathbf{r},\ \tau)d\mathbf{r}} \times e^{\frac{1}{4}\int_0^\beta d\tau\int V(\mathbf{r}_1-\mathbf{r}_2)\overline{\psi}_\alpha(\mathbf{r}_1,\ \tau)(\sigma_i\sigma_2)_{\alpha\beta}\overline{\psi}_\beta(\mathbf{r}_2,\ \tau)\psi_\gamma(\mathbf{r}_2,\ \tau)(\sigma_2\sigma_i)_{\gamma\Delta}\psi_\Delta(\mathbf{r}_1,\ \tau)d\mathbf{r}_1 d\mathbf{r}_2}\,. \tag{36}$$

We introduce the bi-local vector fields $\Phi^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)$ and $\overline{\Phi}^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)$, the functional integral over these fields

$$C^{(1)} = \int [D\Phi^i][D\overline{\Phi}^i]e^{-\int_0^\beta d\tau\int V(\mathbf{r}_1-\mathbf{r}_2)\overline{\Phi}^i(\mathbf{r}_1,\ \mathbf{r}-2,\ \tau)\Phi^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)d\mathbf{r}_1 d\mathbf{r}_2}\,, \tag{37}$$

and shift the integration variables

$$\Phi^i(\mathbf{r}_1, \mathbf{r}_2, \tau) \to \Phi^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau) + \tfrac{1}{2}\psi_\gamma(\mathbf{r}_2,\ \tau)(\sigma_2\sigma_i)_{\gamma\Delta}\psi_\Delta(\mathbf{r}_1,\ \tau)\,,$$
$$\overline{\Phi}^i(\mathbf{r}_1,\ \mathbf{r}_2, \tau) \to \overline{\Phi}^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau) + \tfrac{1}{2}\overline{\psi}_\alpha(\mathbf{r}_1,\ \tau)(\sigma_i\sigma_2)_{\alpha\beta}\overline{\psi}_\beta(\mathbf{r}_2,\ \tau)\,. \tag{38}$$

Then we obtain following Hubbard–Stratonovich transformation

$$e^{\frac{1}{4}\int_0^\beta d\tau \int V(\mathbf{r}_1-\mathbf{r}_2)\overline{\psi}_\alpha(\mathbf{r}_1,\ \tau)(\sigma_i\sigma_2)_{\alpha\beta}\overline{\psi}_\beta(\mathbf{r}_2,\ \tau)\psi_\gamma(\mathbf{r}_2,\ \tau)(\sigma_2\sigma_i)_{\gamma\Delta}\psi_\Delta(\mathbf{r}_1,\ \tau)d\mathbf{r}_1 d\mathbf{r}_2}$$
$$= \frac{1}{C^{(1)}}\int [D\Phi^i][D\overline{\Phi}^i] e^{-\int_0^\beta d\tau \int V(\mathbf{r}_1-\mathbf{r}_2)\overline{\Phi}^i(\mathbf{r}_1,\ \mathbf{r}-2,\ \tau)\Phi^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)d\mathbf{r}_1 d\mathbf{r}_2}$$
$$\times\, e^{-\frac{1}{2}\int_0^\beta d\tau \int V(\mathbf{r}_1-\mathbf{r}_2)[\overline{\Phi}^i(\mathbf{r}_1,\mathbf{r}_2,\ \tau)\psi_\beta(\mathbf{r}_2,\ \tau)(\sigma_2\sigma_i)_{\beta\alpha}\psi_\alpha(\mathbf{r}_1,\ \tau)+h.c.]d\mathbf{r}_1 d\mathbf{r}_2}\,. \tag{39}$$

Applying this transformation to the last exponential on the right side of (36) and performing the functional integration over the fermionic variables $\psi(\mathbf{r},\ \tau)$ and $\overline{\psi}(\mathbf{r},\ \tau)$, we obtain the partition function in the form of a functional integral over the bi-local vector composite fields $\Phi^i(\mathbf{r}_1,\ \mathbf{r}_2, \tau)$ and $\overline{\Phi}^i(\mathbf{r}_1,\ \mathbf{r}_2, \tau)$:

$$Z^{(1)} = \text{const}\int [D\Phi^i][D\overline{\Phi}^i] e^{-S_{\text{eff}}[\Phi^i, \overline{\Phi}^i]}\,, \tag{40}$$

$$S_{\text{eff}}[\Phi^i,\ \overline{\Phi}^i] = \int_0^\beta d\tau \int V(\mathbf{r}_1-\mathbf{r}_2)\overline{\Phi}^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)\Phi^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)d\mathbf{r}_1 d\mathbf{r}_2 - W[\Phi^i,\ \overline{\Phi}^i]\,, \tag{41}$$

$$e^{W[\Phi^i,\ \overline{\Phi}^i]} = \frac{1}{Z_0}\int [D\psi][D\overline{\psi}] e^{-\int_0^\beta d\tau \int \overline{\psi}_\alpha(\mathbf{r},\ \tau)[\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu]\psi_\alpha(\mathbf{r},\ \tau)d\mathbf{r}} \tag{42}$$
$$\times\, e^{-\frac{1}{2}\int_0^\beta d\tau \int V(\mathbf{r}_1-\mathbf{r}_2)[\overline{\Phi}^i(\mathbf{r}_1,\mathbf{r}_2,\ \tau)\psi_\beta(\mathbf{r}_2,\ \tau)(\sigma_2\sigma_i)_{\beta\alpha}\psi_\alpha(\mathbf{r}_1,\ \tau)+h.c.]d\mathbf{r}_1 d\mathbf{r}_2}\,.$$

Here $W[\Phi^i,\ \overline{\Phi}^i]$ is a functional power series in the fields $\Phi^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)$ and $\overline{\Phi}^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)$ of the form

$$W[\Phi^i,\ \overline{\Phi}^i] = 1 + \sum_{n=1}^{\infty} W^{(2n)}[\Phi^i,\ \overline{\Phi}^i]\,, \tag{43}$$

where $W^{(2n)}[\Phi^i,\ \overline{\Phi}^i]$ is some functional of order n with respect to the fields of each type $\Phi^i(\mathbf{r}_1, \mathbf{r}_2,\ \tau)$ and $\overline{\Phi}^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)$. The field equation is derived from the variational principle:

$$\frac{\Delta S_{\text{eff}}[\Phi^i,\ \overline{\Phi}^i]}{\Delta\overline{\Phi}^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)} = V(\mathbf{r}_1-\mathbf{r}_2)\Phi^i(\mathbf{r}_1,\ \mathbf{r}_2, \tau) - \frac{\Delta W[\Phi^i,\ \overline{\Phi}^i]}{\Delta\overline{\Phi}^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)} = 0\,. \tag{44}$$

Consider again the equation of motion (44) for the special class of τ-independent bi-local fields

$$\Phi^i(\mathbf{r},\ \mathbf{r}',\ \tau) = \Phi^i(\mathbf{r},\ \mathbf{r}') = \Phi^i(\mathbf{R};\ \boldsymbol{\rho})\,, \tag{45}$$

and set

$$\Delta^i(\mathbf{R};\ \boldsymbol{\rho}) = V(\boldsymbol{\rho})\Phi^i(\mathbf{R};\ \boldsymbol{\rho}) = V(\mathbf{r}-\mathbf{r}')\Phi^i(\mathbf{r},\ \mathbf{r}')\,. \tag{46}$$

We denote $\tilde{\Delta}^i(\mathbf{R},\ \boldsymbol{\rho})$ as the Fourier transform of $\Delta^i(\mathbf{R};\ \boldsymbol{\rho})$,

$$\Delta^i(\mathbf{R};\ \boldsymbol{\rho}) = \frac{1}{(2\pi)^3}\int e^{i\mathbf{p}\boldsymbol{\rho}}\tilde{\Delta}^i(\mathbf{R};\ \mathbf{p})d\mathbf{p}\,,$$

and neglect again the derivative $\nabla\tilde{\Delta}^i(\mathbf{R};\ \mathbf{p})$ in the nonlinear terms. Then we obtain the generalised GL equation for the triplet superconducting order parameter

$$\begin{aligned}\tilde{\Delta}^i(\mathbf{R};\mathbf{p}) = \frac{1}{(2\pi)^3}\int \tilde{V}(\mathbf{p}-\mathbf{q})\Bigg\{&\frac{1}{2E_0(\mathbf{q})}th\frac{\beta E_0(\mathbf{q})}{2}\left[\frac{1}{K(-i\nabla;\ \mathbf{q})}-1\right]\tilde{\Delta}^i(\mathbf{R};\ \mathbf{q})\\ &+\tfrac{1}{2}\left[\frac{1}{2E^{(+)}(\mathbf{R};\ \mathbf{q})}th\frac{\beta E^{(+)}(\mathbf{R};\ \mathbf{q})}{2}+\frac{1}{2E^{(-)}(\mathbf{R};\ \mathbf{q})}th\frac{\beta E^{(-)}(\mathbf{R};\ \mathbf{q})}{2}\right]\\ &\times\tilde{\Delta}^i(\mathbf{R};\ \mathbf{q})\\ &+\tfrac{1}{2}\left[\frac{1}{2E^{(+)}(\mathbf{R};\mathbf{q})}th\frac{\beta E^{(+)}(\mathbf{R};\ \mathbf{q})}{2}-\frac{1}{2E^{(-)}(\mathbf{R};\ \mathbf{q})}th\frac{\beta E^{(-)}(\mathbf{R};\ \mathbf{q})}{2}\right]\\ &\times i\epsilon_{ijk}\frac{\tilde{\Delta}^j(\mathbf{R};\ \mathbf{q})B^k(\mathbf{R};\ \mathbf{q})}{B(\mathbf{R};\ \mathbf{q})}\Bigg\}d\mathbf{q}\,,\end{aligned} \tag{48}$$

where

$$B^i(\mathbf{R};\ \mathbf{q}) = i\epsilon_{ijk}\tilde{\Delta}^j(\mathbf{R};\ \mathbf{q})^*\tilde{\Delta}^k(\mathbf{R};\ \mathbf{q})\,, \tag{49}$$

$$B(\mathbf{R};\ \mathbf{q}) = \left[\sum_i |B^i(\mathbf{R};\ \mathbf{q}|^2\right]^{\frac{1}{2}}\,, \tag{50}$$

$$E^{\pm}(\mathbf{R};\ \mathbf{q})^2 = E_0(\mathbf{q})^2 + \sum_i|\tilde{\Delta}^i(\mathbf{R};\ \mathbf{q})|^2 \pm B(\mathbf{R};\ \mathbf{q})\,. \tag{51}$$

Thus triplet pairing might exist even in the system with an attractive spin-independent potential.

4. General Case of Both Singlet and Triplet Pairings in the System with Two-body Attractive Potential $V(\mathbf{r_1}-\mathbf{r_2})$

After the study of two special cases now we consider the general case when both terms in the interaction Hamiltonian (12) representing both singlet and triplet pairing interactions are taken into account. We start from the partition function

$$Z = \int [D\psi][D\overline{\psi}] e^{-\int_0^\beta d\tau \{\int \overline{\psi}_\alpha(\mathbf{r},\ \tau)[\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu]\psi_\alpha(\mathbf{r},\ \tau)d\mathbf{r} + H_{\rm int}(\tau)\}}\,, \qquad (52)$$

and apply the Hubbard–Stratonovich transformation to the exponential

$$e^{-\int_0^\beta d\tau H_{\rm int}(\tau)}\,.$$

In order to establish this transformation we introduce the bi-local composite scalar and vector fields $\Phi(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)$ and $\Phi^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)$, the functional integral over these fields

$$C = \int [D\Phi][D\overline{\Phi}][D\Phi^i][D\overline{\Phi}^i] \times e^{-\int_0^\beta d\tau \int V(\mathbf{r}_1 - \mathbf{r}_2)[\overline{\Phi}(\mathbf{r}_1, \mathbf{r}_2,\ \tau)\Phi(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau) + \overline{\Phi}^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)\Phi^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)]\ d\mathbf{r}_1 d\mathbf{r}_2}\,, \qquad (53)$$

and the simultaneous shifts (16) and (38) of all integration variables. Then we obtain the corresponding Hubbard–Stratonovich transformation

$$e^{-\int_0^\beta d\tau H_{\rm int}(\tau)} = \frac{1}{C}\int [D\Phi][D\overline{\Phi}][D\Phi^i][D\overline{\Phi}^i] \times e^{-\int_0^\beta d\tau \int V(\mathbf{r}_1 - \mathbf{r}_2)[\overline{\Phi}(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)\Phi(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau) + \overline{\Phi}^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)\Phi^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)]\ d\mathbf{r}_1 d\mathbf{r}_2} \times e^{-\frac{1}{2}\int_0^\beta d\tau \int V(\mathbf{r}_1 - \mathbf{r}_2)\left\{\overline{\psi}_\alpha(\mathbf{r}_1,\ \tau)\ \left[(\sigma_2)_{\alpha\beta}\Phi(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau) + (\sigma_i\sigma_2)_{\alpha\beta}\Phi^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)\right]\overline{\psi}_\beta(\mathbf{r}_2,\ \tau) + h.c.\right\} d\mathbf{r}_1 d\mathbf{r}_2}\,. \qquad (54)$$

Inserting this expression into the right side of (52) and performing the functional integration over the fermionic variables $\psi(\mathbf{r},\ \tau)$ and $\overline{\psi}(\mathbf{r},\ \tau)$, we obtain the partition function Z of the system in the form of some functional integral over the bi-local scalar and vector composite fields $\Phi(\mathbf{r}_1,\ \mathbf{r}_2, \tau)$, $\overline{\Phi}(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)$ and $\Phi^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)$, $\overline{\Phi}^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)$:

$$Z = \mathrm{const}\int [D\Phi][D\overline{\Phi}][D\Phi^i][D\overline{\Phi}^i] e^{-S_{\rm eff}[\Phi, \overline{\Phi}, \Phi^i,\ \overline{\Phi}^i]}\,, \qquad (55)$$

$$S_{\rm eff}[\Phi,\ \overline{\Phi},\ \Phi^i,\ \overline{\Phi}^i] = \int_0^\beta d\tau \int V(\mathbf{r}_1 - \mathbf{r}_2)[\overline{\Phi}(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)\Phi(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau) + \overline{\Phi}^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)\Phi^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)]\ d\mathbf{r}_1 d\mathbf{r}_2 - W[\Phi,\ \overline{\Phi},\ \Phi^i,\ \overline{\Phi}^i]\,, \qquad (56)$$

$$e^{W[\Phi,\ \overline{\Phi},\ \Phi^i,\ \overline{\Phi}^i]} = \frac{1}{Z_0}\int [D\psi][D\overline{\psi}] e^{-\int_0^\beta d\tau \int \overline{\psi}_\alpha(\mathbf{r},\ \tau)\ \left[\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu\right]\psi_\alpha(\mathbf{r},\ \tau)d\mathbf{r}} \qquad (57)$$

$$\times\, e^{-\frac{1}{2}\int_0^\beta d\tau \int V(\mathbf{r}_1 - \mathbf{r}_2)\left\{\overline{\psi}_\alpha(\mathbf{r}_1,\ \tau)\left[(\sigma_2)_{\alpha\beta}\Phi(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau) + (\sigma_i\sigma_2)_{\alpha\beta}\Phi^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)\right]\overline{\psi}_\beta(\mathbf{r}_2,\ \tau) + h.c.\right\} d\mathbf{r}_1 d\mathbf{r}_2}\,.$$

It is straightforward to extend our calculations to the general case of the partition function determined by the formulae (55)–(57). For that purpose we introduce the 4-dimensional vector field $\Phi^\mu(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)$, $\mu = 0,1,2,3$, with $\Phi^0(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)$ being the scalar field $\Phi(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)$ and $\Phi^i(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)$ being the components of the previous 3-dimensional vector field, and four matrices σ_μ, $\mu = 0,1,\ 2,\ 3$, with $\sigma_0 = 1$ and σ_i being three Pauli matrices. Then we can write

$$Z = \mathrm{const} \int [D\Phi^\mu][D\overline{\Phi}^\mu] e^{-S_{\mathrm{eff}}[\Phi^\mu,\ \overline{\Phi}^\mu]}\,, \tag{58}$$

$$S_{\mathrm{eff}}[\Phi^\mu,\ \overline{\Phi}^\mu] = \int_0^\beta d\tau \int V(\mathbf{r}_1 - \mathbf{r}_2)\overline{\Phi}^\mu(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)\Phi^\mu(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)d\mathbf{r}_1 d\mathbf{r}_2 - W[\Phi^\mu,\ \overline{\Phi}^\mu]\,, \tag{59}$$

$$\begin{aligned} e^{W[\Phi^\mu,\ \overline{\Phi}^\mu]} &= \frac{1}{Z_0}\int [D\psi][D\overline{\psi}] e^{-\int_0^\beta d\tau \int \overline{\psi}_\alpha(\mathbf{r},\ \tau)\left[\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu\right]\psi_\alpha(\mathbf{r},\ \tau)d\mathbf{r}} \\ &\quad\times e^{-\frac{1}{2}\int_0^\beta d\tau \int V(\mathbf{r}_1 - \mathbf{r}_2)\left[\overline{\psi}_\alpha(\mathbf{r}_1,\ \tau)(\sigma_\mu\sigma_2)_{\alpha\beta}\overline{\psi}_\beta(\mathbf{r}_2,\ \tau)\Phi^\mu(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau) + h.c.\right]d\mathbf{r}_1 d\mathbf{r}_2}\,. \end{aligned} \tag{60}$$

From the variational principle we derive the field equations

$$\frac{\Delta S_{\mathrm{eff}}[\Phi^\mu,\ \overline{\Phi}^\mu]}{\Delta\overline{\Phi}^\mu(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau)} = V(\mathbf{r}_1 - \mathbf{r}_2)\Phi^\mu(\mathbf{r}_1,\ \mathbf{r}_2,\ \tau) - \frac{\Delta W[\Phi^\mu, \overline{\Phi}^\mu]}{\Delta\overline{\Phi}^\mu(\mathbf{r}_1,\ \mathbf{r}_2, \tau)} = 0\,. \tag{61}$$

Consider again these equations of motion for the class of τ-independent bi-local fields (24) and (45) which are weakly $\mathbf{R}$-dependent. Neglecting the derivatives $\nabla\tilde{\Delta}(\mathbf{R},\ \mathbf{p})$ and $\nabla\tilde{\Delta}^i(\mathbf{R},\ \mathbf{p})$ in the nonlinear terms, we obtain the following system of generalised GL equations for the wave functions of singlet and triplet Cooper pairs

$$\begin{aligned} \tilde{\Delta}(\mathbf{R};\ \mathbf{p}) &= \frac{1}{(2\pi)^3}\int \tilde{V}(\mathbf{p}-\mathbf{q})\Bigg\{\frac{1}{2E_0(\mathbf{q})} th\frac{\beta E_0(\mathbf{q})}{2}\left[\frac{1}{K(-i\nabla;\mathbf{q})} - 1\right]\tilde{\Delta}(\mathbf{R};\ \mathbf{q}) \\ &\quad + \tfrac{1}{2}\left[\frac{1}{2E^{(+)}(\mathbf{R};\ \mathbf{q})} th\frac{\beta E^{(+)}(\mathbf{R};\mathbf{q})}{2} + \frac{1}{2E^{(-)}(\mathbf{R};\ \mathbf{q})} th\frac{\beta E^{(-)}(\mathbf{R};\ \mathbf{q})}{2}\right]\tilde{\Delta}(\mathbf{R};\ \mathbf{q}) \\ &\quad + \tfrac{1}{2}\left[\frac{1}{2E^{(+)}(\mathbf{R};\ \mathbf{q})} th\frac{\beta E^{(+)}(\mathbf{R};\mathbf{q})}{2} - \frac{1}{2E^{(-)}(\mathbf{R};\ \mathbf{q})} th\frac{\beta E^{(-)}(\mathbf{R};\ \mathbf{q})}{2}\right] \\ &\quad \times \frac{\tilde{\Delta}^i(\mathbf{R};\ \mathbf{q})B^i(\mathbf{R};\mathbf{q})}{B(\mathbf{R};\ \mathbf{q})}\Bigg\} d\mathbf{q}\,, \end{aligned} \tag{62}$$

$$
\begin{aligned}
\tilde{\Delta}^i(\mathbf{R};\ \mathbf{p}) = \frac{1}{(2\pi)^3}\int \tilde{V}(\mathbf{p}-\mathbf{q})\Bigg\{ & \frac{1}{2E_0(\mathbf{q})} th\frac{\beta E_0(\mathbf{q})}{2}\left[\frac{1}{K(-i\nabla;\ \mathbf{q})}-1\right]\tilde{\Delta}^i(\mathbf{R};\ \mathbf{q}) \\
& + \tfrac{1}{2}\left[\frac{1}{2E^{(+)}(\mathbf{R};\ \mathbf{q})} th\frac{\beta E^{(+)}(\mathbf{R};\mathbf{q})}{2} + \frac{1}{2E^{(-)}(\mathbf{R};\ \mathbf{q})} th\frac{\beta E^{(-)}(\mathbf{R};\mathbf{q})}{2}\right]\tilde{\Delta}^i(\mathbf{R};\ \mathbf{q}) \\
& + \frac{1}{2}\left[\frac{1}{2E^{(+)}(\mathbf{R};\ \mathbf{q})} th\frac{\beta E^{(+)}(\mathbf{R};\mathbf{q})}{2} - \frac{1}{2E^{(-)}(\mathbf{R};\ \mathbf{q})} th\frac{\beta E^{(-)}(\mathbf{R};\ \mathbf{q})}{2}\right] \\
& \times \frac{\tilde{\Delta}(\mathbf{R};\ \mathbf{q})B^i(\mathbf{R};\ \mathbf{q}) + i\epsilon_{ijk}\tilde{\Delta}^j(\mathbf{R};\ \mathbf{q})B^k(\mathbf{R};\ \mathbf{q})}{B(\mathbf{R};\ \mathbf{q})}\Bigg\} d\mathbf{q}\,,
\end{aligned}
\tag{63}
$$

where

$$
B^i(\mathbf{R};\ \mathbf{q}) = \tilde{\Delta}(\mathbf{R};\ \mathbf{q})\tilde{\Delta}^i(\mathbf{R};\ \mathbf{q})^* + \tilde{\Delta}(\mathbf{R};\ \mathbf{q})^*\tilde{\Delta}^i(\mathbf{R};\ \mathbf{q}) + i\epsilon_{ijk}\tilde{\Delta}^j(\mathbf{R};\ \mathbf{q})^*\tilde{\Delta}^k(\mathbf{R};\ \mathbf{q})\,,
\tag{64}
$$

and where $B(\mathbf{R};\mathbf{q})$ is determined by the relation (50) and

$$
E^{(\pm)}(\mathbf{R};\ \mathbf{q})^2 = E_0(\mathbf{q})^2 + |\tilde{\Delta}(\mathbf{R};\ \mathbf{q})|^2 + \sum_i |\tilde{\Delta}^i(\mathbf{R};\ \mathbf{q})|^2 \pm B(\mathbf{R};\ \mathbf{q})\,.
\tag{65}
$$

For the $\mathbf{R}$-independent superconducting order parameters, equations (62) and (63) are reduced to the system of BCS gap equations in the case of simultaneous and triplet pairings.

5. Applications to the t-J Model of High-T_c Superconductivity

In the t-J model of high-T_c superconductivity besides the Coulomb interaction between the spin $\frac{1}{2}$ fermions there exists also the quantum Heisenberg magnetic (ferromagnetic and antiferromagnetic) coupling. We denote $c_{i\alpha}$ and $c^+_{i\alpha}$ the destruction and creation operators for the fermions at the site i. We have following total Hamiltonian:

$$
\begin{aligned}
& H = \sum_i E c^+_{i\alpha}c_{i\alpha} + \tfrac{1}{2}\sum_{i\neq j} t_{ij}c^+_{i\alpha}c_{j\alpha} + \tfrac{1}{2}\sum_{i\neq j} J_{ij}\mathbf{S}_i\mathbf{S}_j + \tfrac{1}{2}\sum_{i,j} U_{ij}n_i n_j\,, \\
& \mathbf{S}_i = \tfrac{1}{2}\ c^+_{i\alpha}(\boldsymbol{\sigma})_{\alpha\beta}c_{i\beta}, \quad n_i = c^+_{i\alpha}c_{i\alpha}\,, \\
& t_{ji} = t_{ij},\ \ J_{ji} = J_{ij},\ \ U_{ji} = U_{ij}\,.
\end{aligned}
\tag{66}
$$

In the simple case when the on-site Coulomb repulsion potential U_{ii} is not very strong we can apply the perturbation expansion to both types of couplings J_{ij} and U_{ij}. Denote $\tilde{t}(\mathbf{p})$, $\tilde{J}(\mathbf{p})$ and $\tilde{U}(\mathbf{p})$ as the Fourier transforms of the t_{ij}, J_{ij} and U_{ij}. Applying the reasoning presented in Section 4 we derive the following generalised GL equations:

$$
\begin{aligned}
\tilde{\Delta}(\mathbf{R};\mathbf{p}) = \frac{1}{\Omega}\int_{\Omega} [\tfrac{3}{4}\tilde{J}(\mathbf{p}-\mathbf{q}) - \tilde{U}(\mathbf{p}-\mathbf{q})]\Bigg\{ & \frac{1}{2E_0(\mathbf{q})}\ th\frac{\beta E_0(\mathbf{q})}{2}\ \left[\frac{1}{K(-i\nabla;\ \mathbf{q})} - 1\right]\tilde{\Delta}(\mathbf{R};\ \mathbf{q}) \\
& + \tfrac{1}{2}\left[\frac{1}{2E^{(+)}(\mathbf{R};\ \mathbf{q})}\ th\frac{\beta E^{(+)}(\mathbf{R};\mathbf{q})}{2} + \frac{1}{2E^{(-)}(\mathbf{R};\ \mathbf{q})}\ th\frac{\beta E^{(-)}(\mathbf{R};\ \mathbf{q})}{2}\ \right]\tilde{\Delta}(\mathbf{R};\ \mathbf{q}) \\
& + \frac{1}{2}\left[\frac{1}{2E^{(+)}(\mathbf{R};\ \mathbf{q})}\ th\frac{\beta E^{(+)}(\mathbf{R};\ \mathbf{q})}{2} - \frac{1}{2E^{(-)}(\mathbf{R};\ \mathbf{q})}\ th\frac{\beta E^{(-)}(\mathbf{R};\ \mathbf{q})}{2}\ \right] \\
& \times \frac{\tilde{\Delta}^i(\mathbf{R};\ \mathbf{q})B^i(\mathbf{R};\ \mathbf{q})}{B(\mathbf{R};\ \mathbf{q})}\Bigg\} d\mathbf{q}\,,
\end{aligned} \tag{67}
$$

$$
\begin{aligned}
\tilde{\Delta}^i(\mathbf{R};\ \boldsymbol{p}) = -\frac{1}{\Omega}\int_{\Omega}[\tfrac{1}{4}\tilde{J}(\mathbf{p}-\mathbf{q}) + \tilde{U}(\mathbf{p}-\mathbf{q})]\Bigg\{ & \frac{1}{2E_0(\mathbf{q})} th\frac{\beta E_0(\mathbf{q})}{2} \\
& \times \left[\frac{1}{K(-i\nabla;\mathbf{q})} - 1\right]\tilde{\Delta}^i(\mathbf{R};\ \mathbf{q}) \\
& + \tfrac{1}{2}\left[\frac{1}{2E^{(+)}(\mathbf{R};\ \mathbf{q})} th\frac{\beta E^{(+)}(\mathbf{R};\ \mathbf{q})}{2} + \frac{1}{2E^{(-)}(\mathbf{R};\ \mathbf{q})} th\frac{\beta E^{(-)}(\mathbf{R};\ \mathbf{q})}{2}\right]\tilde{\Delta}^i(\mathbf{R};\ \mathbf{q}) \\
& + \tfrac{1}{2}\left[\frac{1}{2E^{(+)}(\mathbf{R};\ \mathbf{q})} th\frac{\beta E^{(+)}(\mathbf{R};\ \mathbf{q})}{2} - \frac{1}{2E^{(-)}(\mathbf{R};\ \mathbf{q})} th\frac{\beta E^{(-)}(\mathbf{R};\ \mathbf{q})}{2}\right] \\
& \times \frac{\tilde{\Delta}(\mathbf{R};\ \mathbf{q})B^i(\mathbf{R};\ \mathbf{q}) + i\epsilon_{ijk}\tilde{\Delta}^j(\mathbf{R};\ \mathbf{q})B^k(\mathbf{R};\ \mathbf{q})}{B(\mathbf{R};\ \mathbf{q})}\Bigg\} d\mathbf{q}\,,
\end{aligned} \tag{68}
$$

where Ω is the volume of the Brillouin zone,

$$E_0(\mathbf{q}) = E + \tfrac{1}{2}\tilde{t}(\mathbf{q}) - \mu\,.$$

In the case of systems with very strong on-site Coulomb repulsion, the constant

$$U = U_{ii}$$

is very large and must be taken into account by means of some nonperturbative calculation method. We write the Hamiltonian in the form

$$
\begin{aligned}
H = & \sum_i E c^+_{i\alpha} c_{i\alpha} + \tfrac{1}{2}\sum_{i\neq j} t_{ij} c^+_{i\alpha} c_{j\alpha} + \tfrac{1}{2}\sum_{i\neq j} J_{ij}\mathbf{S}_i\mathbf{S}_j + \tfrac{1}{2}\sum_{i\neq j} U_{ij} n_i n_j \\
& + U\sum_i n_{i\uparrow} n_{i\downarrow}\,,
\end{aligned} \tag{69}
$$

and apply the perturbation expansion only with respect to the constants J_{ij} and U_{ij} with $i \neq j$. Denote $\tilde{J}(\mathbf{p})$ and $\tilde{U}(\mathbf{p})$ as their Fourier transforms. We can show that in the system with a large constant U the energy spectrum of fermions consists of the low energy bands $E^{(\pm)}(\mathbf{R};\ \mathbf{p})$ of singly occupying fermions and high energy bands $E^{\pm}(\mathbf{R};\ \mathbf{p}) + U$ of doubly occupying ones. Let the mean value of the number of itinerant fermions at each site be equal to n,

$$n = \langle c^+_{i\alpha} c_{i\alpha} \rangle \,.$$

The generalised Ginzburg–Landau equations have forms similar to relations (67) and (68) with the following substitutions:

$$\frac{1}{E^{(\pm)}(\mathbf{R};\ \mathbf{q})}\ th\frac{\beta E^{(\pm)}(\mathbf{R};\ \mathbf{q})}{2} \rightarrow (1-\frac{n}{2})\frac{1}{E^{(\pm)}(\mathbf{R};\ \mathbf{q})}\ th\frac{\beta E^{(\pm)}(\mathbf{R};\mathbf{q})}{2} + \frac{n}{2}\frac{1}{E^{(\pm)}(\mathbf{R};\ \mathbf{q})+U}\ th\frac{\beta}{2}[E^{(\pm)}(\mathbf{R};\ \mathbf{q})+U]\,. \tag{70}$$

If the coupling constants J_{ij} and U_{ij}, $i \neq j$, are nonzero only for the nearest neighbour sites i and j, then in the two-dimensional square lattice with lattice spacing a we have

$$\tilde{J}(\mathbf{p}) = 2J_0(\cos ap_x + \cos ap_y), \qquad \tilde{U}(\mathbf{p}) = 2U_0(\cos ap_x + \cos ap_y)\,.$$

From the extended GL equations (67) and (68) it follows that the superconducting order parameters $\tilde{\Delta}(\mathbf{R};\ \mathbf{p})$ and $\tilde{\boldsymbol{\Delta}}(\mathbf{R};\ \mathbf{p})$ have the forms

$$\tilde{\Delta}(\mathbf{R};\ \mathbf{p}) = \Delta_s(\mathbf{R};\ p)(\cos ap_x + \cos ap_y) + \Delta_d(\mathbf{R};\ p)(\cos ap_x - \cos ap_y)\,, \tag{71}$$

$$\tilde{\boldsymbol{\Delta}}(\mathbf{R};\ \mathbf{p}) = \boldsymbol{\Delta}_x(\mathbf{R};\ p)\sin ap_x + \boldsymbol{\Delta}_y(\mathbf{R};\ p)\sin ap_y\,. \tag{72}$$

In particular, the superconducting order parameter $\tilde{\Delta}(\mathbf{R};\ \mathbf{p})$ for the singlet pairing consists of the s- wave and d-wave. Inserting expressions (71) and (72) into both sides of (67) and (68) we obtain the systems of equations for the coefficients $\Delta_s(\mathbf{R};p), \Delta_d(\mathbf{R};p)$ and $\boldsymbol{\Delta}_x(\mathbf{R};p)$, $\boldsymbol{\Delta}_y(\mathbf{R};\ p)$. The physical consequences of these equations are being widely investigated.

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