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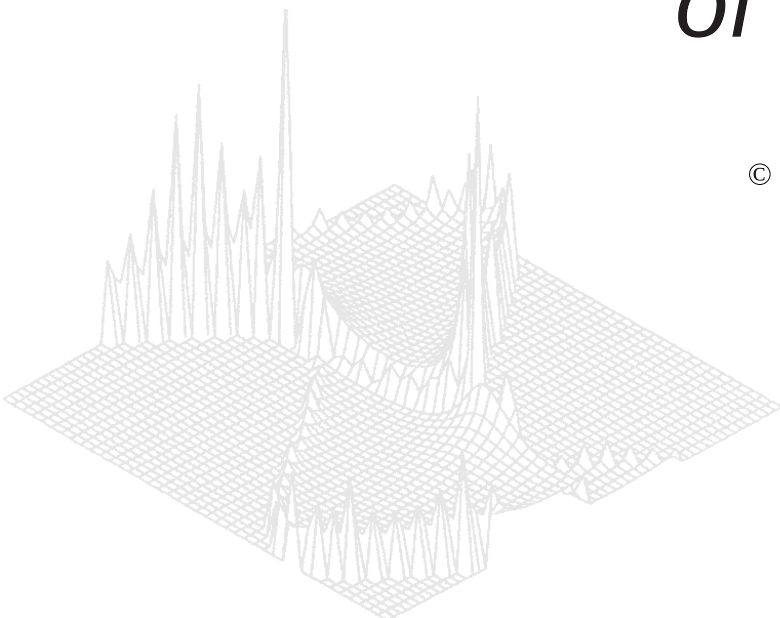

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Large Amplitude Alfvén Soliton in the Solar Wind and Painlevé Analysis

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Abstract

A detailed analysis of complete integrability is performed for a new set of equations for large amplitude Alfvén waves in the solar wind recently derived by Hada. It is observed that there exist two branches of the Painlevé expansion and the number of resonances is less than the degree of the equation, thus indicating that the system is not completely integrable. We have found the exact one soliton solution of the system. This one solution has the distinctive feature that even the phase part of the complex is a nonlinear wave packet.

1. Introduction

Generation and propagation of nonlinear waves in a plasma has become a subject of prime interest over the last two decades. Both the cases of large and small amplitude waves have been treated separately (Schamel 1972; Taniuti and Washimi 1968). Recently Hada (1993) has deduced an equation describing the propagation of large amplitude Alfvén waves in the solar wind. Its importance arises from the fact that the solar wind can serve as a natural setting for a test of the nonlinear theory of finite amplitude Alfvén waves. This is because the scale lengths of plasma inhomogeneities in the solar wind are far greater than the typical Alfvén wavelength. Another important aspect of the solar wind is that it is free from any boundary effects. It is a fact that various electromagnetic instabilities can amplify the Alfvén wave amplitude and for which the nonlinearity can play an important role. It is known that for propagation at a sufficiently large angle to the magnetic field, the fast and slow magnetosonic waves follow the KdV equation (Morton 1964; Kever and Morikawa 1969; Kakutani and Ono 1969), whereas the quasi-parallel Alfvén waves are described by the derivative nonlinear Schrödinger equations (DNLS) (Rogister 1971; Mjolhus 1974; Mio *et al.* 1976). In the latter case, it is usually assumed that $|1 - c_s^2/c_i^2| \gg \delta B/B_0$, where c_i and c_s are the intermediate and sound speeds. On the other hand, in the situation discussed by Hada (1993), the opposite condition is assumed, that is, $|1 - c_s^2/c_i^2| < \delta B/B_0$. Now it is well known that both KdV and DNLS are completely integrable systems and are solvable by the inverse spectral tranform (Ablowitz *et al.* 1974; Zakharov *et al.* 1971). Thus, they can sustain multi-soliton solutions, and one can obtain the full properties of these waves. So we presume that it will be a very interesting task to study the complete integrability of this

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new set of equations deduced by Hada and to study its solitary waves. We find that due to the lack of a sufficient number of resonances this set is not completely integrable, but one can always utilise the translational invariance to obtain the one soliton solution in terms of hyperelliptic functions.

2. Formulation

The equations deduced by Hada (1993) can be written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho^2 + \Delta\rho + \frac{|b|^2}{2}\right) = 0\,, \tag{1}$$

$$\frac{\partial b}{\partial t} + \frac{\partial}{\partial x}(\rho b) + \mathrm{i}\ \frac{\partial^2 b}{\partial x^2} = 0\,, \tag{2}$$

where b represents the magnetic field normalised to B_0 and ρ is the density inside the plasma. To proceed with the Painlevé analysis we set (Weiss 1984; Chanda and Roy Chowdhury 1988; Roy Chowdhury and Chanda 1987) $b = \psi\, e^{i\phi}$, so that equation (2) yields

$$\psi_t + (\rho\psi)_x - 2\psi_z\,\phi_x - \psi\phi_{xx} = 0\,, \tag{3}$$

$$\psi\phi_t + \rho\psi\phi_x + \psi_{xx} - \psi\phi_x^2 = 0\,. \tag{4}$$

We now consider the set (1), (3) and (4). It is customary to put

$$\psi = \sum_n a_n\,\chi^{n+a}; \quad \phi = \sum_n \chi^{n+\beta}; \quad \rho = \sum_n c_n\,\chi^{n+\gamma}\,, \tag{5}$$

where α, β and γ are negative integers, $\chi = X(x,t)$ is the singular manifold, and a_n, b_n, c_n are still arbitrary functions of (x,t). We substitute the expansions (5) in (1), (3) and (4), and match the most singular terms to determine the coefficients α, β and γ. It is not difficult to observe that we have two choices:

$$\alpha = -2, \quad \beta = -1, \quad \gamma = -2\,,$$

along with

$$\chi_t\, b_0 = 6, \quad c_0 = -\tfrac{3}{2}b_0, \quad \alpha_0^2 = -2\Gamma c_0^2\,,$$

or

$$\alpha = -1, \quad \beta = -1, \quad \gamma = -2\,,$$

along with

$$a_0^2 + 2\Delta c_0 = 0, \quad b_0 = 2a_0\,.$$

These two possibilities actually indicate two different branches of the Painlevé analysis. We now proceed with the whole series in (5) and determine the recursion relation of the coefficients a_n, b_n, c_n. From equation (3) we get

$$a_s[c_0(s-4)+2b_0(s-3)]+b_s\,a_0(s-1)(4-s)+c_s\,a_0(s-4)$$
$$=F(a_{s-1},b_{s-1},\,c_{s-1})\,. \tag{6}$$

Similarly equations (4) and (1) respectively lead to

$$-a_s(b_0+c_0)b_0+b_s(s-1)(2b_0+c_0)a_0-c_s\,b_0\,c_0=G(a_{s-1},\,b_{s-1},\,c_{s-1})\,, \tag{7}$$

$$c_s\,2\Gamma(s-4)c_0+a_s\,a_0(s-4)=H(a_{s-1},\,b_{s-1},c_{s-1})\,. \tag{8}$$

In equations (6), (7) and (8), F, G and H represent certain polynomials in the coefficients a_{s-1}, b_{s-1} and c_{s-1}. Now the secular determinant for the determination of the resonances reads

$$\begin{vmatrix} a_0(s-4) & 0 & 2\Gamma c_0(s-4) \\ -b_0(b_0+c_0) & (s-1)a_0(c_0+2b_0) & -b_0\,c_0 \\ c_0(s-4)+2b_0(s-3) & a_0(s-1)(4-s) & a_0(s-4) \end{vmatrix}=0$$

or

$$(s-4)\begin{vmatrix} a_0 & 0 & 2\Gamma c_0 \\ -b_0(b_0+c_0) & (s-1)(c_0+2b_0)a_0 & -b_0\,c_0 \\ c_0(s-4)+2b_0(s-3) & a_0(s-1)(4-s) & a_0(s-4) \end{vmatrix}=0\,. \tag{9}$$

So $s=4$ is one resonance position. The determinant in equation (9) is a quadratic function of s, not having any integer (positive) root, nor does it possess the essential zero at $s=-1$. We conclude that unlike the KdV and DNLS equations the set (1), (3) and (4) is not completely integrable. This is because our coupled set (1), (3) and (4) is of degree >4, but we have only one resonance. In our computation we have assumed that the Kruskal simplification (Clarkson and Kruskal 1992) is valid, i.e. χ is of the form $\chi=x-f(t)$. A similar analysis with the second branch also shows that we cannot meet the full requirement of the Painlevé analysis.

3. One Soliton Solution

Although our equation set does not pass the Painlevé test yet, due to the existence of translational invariance we can find the explicit soliton solution by direct quadrature. To obtain the solution we assume

$$\psi=\psi(x-vt)\,,\quad \rho=\rho(x-vt),\quad \phi=kx-\omega t+\theta(x-vt)\,. \tag{10}$$

The third equation in (10) for ϕ shows that the phase function is not at all linear. Substitution of (10) in (1), (3) and (4) yields after integration:

$$\rho(\Delta-v)+\Gamma\rho^2+\tfrac{1}{2}\psi^2=\alpha\,,$$
$$\psi^2\theta_\xi=-(\tfrac{1}{2}v+k)\psi^2+\int\psi(\rho\psi)_\xi\ \mathrm{d}\xi$$
$$-\psi(\omega+v\theta_\xi)+p\psi(k+\theta_\xi)+\psi_{\xi\xi}-\psi(k+\theta_\xi)^2=0\,. \tag{11}$$

To start with, if we assume $\Gamma = 0$ (which may not be important in the case of the solar wind), then (11) leads to

$$\rho = \frac{\psi^2}{2(v-\Delta)} = \beta\psi^2 \quad \text{(say)}\,.$$

Then the second equation of (11) gives

$$\theta_\xi = -(\tfrac{1}{2}v + k) + \tfrac{3}{4}\beta\psi^2\,,$$

and when used in the third equation leads to

$$\psi_{\xi\xi} + a\psi + b\psi^2 + c\psi^3 + c\psi^4 + d\psi^5 = 0\,. \tag{12}$$

Here $\xi = x - vt$. Equation (12) is a second order nonlinear ordinary differential equation which can be solved by hyperelliptic functions.

The second case is $\Gamma \neq 0$, but we may assume $\Delta = v$, whence

$$\psi = \pm\,(2\Gamma')^{\frac{1}{2}}\rho = \sigma\rho\,,$$

with $\Gamma' = -\Gamma$ and in the first equation we have set $\alpha = 0$, by the choice of boundary condition. In this case we get

$$\theta_\xi = -\gamma + \tfrac{2}{3}\rho; \qquad \gamma = \tfrac{1}{2}v + k\,,$$

and the equation for ρ is

$$\rho_{\xi\xi} + \lambda\rho + \mu\rho^2 + \nu\rho^3 = 0\,. \tag{13}$$

Again, we can solve for ρ by elliptic functions.

Lastly, in the general situation where $\alpha \neq 0$, $\Delta \neq 0$ and $F \neq 0$, we observe from equation (11) that

$$\psi = F(\rho) = \pm\,[\alpha + \Gamma\rho^2 + \rho(v-\Delta)]^{\frac{1}{2}}\,,$$

whence

$$\frac{\mathrm{d}\rho}{\mathrm{d}\xi} = \frac{[2\int Q(\rho)\,F\rho^2\ \mathrm{d}\rho]^{\frac{1}{2}}}{F(\rho)}$$

or

$$\int \frac{F(\rho)\ \mathrm{d}\rho}{[2\int Q(\rho)\,F(\rho)\ \mathrm{d}\rho]^{\frac{1}{2}}} = \xi + \alpha\,,$$

where we have set

$$Q(\rho) = (\rho k - \omega)F + (\rho - v)GF - F(k+G)^2\,,$$

$$D(\rho) = \frac{\Gamma - p^3[(v-\Delta) - 2\gamma\Gamma]\rho^2 - \gamma(v-\Delta)\rho^2}{2\Gamma\rho^2 + (v-\Delta)\rho}\,.$$

Thus, even in the general case we can reduce the problem to single quadrature. Here the parameters a, b, λ_1, μ, etc. are all simple functions of the plasma parameters v, k, Δ, Γ, etc.

4. Discussion

In our analysis we have shown that, unlike the case of Alfvén waves with the condition $|1 - c_s^2/c_i^2| \gg \delta B/B_0$, the wave equation obtained in the reverse situation is not completely integrable due to a lack of resonances. But the one solution can be obtained explicitly, though it is expressible in terms of elliptic functions only. It may be noted that in the special situation $\Delta = v$, we can derive a relation between (k, ω) similar to the one obtained by Hada (1993) in the linear analysis. This can be seen by a special choice of the coefficients in (13). Written explicitly this equation reads

$$\rho'' - \rho[\omega + v\gamma + (k-\gamma)^2] + \rho^2[k + \gamma - \tfrac{2}{3}v - \tfrac{4}{3}(v-\gamma)] - \tfrac{4}{9}\rho^3 = 0\,.$$

The solution of this equation is written as

$$\rho = \mathrm{Cdn}(p\psi,\, q)\,,$$

where C, p, q are given by

$$q^2 = \frac{2p^2 - A}{\rho^2}; \qquad C^2 = -\frac{2p^2}{B}\,,$$

$$A = \omega + v\gamma + (k-\gamma)^2, \qquad B = \tfrac{4}{9}\,,$$

when

$$[k + \gamma - \tfrac{2}{3}v - \tfrac{4}{3}(v-\gamma)] = 0\,.$$

Thus, a complete solution can be written as

$$\psi = \pm (2\Gamma)^{\frac{1}{2}}\,\mathrm{Cdn}[p(x-vt),\, q]\,,$$

$$\theta = \gamma(x-vt) + \tfrac{2}{3}\int \mathrm{d}n(p\psi,\, q)\mathrm{d}\psi\,,$$

$$\phi = kx - \omega t + \theta(x - vt)\,.$$

So this particular solution yields the cnoidal wave solution. Lastly, as noted by Hada (1993), this set of equations contains the DNLS and KdV cases as a special situation and this generalised case may not be integrable completely.

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