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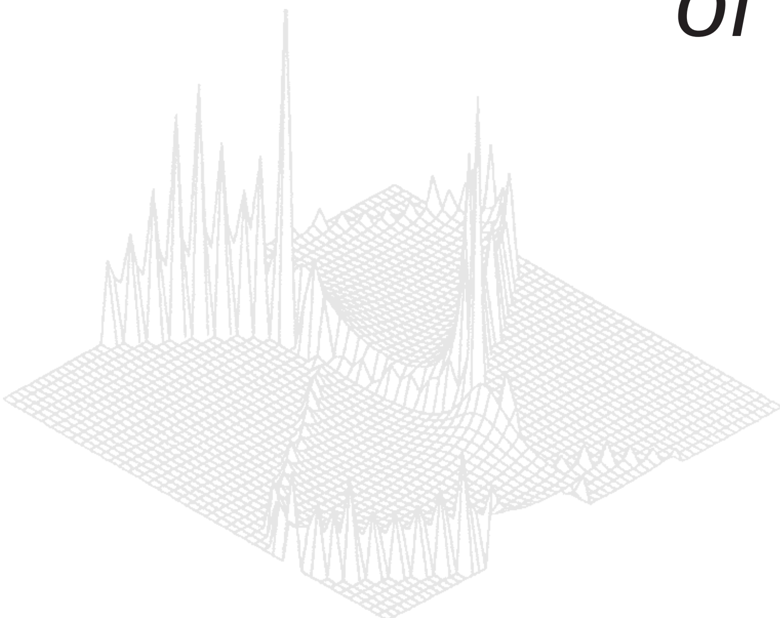

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Higher Dimensional Exact Solutions of Einstein's Field Equations for a Massive Point Particle with Scalar Point Charge

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Abstract

The exact spherically symmetric static solution of Einstein's field equations in higher dimensions for a massive point particle with a scalar point charge as the source of a massless scalar field is derived in Schwarzschild coordinates. There exists no Schwarzschild horizon. This result is an extension in higher dimensions of a similar one obtained by Hardell and Dehnen (1993) earlier for 4D space–time.

1. Introduction

Recently, there has been renewed interest in discussing higher dimensional space times both in localised and cosmological domains (Emelyanov *et al*. 1986). For localised distributions all the solutions obtained so far are arbitrary dimensional generalisations of the usual Schwarzschild or Kerr solutions (Myer *et al*. 1986; Chatterjee 1987). In quantum field theory, the long term objective of unification of gravity and other forces in nature seems to remain as far away as ever. The advance of super gravity in 11D and that of super strings in 10D indicates that multidimensionality of space is required for interactions over a distance $R \ll 10^{-16}$ cm, where unification of all forces is possible (Witten 1984).

The higher dimensional space times have been studied by various researchers such as Liddle *et al*. (1990), Bannerjee *et al*. (1992), etc. In the present paper, we investigate exact solutions of Einstein's field equations for a massive point particle with scalar point charge in a higher dimensional spherically symmetric metric. It is seen that in the Schwarzschild coordinates, the exact static solution of a point mass with a scalar point charge does not possess a Schwarzschild horizon. The metric and the scalar field are regular with the exception of the point $R = 0$. The problem has been already discussed in four dimensions by Hardell *et al*. (1993).

2. Field Equations in Higher Dimensions and Their Solutions

We consider the higher dimensional static spherically symmetric space–time to have the form (Banerjee *et al*. 1992):

$$ds^2 = e^{\beta} dt^2 - e^{\alpha} dR^2 - R^2 (X_n^2) \tag{1}$$

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where

$$X_n^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \sin^2\theta_1 \sin^2\theta_2 d\theta_3^2 + ... + \left[\prod_{i=1}^{n-1}(\sin^2\theta_i)\right] d\theta_n^2 \,, \tag{2a}$$

$$g_{11} = -e^{\alpha},\;\; g_{22} = -R^2,\;\; g_{33} = -R^2\sin^2\theta_1,\;\; g_{44} = -R^2\sin^2\theta_1\sin^2\theta_2,$$

$$...,\;\; g_{(n+1)(n+1)} = -R^2\left[\prod_{i=1}^{n-1}(\sin^2\theta_i)\right],\;\; g_{(n+2)(n+2)} = e^{\beta} \,. \tag{2b}$$

The convention is

$$x^1 = R,\;\; x^2 = \theta_1,\;\; x^3 = \theta_2,\;\; ...,\;\; x^{(n+1)} \;=\; \theta_n,\;\; x^{(n+2)} = t \,.$$

We assume that α, β and the scalar field are functions of R alone.

It can be easily seen that the line element (1) represents the usual 4D spherically symmetric space–time for $n = 2$. For determination of the gravitational field of a point particle with a scalar point charge as source for a massless scalar field ϕ, we start from the coupled field equations:*

$$\phi^{/i}_{//i} = \frac{1}{\sqrt{-g}}(\sqrt{-g}\phi^{/i})_{/i} = 0 \,, \tag{3}$$

$$R_{ij} = -\kappa\left(T_{ij} - \frac{1}{n}T g_{ij}\right) , \tag{4}$$

$$T_{ij} = \phi_{/i}\phi_{/j} - \tfrac{1}{2}\phi_{/a}\phi^{/a} g_{ij} \,, \tag{5}$$

where $\kappa = 8\pi G$ (G is the Newtonian gravitational constant and $c = 1$). From (5), we get

$$T_{ij} - \tfrac{1}{2}T g_{ij} = \phi_{/i}\phi_{/j} \;+ \frac{n-2}{4}\;\phi_{/k}\phi^{/k}\;g_{ij}. \tag{6}$$

Thus equation (4) reduces to

$$R_{ij} = -\kappa\phi_{/i}\phi_{/j} \,. \tag{7}$$

For the line element (1), we get from (3)

$$\phi_{/R} = -\frac{\epsilon}{R^n}e^{(\alpha-\beta)/2}, \qquad \phi_{/i} = 0, \qquad \text{otherwise.} \tag{8}$$

* $//i$ is the covariant derivative and $/i$ is the usual partial derivative with respect to x^i.

Here, ϵ is an integration constant, giving the scalar point charge at $R = 0$. The field equation (7) then, along with (1), yields

$$\frac{\beta^{''}}{2} - \frac{n\alpha^{'}}{2R} - \frac{\alpha^{'}\beta^{'}}{4} + \frac{\beta^{'2}}{4} = -\kappa\frac{\epsilon^2}{R^{2n}}e^{\alpha-\beta}\,, \tag{9a}$$

.

$$\left[(n-1) - \frac{R}{2}(\alpha^{'} - \beta^{'})\right]e^{-\alpha} - (n-1) \;=\; 0\,, \tag{9b}$$

$$\left[-\frac{\beta^{''}}{2} + \frac{\alpha^{'}\beta^{'}}{4} - \frac{n}{2}\frac{\beta^{'}}{R} - \frac{\beta^{'2}}{4}\right]e^{\beta-\alpha} = 0\,, \tag{9c}$$

where the prime denotes differentiation with respect to R.

From (9c) and (9a) we get

$$\alpha^{'} + \beta' = 2\frac{\kappa\epsilon^2}{n}\frac{e^{\alpha-\beta}}{R^{(2n-1)}}\,, \tag{10}$$

and (9c) gives

$$\frac{\beta^{''}}{\beta^{'}} - \frac{(\alpha^{'} - \beta^{'})}{2} + \frac{n}{R} = 0\,, \tag{11}$$

which on integration yields

$$\beta^{'} = \frac{A}{R^n}e^{(\alpha-\beta)/2}\,. \tag{12}$$

Let

$$\alpha + \beta = u \quad \text{and} \quad \alpha - \beta = v\,,$$

which means

$$\alpha = \frac{u+v}{2} \quad \text{and} \quad \beta = \frac{u-v}{2}\,. \tag{13}$$

From (9b), (10) and (12), we get using (13)

$$(n-1) - \frac{Rv^{'}}{2} = (n-1)e^{(u+v)/2}\,, \tag{14}$$

$$u^{'} = 2\frac{\kappa\epsilon^2}{n}\frac{e^v}{R^{(2n-1)}}\,, \tag{15}$$

$$\tfrac{1}{2}(u^{'} - v^{'}) = \frac{A}{R^n} e^{v/2} \,. \tag{16}$$

Equations (15) and (16) give

$$v^{'} = 2\frac{\kappa\epsilon^2}{n}\frac{e^v}{R^{(2n-1)}} - 2\frac{A}{R^n} e^{v/2} \,, \tag{17}$$

which along with (14) gives

$$(n-1)e^{u/2} \;=\; \frac{A}{R^{(n-1)}} + (n-1)e^{-v/2} - \frac{\kappa\epsilon^2}{n}\frac{e^{v/2}}{R^{2(n-1)}} \,. \tag{18}$$

From (8) and (12), for $A \neq 0$, we have

$$\phi^{'} = -\frac{\epsilon}{R^n} e^{(\alpha-\beta)/2} = -\frac{\epsilon}{A}\beta^{'} \,,$$

$$\Rightarrow \quad \phi = -\frac{\epsilon}{A}\beta = -\frac{\epsilon}{2A}(u - v) \,. \tag{18a}$$

It is postulated that in the asymptotic case $R \to \infty$, i.e. at the boundary the metric converts to Minkowski metric, i.e.

$$ds^2 = dt^2 - dR^2 - R^2(X_n^2) \,,$$

i.e. as $R \to \infty$, $\alpha, \beta \to 0$ and also $u, v \to 0$, which means $|u| \ll 1$ and $|v| \ll 1$. Thus, as $R \to \infty$, we get from (17)

$$v^{'} \;\approx\; 2\frac{\kappa\epsilon^2}{n}\frac{1}{R^{(2n-1)}} - \frac{2A}{R^n} \,,$$

$\Rightarrow$

$$v \;\approx -\frac{\kappa\epsilon^2}{n(n-1)}\frac{1}{R^{2(n-1)}} + \frac{2A}{(n-1)}\frac{1}{R^{(n-1)}} \,, \tag{19a}$$

(15) $\Rightarrow$

$$u^{'} \;\approx \frac{2\kappa\epsilon^2}{n}\frac{1}{R^{(2n-1)}} \quad \Rightarrow \quad u \;\approx -\frac{\kappa\epsilon^2}{n(n-1)}\frac{1}{R^{2(n-1)}} \,. \tag{19b}$$

Thus

$$\alpha \;\approx \frac{A}{(n-1)}\frac{1}{R^{(n-1)}} - \frac{\kappa\epsilon^2}{n(n-1)}\frac{1}{R^{2(n-1)}} \,, \tag{19c}$$

$$\beta \approx -\frac{A}{(n-1)}\frac{1}{R^{(n-1)}}\,. \tag{19d}$$

Thus, for the Schwarzschild mass M of the particle, $A = 2\,MG$ is valid in view of the line element (1). On substituting

$$e^{v/2} = R^{(n-1)}g(R)\,, \tag{20}$$

we get from (18)

$$e^{u/2} = \frac{1}{R^{n-1}}\left[\frac{A}{(n-1)} + \frac{1}{g} \;-\; \frac{\lambda g}{(n-1)}\right]. \tag{20a}$$

From (20) and (17), we get

$$g^{'} = \frac{1}{R}[\lambda g^3 - Ag^2 - (n-1)g]\,, \tag{21}$$

where

$$\lambda = \frac{\kappa\epsilon^2}{n} = \frac{8\pi G\epsilon^2}{n}\,. \tag{22}$$

From (21) we get on integration

$$\ln\left|\frac{g^2}{(n-1)+Ag-\lambda g^2}\right| + \frac{A}{\sqrt{A^2+4(n-1)\lambda}}\ln\left|\frac{\sqrt{A^2+4(n-1)\lambda}+A-2\lambda g}{\sqrt{A^2+4(n-1)\lambda}-A+2\lambda g}\right| = \ln\frac{C}{R^{2(n-1)}}\,, \tag{23a}$$

$\Rightarrow$

$$\left|\frac{g^2}{(n-1)+Ag-\lambda g^2}\right|\left|\frac{\sqrt{A^2+4(n-1)\lambda}+A-2\lambda g}{\sqrt{A^2+4(n-1)\lambda}-A+2\lambda g}\right|^{A/\sqrt{A^2+4(n-1)\lambda}} = \frac{C}{R^{2(n-1)}}\,, \tag{23b}$$

where C is an integration constant.

The metric components can be expressed in terms of g as

$$e^{\alpha} = \frac{(n-1)+Ag-\lambda g^2}{(n-1)}\,, \tag{24a}$$

$$e^{\beta} = \frac{1}{(n-1)R^{2(n-1)}g^2}[(n-1)+Ag-\lambda g^2]\,. \tag{24b}$$

From (18a) we get

$$\phi = -\frac{\epsilon}{A}\beta = -\frac{\epsilon}{A}\ln\left[\frac{(n-1)+Ag-\lambda g^2}{(n-1)R^{2(n-1)}g^2}\right]. \tag{24c}$$

For $A = 0$, ϕ must be evaluated by using (23a) and (23b).

Equations (23) and (24) give the exact solution for the metric and the scalar field. The integration constant C can be found by considering the fact that for $R \to \infty$, α and β take up the Minkowski values, i.e. zero. Thus, for $R \to \infty$, we find

$$g^2 \approx \frac{(n-1)C}{R^{2(n-1)}}\left[\frac{\sqrt{A^2+4(n-1)\lambda}-A}{\sqrt{A^2+4(n-1)\lambda}+A}\right]^{A/\sqrt{A^2+4(n-1)\lambda}}, \tag{25}$$

$$C \approx \frac{1}{(n-1)}\left[\frac{\sqrt{A^2+4(n-1)\lambda}+A}{\sqrt{A^2+4(n-1)\lambda}-A}\right]^{A/\sqrt{A^2+4(n-1)\lambda}}. \tag{26}$$

which are totally dependent only on the mass and the scalar point charge functions A and λ respectively.

3. Discussion of the Solution

(i) Let $A = 0$, $\lambda \neq 0$

(23)$\Rightarrow$

$$g^2 = \frac{(n-1)/R^{2(n-1)}}{1+[\lambda/R^{2(n-1)}]}, \tag{27}$$

$$e^{\alpha} = \frac{1}{1+\lambda/R^{2(n-1)}}, \qquad e^{\beta} = \frac{1}{(n-1)}, \tag{28}$$

$$\phi = [\epsilon/\sqrt{(n-1)\lambda}]\sinh^{-1}[\sqrt{\lambda}/R^{2(n-1)}].$$

(ii) If $A \neq 0$, $\lambda = 0$,

$$\frac{(n-1)^2 g^2}{[(n-1)+Ag]^2} = \frac{1}{R^{2(n-1)}}, \tag{29}$$

$\Rightarrow$

$$g = \frac{\pm 1/R^{n-1}}{1 \mp [A/(n-1)R^{(n-1)}]}. \tag{30}$$

Neglecting the lower sign of the numerator which corresponds to negative mass, we get

$$e^{\alpha} = \frac{1}{1 \mp [\ A/(n-1)R^{(n-1)}\ ]}\ , \qquad e^{\beta} = 1 \mp \frac{A}{(n-1)R^{(n-1)}}\,. \tag{31}$$

(iii) Lastly, if $A \neq 0,\ \lambda \neq 0$. Under restriction to positive masses, we consider only the positive values of $g(R)$. Then, in the limiting case $R \to 0$, we find from (23b),

$$g(R) = g_0 - \gamma(R)\,, \tag{32}$$

where

$$g_0 = \frac{A + \sqrt{A^2 + 4(n-1)\lambda}}{2\lambda} > 0\,, \tag{33}$$

$$C = \frac{1}{(n-1)} \left[\frac{2\lambda g_0}{\sqrt{A^2 + 4(n-1)\lambda} - A} \right]^{A/\sqrt{A^2+4(n-1)\lambda}}\,,$$

$$\begin{aligned}\gamma(R) = \ & (n-1)\left[\frac{g_0^2}{\sqrt{A^2+4(n-1)\lambda}}\right]^{\sqrt{A^2+4(n-1)\lambda}/(\sqrt{A^2+4(n-1)\lambda}-A)} \\ & \times \left(\frac{\sqrt{A^2+4(n-1)\lambda}-A}{2g_0\sqrt{A^2+4(n-1)\lambda}}\right)^{A/(\sqrt{A^2+4(n-1)\lambda}-A)} \\ & \times R^{[2(n-1)\sqrt{A^2+4(n-1)\lambda}/(\sqrt{A^2+4(n-1)\lambda}-A)]} \\ & > 0\,,\end{aligned} \tag{34}$$

and from (24)

$$e^{\alpha} = \frac{\sqrt{A^2+4(n-1)\lambda}}{(n-1)}\gamma \sim R^{2(n-1)(\sqrt{A^2+4(n-1)\lambda}/(\sqrt{A^2+4(n-1)\lambda}-A)} \geq 0\,, \tag{35a}$$

$$e^{\beta} = \frac{\sqrt{A^2+4(n-1)\lambda}}{(n-1)g_0^2}\frac{\gamma}{R^{2(n-1)}} \sim R^{2(n-1)(A/(\sqrt{A^2+4(n-1)\lambda}-A)} \geq 0\,. \tag{35b}$$

The asymptotic value of $g(R)$ in the case $R \to \infty$ is given by (25) and those for e^{α} and e^{β} by (19c) and (19d) respectively. Obviously, there exists no Schwarzschild horizon. This appears only in case of $\lambda = 0$ for vanishing scalar point charge (cf. equation 31). Only at point $R = 0$ does the metric become singular.

For the scalar field from (24c), we obtain

$$\phi = \frac{-\epsilon g_0}{(n-1)} \ln R + \text{const.} \tag{36a}$$

For $R \to \infty$, we find from (5) using (24) and (25),

$$\phi_{/R} = \frac{-\epsilon}{R^n} \qquad \Rightarrow \qquad \phi = \frac{\epsilon}{(n-1)R^{(n-1)}} \,. \tag{36b}$$

4. Conclusion

It has been shown that, in Schwarzschild coordinates, any scalar field will influence and modify the metric independently from its strength in such a way that a simultaneous solution of a massless scalar field equation and Eintein's field equations always exists in higher dimensions for the static case of a massive point particle with a scalar point charge. The metric and the scalar field are regular with the exception of the point $R = 0$. For any value of the scalar point charge different from zero, no Schwarzschild horizon appears.

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References

Banerjee, A., Duttachoudhury, S. B., and Chatterjee, S. (1992). *Gen. Relat. Gravitat.* **24**, 991.
Chatterjee, S. (1987). *Astron. Astrophys.* **179**, 1.
Emelyanov, V. M., *et al.* (1986). *Phys. Rep.* **43**, 1
Hardell, A., and Dehnen, H. (1993). *Gen. Relat. Gravitat.* **25**, 1165.
Liddle, A. R., Moorhouse, R. G., and Henriques, A. B. (1990). *Class. Quant. Grav.* **7**, 1009.
Myer, R. C., and Perry, M. J. (1986). *Ann. Phys. (NY)* **172**, 304.
Witten, E. (1984) *Phys. Lett.* B**149**, 351.

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