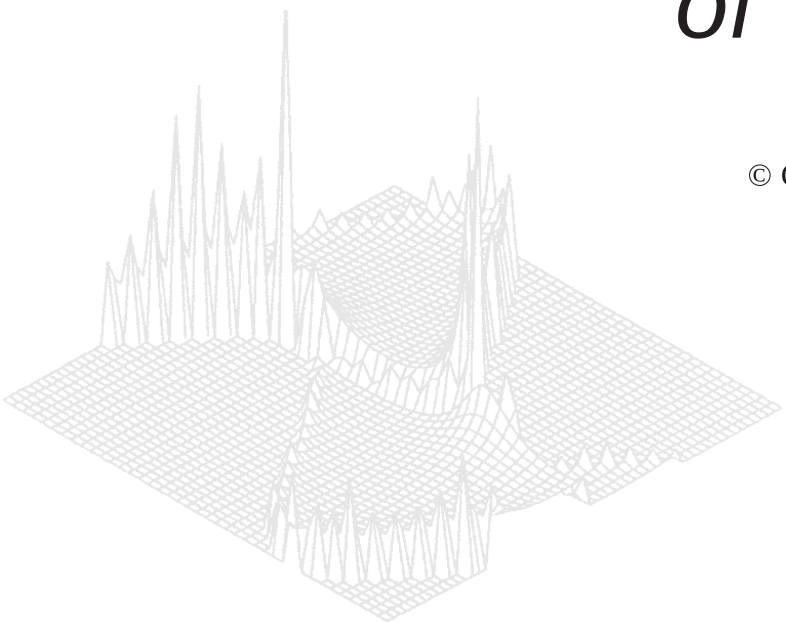
CSIRO PUBLISHING

Australian Journal of Physics

Volume 52, 1999



A journal for the publication of
original research in all branches of physics

www.publish.csiro.au/journals/ajp

All enquiries and manuscripts should be directed to
Australian Journal of Physics
CSIRO PUBLISHING
PO Box 1139 (150 Oxford St)
Collingwood
Vic. 3066
Australia
Telephone: 61 3 9662 7626
Facsimile: 61 3 9662 7611
Email: peter.robertson@publish.csiro.au

CSIRO AUSTRALIA

Published by **CSIRO** PUBLISHING
for CSIRO Australia and
the Australian Academy of Science

Academy of Science

Aust. J. Phys., 1999, **52**, 59–70

Radial Structure of Electron Drift Waves in Tokamak Geometry

J. L. V. Lewandowski

Department of Theoretical Physics, and Plasma Research Laboratory,
RSPhysSE, Australian National University,
Canberra, ACT 0200, Australia.

Abstract

The radial structure of electron drift waves in a low-pressure tokamak plasma is presented. The ions are cold and an electrostatic approximation for the fluctuating potential is used. It is shown that problem of the radial structure of drift waves in toroidal geometry is amenable to a two-step solution; in the first approximation, the radial structure of the mode is neglected and the problem to be solved is the usual eigenmode equation along the (extended) poloidal angle; in the second approximation, the mode amplitude is expanded in ascending powers of the parameter $(k_\perp L_n)^{-1/2}$, where $k_\perp$ is the magnitude of the lowest-order wavevector and L_n is the radial density scalelength. The implications of these radially-extended drift-type modes for the anomalous cross-field diffusion are discussed.

1. Introduction

It is now generally accepted in the fusion community that even if fast, large-scale magnetohydrodynamic (MHD) instabilities can be suppressed, magnetically-confined plasmas always contain sufficient free energy to drive slow, short-scale instabilities. The cross-field 'anomalous' transport generated by drift-type instabilities is typically two orders of magnitude larger than the neoclassical transport (Horton 1989; Tang 1978; Liewer 1985; Wagner and Stroth 1993). Using the heuristic 'derivation' of Kadomtsev (1965) one can estimate the perpendicular anomalous diffusion coefficient as

$$D_\perp \approx \frac{\gamma}{(k_r^{(\mathrm{eff})})^2}\,, \tag{1}$$

where γ is the linear growth rate and $k_r^{(\mathrm{eff})}$ is the effective radial wavevector. Most theoretical studies deal with detailed calculations to determine the linear growth rate γ. Typically the most unstable mode is found for ${k_\perp}^{-1} \sim {k_r}^{-1} = \xi\rho_s$ where ξ is a nondimensional quantity of the order unity (typically between 3 and 10) and $\rho_s = c_s/\omega_{\mathrm{ci}}$, and where $c_s = (T_{\mathrm{e}}/m_{\mathrm{i}})^{\frac{1}{2}}$ is the ion sound speed and $\omega_{\mathrm{ci}} = eB/(m_{\mathrm{i}}c)$ is the ion cyclotron frequency. Although it is important to calculate the linear growth rate accurately, we note that the (heuristic) diffusion coefficient (1) has a strong dependence on $k_r^{(\mathrm{eff})}$.

The aim of this paper is to study the *radial* localisation of drift waves in tokamak geometry and to estimate $k_r^{(\mathrm{eff})}$ in equation (1). In order to clarify the

 10.1071/PH98067 0004-9506/99/010059$05.00

method used to calculate $k_r^{(\mathrm{eff})}$, we consider a simple model for electron drift waves with cold ions. A simple $i\delta$ model is used for the nonadiabatic part of the electron response and the geometry is the one of a low-β tokamak plasma with circular, concentric magnetic surfaces. In Section 2, we discuss the representation of the fluctuating electrostatic potential. The method used by Romanelli and Zonca (1993) to estimate the radial extension of ion-temperature-gradient (ITG) driven modes is briefly discussed. In Section 3, we derive a set of two coupled equations; one equation describes the structure of the mode along the magnetic field line whereas the second equation is a differential equation (DE) describing the radial structure of the mode. In Section 4, we discuss the implication of our results and suggest possible improvements for future work.

2. Representation of the Fluctuating Potential

In this section, we discuss two different, but complementary, representations for the fluctuating electrostatic potential Φ. Since the density fluctuation is related to Φ by a simple $i\delta$ correction, our model is essentially a one-field (Φ) model. The confining magnetic field can be written in straight-field-line coordinates (Boozer 1981, 1982) as

$$\mathbf{B} = \nabla\alpha \times \nabla\alpha \times \nabla\psi\,, \tag{2}$$

where α is the field line label and $2\pi\psi$ is the enclosed poloidal flux. For a low-β tokamak plasma with circular, concentric magnetic surfaces we can also write $\mathbf{B}$ as

$$\mathbf{B} = \frac{B_0 R_0}{R}\ \widehat{\phi} + \frac{rqB_0}{R}\ \widehat{\boldsymbol{\theta}}\,, \tag{3}$$

where $R = R_0 + r\cos\theta$ is the local major radius, B_0 is the magnetic field strength at the magnetic axis ($R = R_0$), θ is the poloidal angle measured from the outside of the torus, and $\widehat{\boldsymbol{\theta}}$ and $\widehat{\boldsymbol{\phi}}$ are unit vectors in the poloidal and toroidal directions, respectively. For the magnetic field given in (3), the field line label is $\alpha = \phi - q(r)\theta$ and the poloidal flux function $\psi(r)$ satisfies

$$\psi(r) = B_0 \int_0^r \frac{r'}{q(r')}\ dr'\,. \tag{4}$$

In the local coordinate system $\{r,\theta,\phi\}$, the contravariant basis vectors are $\nabla r = \widehat{\mathbf{r}}$, $\nabla\theta = \widehat{\boldsymbol{\theta}}/r$ and $\nabla\phi = \widehat{\boldsymbol{\phi}}/R$. For low-frequency modes with $k_{||}/k_\perp \ll 1$, it is convenient to use the eikonal representation for the fluctuating electrostatic potential $\widetilde{\Phi} \equiv e\Phi/T_{\mathrm{e}}$. Following Antonsen and Lane (1980), we write

$$\widetilde{\Phi} = \widehat{\Phi}\exp(-i\omega t)\exp(iNS)\,, \tag{5}$$

where ω is the mode frequency and $N \gg 1$ is the toroidal mode number. The amplitude $\widehat{\Phi}$, and the eikonal S, vary on the equilibrium scale length. We demand that $\mathbf{B}\,\boldsymbol{.}\,\nabla S \equiv 0$. Then, taking into account equation (2), the general solution for the eikonal is $S = S(\alpha,\psi)$ or, equivalently, $S = S(\alpha,q)$ since the safety factor is a flux surface quantity, $q = q(\psi)$. The lowest-order perpendicular wavevector is defined as (Antonsen and Lane 1980)

$$\mathbf{k}_\perp \equiv N\nabla S \,. \tag{6}$$

For simplicity, we write the eikonal as $S = \alpha + \overline{S}(q)$ and we can write the above equation as

$$\begin{aligned}\mathbf{k}_\perp &= N(\nabla\alpha + \theta_k \nabla q) \\ &= N\left(\nabla\alpha + \frac{dq}{dr}\;\theta_k \nabla r\right),\end{aligned} \tag{7}$$

where $\theta_k \equiv d\overline{S}/dq$ is the radial mode number. In the standard eikonal representation, the amplitude is written as $\widehat{\Phi} = \widehat{\Phi}(x_{||})$ where $x_{||}$ is the length along the magnetic field. For our simple equilibrium magnetic field (3), we have $x_{||} \simeq qR_0\theta$ where, now, θ is the *extended* poloidal angle. When representation (5) is used in the model equation, one gets a second-order differential in θ and the mode frequency is $\omega = \omega(r, \theta_k)$.

As shown by Romanelli and Zonca (1993), the radial structure of the eigenmode is related to the radial mode number θ_k which is then considered as an *operator* in the radial direction. In this paper, we suggest another approach to the problem of the radial structure of eigenmodes in toroidal geometry. We simply set the radial mode number to zero but retain the radial variation of the mode in the amplitude. Therefore, we write the potential fluctuation as

$$\widetilde{\Phi} = \widehat{\Phi}(x_{||}, r)\exp(iN\alpha - i\omega t)\,. \tag{8}$$

The lowest-order perpendicular wavevector is now $\mathbf{k}_\perp = N\nabla\alpha$. Typically, $k_\perp \approx k_\theta \equiv Nq/r$. In order to decouple $\partial\widehat{\Phi}/\partial r$ from the equilibrium, we write $\partial\widehat{\Phi}/\partial r \sim \widehat{\Phi}/\lambda$ and assume

$$\lambda \ll L \sim L_n \,, \tag{9}$$

where L is a typical equilibrium scale length and L_n is the radial density scale length. Since the rapid variation is included in the phase factor $\exp(iN\alpha - i\omega t)$ we assume

$$\lambda \gg \lambda_\perp \equiv \frac{2\pi}{k_\perp} \;\sim \frac{r}{Nq}\,. \tag{10}$$

Taking into account orderings (9) and (10), it is natural to adopt the meso-scale ordering

$$\lambda \sim \sqrt{L\lambda_\perp}\,. \tag{11}$$

We note that ordering (11) is an *assumption* and it must be verified that this ordering yields a physically correct solution. In the coming sections, we verify that ordering (11) is consistent with a broad radial extension of drift waves in toroidal geometry. For clarity, we write the ordering that we shall use throughout this paper:

$$k_\perp^{-1}/a = \mathcal{O}(N^{-1})\,, \quad \lambda/a = \mathcal{O}(N^{-1/2})\,, \quad L/a = \mathcal{O}(1)\,. \tag{12}$$

3. Two-dimensional Eigenmode Equation

In this section, we adopt the eikonal representation (8) and derive a two-dimensional (r, θ) eigenmode equation for drift waves in toroidal geometry. Using the meso-scale (11), it is shown that the two-dimensional problem can be reduced to a set of 2 one-dimensional eigenmode equations. One eigenmode equation describes the 'parallel' (along the field line) structure of the mode, whereas the second eigenmode equation describes the radial envelope of the toroidal drift wave. As discussed in the Introduction, we adopt a simple model for electron drift waves with cold ions. Since the typical perpendicular wavelength is much larger the Debye length, the plasma is quasi neutral, $n_\mathrm{e} \simeq n_\mathrm{i} \equiv n$. Our basic equations are the ion continuity equation (Braginskii 1965)

$$\frac{\partial n}{\partial t} + \nabla\boldsymbol{.}(n\mathbf{v}_\mathrm{i}) = 0\,, \tag{13}$$

and the ion momentum equations (with $T_\mathrm{i} \mapsto 0$)

$$m_\mathrm{i} n\left(\frac{\partial}{\partial t} + \mathbf{v}_\mathrm{i}\boldsymbol{.}\nabla\right)\mathbf{v}_\mathrm{i} = en\left(\mathbf{E} + \frac{\mathbf{v}_\mathrm{i}\times\mathbf{B}}{c}\right), \tag{14}$$

where, for a low-β plasma, $\mathbf{E} \simeq -\nabla\Phi$. For low-frequency drift-type modes, we solve equation (14) for the cross-field ion drift velocity. The derivation is standard (Lewandowski 1997*a*, 1997*b*) and we state the result

$$\mathbf{v}_{\mathrm{i}\perp} = \mathbf{v}_E + \mathbf{v}_\mathrm{pi}\,, \tag{15}$$

where $\mathbf{v}_E \equiv c\mathbf{E}\times\mathbf{B}/B^2$ is the lowest-order $\mathbf{E}\times\mathbf{B}$ drift velocity and

$$\mathbf{v}_\mathrm{pi} \equiv \omega_\mathrm{ci}^{-1}\widehat{\mathbf{e}}_{||} \times \left(\frac{\partial}{\partial t} + \mathbf{v}_E\boldsymbol{.}\nabla\right)\mathbf{v}_\mathbf{E} \tag{16}$$

is the ion polarisation drift velocity. Here $\widehat{\mathbf{e}}_{||} \equiv \mathbf{B}/B$ is the unit vector along $\mathbf{B}$ and $\omega_\mathrm{ci} = eB/m_\mathrm{i}c$ is the ion cyclotron frequency. The parallel component of the ion momentum equation yields

$$\frac{\partial\widetilde{v}_{||}}{\partial t} = -c_s\nabla_{||}\widetilde{\Phi}\,, \tag{17}$$

where $\widetilde{v}_{||} \equiv v_{\mathrm{i}||}/c_s$ is the normalised parallel ion velocity. Our model equation is

$$\frac{\partial n}{\partial t} = -\nabla\boldsymbol{.}(n\mathbf{v}_E + n\mathbf{v}_\mathrm{pi} + n\mathbf{v}_{\mathrm{i}||})\,. \tag{18}$$

To obtain a single eigenmode equation, we write the perturbed density as $\widetilde{n} = (1-i\delta)\widetilde{\Phi}$ where δ is a small, positive constant which describes the nonadiabatic response of the electrons. Since the amplitude $\widehat{\Phi} = \widehat{\Phi}(r, x_{||})$ depends on the radius, the eigenmode equation resulting from equation (18) will be two-dimensional. To simplify intermediate calculation, it is convenient to define the following operator:

$$\mathbf{P}_\perp \equiv \left(i\mathbf{k}_\perp + \nabla r \frac{\partial}{\partial r} \right) . \tag{19}$$

The $\mathbf{E} \times \mathbf{B}$ drift velocity is

$$\mathbf{v}_E = c_\mathrm{s}\rho_\mathrm{s} f^\star \widehat{e}_{||} \times \mathbf{P}_\perp(\widehat{\Phi}) \tag{20}$$

and, in the linear approximation, the polarisation drift velocity reads

$$\mathbf{v}_\mathrm{pi} = -f^\star \rho_\mathrm{s}^2 \frac{\partial}{\partial t}\ \mathbf{P}_\perp(\widehat{\Phi}) . \tag{21}$$

In equations (20) and (21), we have defined the phase factor $f^\star \equiv \exp(iN\alpha)$. The divergence of the ion polarisation drift velocity is

$$\nabla \boldsymbol{.}\, \mathbf{v}_\pi = -\rho_\mathrm{s}^2 \frac{\partial}{\partial t} \left[f^\star \frac{\nabla T_\mathrm{e}}{T_\mathrm{e}}\ \boldsymbol{.}\, \mathbf{P}_\perp(\widehat{\Phi}) + \nabla f^\star \boldsymbol{.}\, \mathbf{P}_\perp(\widehat{\Phi}) + f^\star \nabla \boldsymbol{.}\, \mathbf{P}_\perp(\widehat{\Phi}) \right] , \tag{22}$$

where we have assumed $|\nabla B/B| \ll |\nabla T_\mathrm{e}/T_\mathrm{e}|$. When written in explicit form, equation (22) contains many terms. For instance, the last term in the square brackets of equation (22) can be written as

$$\nabla \boldsymbol{.}\, \mathbf{P}_\perp(\widehat{\Phi}) = i\mathbf{k}_\perp \boldsymbol{.}\, \nabla_\perp \widehat{\Phi} + \nabla \boldsymbol{.}\, \nabla r \frac{\partial \widehat{\Phi}}{\partial r} + i\nabla \boldsymbol{.}\, \mathbf{k}_\perp \widehat{\Phi} + \nabla r \boldsymbol{.}\, \nabla \left(\frac{\partial \widehat{\Phi}}{\partial r} \right) . \tag{23}$$

In particular, we have to evaluate $\nabla \boldsymbol{.}\, \nabla\alpha$ [third term on the right-hand side of equation (23)]. Using the expression for the field line label $\alpha = \phi - q(r)\theta$, we have

$$\begin{aligned} \nabla \boldsymbol{.}\, \nabla\alpha &= \nabla \boldsymbol{.}\, \nabla\phi - q\nabla \boldsymbol{.}\, \nabla\theta - \dot{q}\nabla r \boldsymbol{.}\, \nabla\theta - \dot{q}\theta\nabla r \boldsymbol{.}\, \nabla r - \ddot{q}\theta\nabla r \boldsymbol{.}\, \nabla r - \dot{q}\nabla\theta \boldsymbol{.}\, \nabla r \\ &= \nabla \boldsymbol{.}\, \nabla\phi - q\nabla\theta - \dot{q}\theta\nabla \boldsymbol{.}\, \nabla r - \ddot{q}\theta , \end{aligned} \tag{24}$$

since, for an equilibrium with *concentric*, circular magnetic surfaces, we have $\nabla r \boldsymbol{.}\, \nabla r = 1$ and $\nabla r \boldsymbol{.}\, \nabla\theta = 0$. In the above equation, a dot denotes a derivative with respect to r. To calculate $\nabla \boldsymbol{.}\, \nabla r$, we note that $\nabla r = \widehat{\mathbf{r}} = \widehat{\theta} \times \widehat{\phi} = \mathcal{J}\nabla\theta \times \nabla\phi$, where $\mathcal{J} \equiv r(R_0 + r\cos\theta)$ is the Jacobian of the transformation. Then we get

$$\begin{aligned} \nabla \boldsymbol{.}\, \nabla r &= \nabla\mathcal{J} \boldsymbol{.}\, (\nabla\theta \times \nabla\phi) + \mathcal{J}\nabla \boldsymbol{.}\, (\nabla\theta \times \nabla\phi) \\ &= \frac{\partial \mathcal{J}}{\partial r}\ \nabla r \boldsymbol{.}\, (\nabla\theta \times \nabla\phi) \\ &= \frac{1}{\mathcal{J}}\ \frac{\partial \mathcal{J}}{\partial r}\ \simeq \frac{1}{r} . \end{aligned} \tag{25}$$

Similarly, it is easy to show that $\nabla\,\boldsymbol{.}\,\nabla\theta = -\sin\theta/rR_0$ and $\nabla\,\boldsymbol{.}\,\nabla\phi = 0$. Then, equation (24) becomes

$$\nabla\,\boldsymbol{.}\,\nabla\alpha \simeq -\frac{q}{r^2}\;(\widehat{s}+\widehat{c})\theta\,. \tag{26}$$

Here we have defined the (global) magnetic shear $\widehat{s} \equiv r\dot{q}/q$ and $\widehat{c} \equiv r^2\ddot{q}/q$. We term $\widehat{c}$ the *scalar* magnetic curvature (since it is related to the second-order radial derivative of the safety factor), to distinguish this quantity from the *vector* magnetic curvature, $\kappa \equiv (\widehat{e}\,\boldsymbol{.}\,\nabla)\widehat{e}$. After tedious but straightforward algebra, we obtain the divergence of the ion polarisation flux:

$$\begin{aligned}\frac{\nabla\,\boldsymbol{.}\,(n\mathbf{v}_{\mathrm{pi}})}{n_0} = &-f^\star\frac{\partial}{\partial t}\left[\frac{\rho s}{L_n}\;(1+\eta_{\mathrm{e}})\left(i\sqrt{b}\widehat{s}\theta\widehat{\Phi} - \frac{\partial\widehat{\Phi}}{\partial x}\right) - bg(\theta)\widehat{\Phi} - 2i\sqrt{b}\widehat{s}\theta\frac{\partial\widehat{\Phi}}{\partial x}\right.\\ &\left. - i\sqrt{b}(\widehat{s}+\widehat{c})\theta\frac{\widehat{\Phi}}{c} + \frac{1}{x}\frac{\partial\widehat{\Phi}}{\partial x} + \frac{\partial^2\widehat{\Phi}}{\partial x^2}\right],\end{aligned} \tag{27}$$

where $x \equiv r/\rho_{\mathrm{s}}$ is the normalised radial coordinate, $g(\theta) = 1 + (\widehat{s}\theta)^2$ is related to the secular behaviour of $|\nabla\alpha|^2$ and $b \equiv (k_\theta\rho_{\mathrm{s}})^2$. Also we have introduced the temperature gradient parameter $\eta_{\mathrm{e}} \equiv d\ln T_{e0}/d\ln n_0$. Other calculations are similar. Our final two-dimensional eigenmode equation reads

$$\mathcal{L}_{||}(\widehat{\Phi}) = \mathcal{L}_\perp(\widehat{\Phi})\,, \tag{28}$$

where we have defined the following operators,

$$\mathcal{L}_{||} \equiv \frac{\partial^2}{\partial\theta^2}\;+\frac{q^2}{\epsilon_n^2}\;\Omega[\Omega A(\theta) + C(\theta)]\,, \tag{29}$$

and where $\Omega \equiv \omega/\omega_\star$ is the normalised mode frequency ($\omega_\star = c_{\mathrm{s}}/L_n$); $\epsilon_n \equiv L_n/R_0$ is the toroidicity parameter. Also we have

$$\begin{aligned}A(\theta) &\equiv 1 + bg(\theta) - i\delta - i(\rho_{\mathrm{s}}/L_n)(1+\eta_{\mathrm{e}})\sqrt{b}\widehat{s}\theta\\ C(\theta) &\equiv \sqrt{b}[1 - 2\epsilon_n f(\theta)]\,,\end{aligned} \tag{30}$$

where $f(\theta) = \cos\theta + \widehat{s}\theta\sin\theta$ is a combination of the normal and geodesic curvatures; this term arises because in an inhomogeneous magnetic field the divergence of the lowest-order $\mathbf{E}\times\mathbf{B}$ drift velocity does not vanish, $\nabla\,\boldsymbol{.}\,\mathbf{v}_E \simeq -2\mathbf{v}_E\,\boldsymbol{.}\,\nabla B/B$ (in the low-β approximation). The term in ρ_{s}/L_n in equation (30) comes from the term $\nabla n_0\,\boldsymbol{.}\,\mathbf{v}_{\mathrm{pi}}$ which is formally smaller than other terms. We have retained it for sake of completeness. The operator on the right-hand side of equation (28) describes the radial structure of the eigenfunction

$$\begin{aligned}\mathcal{L}_\perp \equiv &\left[2i\epsilon_n\Omega\sin\theta - \Omega^2\left(\rho_{\mathrm{s}}/L_n(1+\eta_{\mathrm{e}}) + 2i\sqrt{b}\widehat{s}\theta + i\sqrt{b}(\widehat{s}+\widehat{c})\theta - \frac{1}{x}\right)\right]\frac{\partial}{\partial x}\\ &+ \Omega^2\frac{\partial^2}{\partial x^2}\,.\end{aligned} \tag{31}$$

We recall that $x \equiv r/\rho_s$ is the normalised radial coordinate.

(3a) Parallel Mode Structure

To lowest order in the smallness parameter $\widehat{\epsilon} \equiv 1/N \ll 1$, we can neglect the right-hand side of equation (28). The envelope of the mode along the magnetic field line, $\widehat{\varphi}(\theta)$, is governed by the usual eigenmode equation

$$\frac{d^2\widehat{\varphi}}{d\theta^2} + \frac{q^2}{gre_n^2} Q(\Omega, \theta)\widehat{\varphi} = 0 \,, \tag{32}$$

where $Q(\Omega, \theta) = A(\theta)\Omega^2 + C(\theta)\Omega$ is the 'effective' potential in the Schrodinger-like equation. Here we note that the $i\delta$ correction due to the nonadiabatic electron population is responsible for the instability. In other words, if $\delta = 0$, both the eigenfunction and the eigenvalue(s) are purely real quantities in equation (32). For small finite δ, the real part of the mode frequency, $\Omega_r \equiv \Re(\Omega)$, will be much larger than its imaginary part, $\gamma \equiv \Im(\Omega)$. In fact, it is not difficult to show that $\gamma \approx \Omega_r\delta \ll \Omega_r$. We exploit the smallness of δ to solve equation (32) perturbatively. In unsheared slab geometry, the mode frequency is obtained by setting $\theta = \epsilon_n = 0$ and by solving $Q = 0$. This yields $\Omega_{\mathrm{SLAB}} = -\sqrt{b}/(1 + b - i\delta)$ or, in physical units, we recover the well-known result (Manheimer and Lashmore-Davies 1989)

$$\omega_{\mathrm{SLAB}} = \frac{\omega_{\star e}}{1 + (k_y\rho_s)^2 - i\delta} \,, \tag{33}$$

where $\omega_{\star e} = -k_y\rho_s c_s/L_n$ is the electron diamagnetic drift frequency. We can estimate the effect of finite toroidicity by solving the local dispersion relation $Q = 0$ with $\epsilon_n \neq 0$. It is easy to show that the mode frequency is weakly affected since $\Omega \simeq \Omega_{\mathrm{SLAB}}(1 - 2\epsilon_n) \approx \Omega_{\mathrm{SLAB}}$. We note that in the edge region of medium-size tokamaks and stellarators (Tsui *et al.* 1993) the toroidicity parameter is a small quantity; typically, $\epsilon_n \sim 10^{-2}$. We shall return to the smallness of ϵ_n in the next section. Following Cordey and Hastie (1977) we use the strong-coupling approximation (Horton *et al.* 1978). The effective potential can then be written as a quadratic polynomial $Q = Q_0 + Q_1\theta + Q_2\theta^2$, where

$$\begin{aligned} Q_0 &\simeq 2\epsilon_n \frac{b}{1+b} \,, \\ Q_1 &= -i\frac{\rho s}{L_n} \; (1 + \eta_e)\frac{b^{\frac{3}{2}}}{(1+b)^2}\widehat{s} \,, \\ Q_2 &\simeq \frac{b^2}{(1+b)^2} \; \left[\widehat{s}^2 + 2\epsilon_n \frac{1+b}{b} \; (\widehat{s} - \tfrac{1}{2})\right] . \end{aligned} \tag{34}$$

Note that Q_1 is roughly $\sqrt{\epsilon_n} \ll 1$ times smaller than Q_2 and, to first approximation (as it is usually done), we may neglect Q_1. Then one can write the eigenmode for $\widehat{\varphi}$ as

$$\frac{d^2\widehat{\varphi}}{d\theta^2} \; + (G - H\theta^2)\widehat{\varphi} = 0 \,, \tag{35}$$

where $G \equiv 2q^2 b/[\epsilon_n(1+b)]$ and $H \equiv -b^2q^2[\widehat{s}^2 + 2\epsilon_n(1+b)(\widehat{s}-\frac{1}{2})/b]/[\epsilon_n(1+b)]^2$ which, when the global magnetic shear is not too large, is a positive-definite quantity. It is convenient to introduce the transformation $\widehat{\varphi}(\theta) = \widehat{F}(\theta)\exp(-\sqrt{H}\theta^2/2)$ so that equation (35) becomes

$$\frac{d^2\widehat{F}}{d\theta^2} - \sqrt{H}\widehat{F} - 2\sqrt{H}\theta\frac{d\widehat{F}}{d\theta} = -G\widehat{F}\,. \tag{36}$$

Then, introducing the new variable $\bar{\theta} = H^{\frac{1}{4}}\theta$, we obtain

$$\frac{d^2\widehat{F}}{d\bar{\theta}}^2 - 2\bar{\theta}\frac{d\widehat{F}}{d\bar{\theta}} - (1-\Lambda)\widehat{F} = 0\,, \tag{37}$$

where $\Lambda \equiv G/\sqrt{H}$. Equation (37) is the Hermite differential equation. Therefore, the general solution for the parallel envelope of the mode can be written as

$$\widehat{\varphi}(\theta, x) = \sum_{l=0}^{\infty} H_l(H^{\frac{1}{4}}\theta)\exp\left(-\frac{\theta^2}{\theta_c^2}\right), \tag{38}$$

where $\theta_c \equiv \sqrt{2}/H^{\frac{1}{4}}$ and H_l is the Hermite polynomial of order l. We have indicated the (slow) radial dependence of the lowest-order eigenfunction (38) since H and θ_c both are functions of the normalised radial coordinate x.

(3b) Radial Mode Structure

We now consider the radial extension of the toroidal drift waves. First we note that the operator (29) is a differential operator in θ alone. Therefore, if $\widehat{\varphi}$ is solution of $\mathcal{L}_{||}(\widehat{\varphi}) = 0$, then, for arbitrary $A_0(x)$, we note that $\widehat{\Phi}_0(\theta, x) = A_0(x)\widehat{\varphi}$ is also a solution to $\mathcal{L}_{||}(\widehat{\Phi}_0) = 0$. Before going further, it is convenient to estimate the order of magnitude of the plasma parameters appearing in the equations. For concreteness, we consider the edge plasma of the Texas Experimental Tokamak (TEXT) (Gentle 1981). The parameters are: $a = 26$ cm (minor radius); $R = 100$ cm (major radius); $L_n = 3{\cdot}0$ cm (radial density scalelength); $B_0 = 2$ T (magnetic field strength at the magnetic axis); $q = 3$ (safety factor at $r = a$); and $T_e = 25$ eV (electron temperature). Experimental measurements show that the turbulence in the edge region of the TEXT tokamak has a long perpendicular wavelength; typically, $k_\perp\rho_s \simeq 0{\cdot}1$. With these parameters, the plasma edge ordering is

$$\epsilon_n \sim \rho_s/L_n \sim b \equiv (k_\theta\rho_s)^2 \sim 10^{-2} \ll 1\,. \tag{39}$$

Also the typical toroidal mode number is in the range $N = 100$ to $N = 200$, so that $\epsilon_n \sim \widehat{\epsilon}$ where $\widehat{\epsilon} \equiv 1/N$ is the expansion parameter. We expand the general eigenfunction in ascending powers of $\widehat{\epsilon}$,

$$\widehat{\Phi} = A_0(x)\widehat{\varphi}(\theta) + \widehat{\epsilon}A_1(x,\theta) + \widehat{\epsilon}^2 A_2(x,\theta) + \cdots\,, \tag{40}$$

and write the radial operator (31) as follows,

$$\mathcal{L}_{\perp} = \Omega(\mu^{(1)} + \mu^{(3/2)})\frac{\partial}{\partial x} + \Omega^2\frac{\partial^2}{\partial x^2}\,, \tag{41}$$

where, taking into account the ordering (39), we have $\Omega \approx \Omega_{\mathrm{r}} \simeq \sqrt{b}/(1+b) = \mathcal{O}(\epsilon^{\frac{1}{2}})$. Here

$$\mu^{(1)} = 2i\epsilon\sin\theta - i\Omega\sqrt{b}\theta(3\widehat{s} + \widehat{c})\,,$$

$$\mu^{(3/2)} = \Omega\left[\frac{1}{x} \quad - \frac{\rho_{\mathrm{s}}}{L_n}\;(1+\eta_{\mathrm{e}})\right], \tag{42}$$

where the superscripts indicate the corresponding order in $\widehat{\epsilon}$. Assuming that A_1 varies slowly along the extended poloidal angle, $\partial A_1/\partial\theta \sim A_1\widehat{\epsilon}^2$, we arrive at the following equations:

$$\mu^{(1)}\frac{dA_0}{dx} + \Omega\frac{d^2A_0}{dx^2} = 0 \quad [\text{corrections } \mathcal{O}(\widehat{\epsilon}^2)]\,, \tag{43}$$

$$A_0\mu^{(1)}\frac{\partial\widehat{\varphi}}{\partial x} = \widehat{\varphi}\mu^{(3/2)}\frac{dA_0}{dx} = 0 \quad [\text{corrections } \mathcal{O}(\widehat{\epsilon}^{\frac{5}{2}})]\,. \tag{44}$$

We solve equations (43) and (44) to obtain

$$\frac{1}{A_0}\;\frac{d^2A_0}{dx^2} = \left\langle \frac{1}{\Omega}\frac{[\mu^{(1)}]^2}{\mu^{(3/2)}}\frac{\partial}{\partial x}\;\ln\widehat{\varphi}\right\rangle, \tag{45}$$

where $\langle\bullet\rangle$ denotes an average over the extended poloidal angle. The definition of $\langle\bullet\rangle$ is given below. It is easy to show that the right-hand side of equation (45) is a negative quantity so that $A_0(x)$ decreases in the radial direction. Equation (45) has two important characteristics that one would expect from A_0; first, the width of A_0 is related to the radial variation of the parallel envelope of the mode; second, A_0 depends on the magnetic shear (and also on the scalar magnetic curvature $\widehat{c}$) through $\mu^{(1)}$ and θ_{c} in the envelope $\widehat{\varphi}$. Although the general solution (38) can be substituted in equation (45), we consider the $l=0$ mode. We note that the $l=0$ mode is the least stable mode since the $l\neq 0$ modes respond more easily to the stabilising effect of the magnetic shear. Therefore, substituting $\varphi = \varphi_0\exp(-\theta^2/\theta_{\mathrm{c}}^2)$ in equation (45) and defining the 'parallel average' of $F(x,\theta)$ as

$$\langle F\rangle(x) \equiv \frac{1}{2\theta_{\mathrm{c}}}\;\int_{-\theta_{\mathrm{c}}}^{+\theta_{\mathrm{c}}} F(x,\theta')d\theta'\,, \tag{46}$$

we obtain after some algebra

$$\frac{1}{A_0}\;\frac{d^2A_0}{dx^2}\;\simeq \frac{(1+b)^2}{b}\;\frac{L_n}{(1+\eta_{\mathrm{e}})a}\;\mathcal{U}(x)\mathcal{W}(x)\,, \tag{47}$$

$$\mathcal{U}(x) \equiv \widehat{s} + \frac{[\widehat{c} + \widehat{s}(1-\widehat{s})][\widehat{s} + \epsilon_n(1+b)/b]}{[\widehat{s}^2 + 2\eta_n(1+b)(\widehat{s}-1/2)/b]}\,, \tag{48}$$

$$\mathcal{W}(x) \equiv 2\eta_n^2\alpha_1 + \frac{(\theta_c b)^2}{5(1+b)^2}\;(3\widehat{s}+\widehat{c})^2 + 2\frac{\epsilon_n b}{\theta_c(1+b)}\;(3\widehat{s}+\widehat{c})\alpha_2\,, \tag{49}$$

where

$$\theta_c = \frac{2\epsilon_n(1+b)}{\sqrt{bq}|\widehat{s}^2 + 2\epsilon_n(1+b)(\widehat{s}-\frac{1}{2})/b|^{\frac{1}{4}}}\,, \tag{50}$$
$$\alpha_1 = \int_{-1}^{+1} y^2\sin(\theta_c y)dy\,,$$
$$\alpha_2 = \int_{-1}^{+1} y^3\sin(\theta_c y)dy\,.$$

The implications of equation (47) for the cross-field anomalous transport are discussed in the next section.

4. Discussion and Conclusion

We can estimate the *effective* radial wavevector in the expression for $D_\perp$, equation (1), from equation (47):

$$\frac{1}{A_0}\,\frac{d^2A_0}{dx^2} \sim [k_r^{(\mathrm{eff})}]^2\rho_s^2\,. \tag{51}$$

In the context of the simple $i\delta$ model, the growth rate in equation (1) is (in unnormalised units)

$$\gamma = \omega_\star\delta\frac{\sqrt{b}}{1+b}\,. \tag{52}$$

If one adopts Kadomtsev's (1965) estimate, $k_r^{(\mathrm{eff})} \sim k_r \approx \sqrt{b}/\rho_s$, one gets the following anomalous cross-field diffusion coefficient:

$$D_\perp \approx \frac{\omega_\star\delta\rho_s^2}{\sqrt{b}(1+b)}\;(\mathrm{Kadomtsev})\,. \tag{53}$$

If one uses equation (1) for $D_\perp$, then one multiplies equation (53) by the enhancement factor $\mathcal{F} \equiv [k_r/k_r^{(\mathrm{eff})}]^2$. Using equation (51) and taking into account (47)–(49) we obtain the enhancement factor

$$\mathcal{F} = (1+\epsilon e)\frac{a}{L_n}\frac{b^2}{(1+b)^2}\;\frac{1}{\mathcal{U}(\widehat{s},\widehat{c})}\;\frac{1}{\mathcal{W}(\widehat{s},\widehat{c})}\,, \tag{54}$$

where the functions $\mathcal{U}$ and $\mathcal{W}$ are given by equations (48) and (49), respectively. Equation (54) is the main result of this paper. To summarise our approach, we have included a radial dependence in the amplitude $\widehat{\Phi}$, and the eikonal representation for the fluctuating electrostatic potential is given by equation (8). The eikonal representation (8) is then substituted in the model equation (18) and a two-dimensional eigenmode equation (28) is readily derived. To lowest order we obtain an eigenmode (32) describing the *parallel* (along the field line) structure of the mode. Then expanding the *general* eigenfunction in ascending powers of $\widehat{\epsilon}$, the radial envelope of the mode $A_0(x)$ is obtained (45), and the *effective* radial wavenumber of electron drift waves in toroidal geometry can be calculated. We would like to point out that the enhancement factor (54) depends strongly on the global magnetic shear $\widehat{s}$, as well as the scalar magnetic curvature $\widehat{c}$, which is related to the second-order derivative of the safety factor.

Let us estimate the enhancement factor for the TEXT edge plasma parameter. In the edge plasma of the TEXT tokamak (Gentle 1981), we have $\widehat{s} \sim \widehat{c} = \mathcal{O}(1)$. Then using equations (39) and (48)–(50), we obtain

$$\mathcal{F} \approx \frac{b}{\epsilon_n (L_n/a)} \approx \frac{a}{L_n} \simeq 10 \;[\text{TEXT edge plasma}]\,. \tag{55}$$

We conclude that the radial extension of (ion and electron) drift waves in toroidal geometry is an important quantity to calculate the cross-field anomalous transport. Using a different approach, Romanelli and Zonca reached a similar conclusion for the ITG mode. We are currently working on a more detailed model for the nonadiabatic electron response (treated as a $i\delta$ term in this paper). We expect to report our estimates for $D_\perp$ in a separate paper.

Acknowledgments

This work was initiated at Chalmers Tekniska Hogskola, Gothenburg, Sweden; the author would like to thank J. Weiland, A. Bondenson, H. Nordman, A. Jarmen and M. Persson for their hospitality. Helpful comments provided by R. Singh are acknowledged with gratitude. The author was supported by a Natural Sciences and Engineering Research Council of Canada (NSERC) Research Fellowship.

References

Antonsen, T. M., and Lane, J. B. (1980). *Phys. Fluids* **23**, 1205.
Boozer, A. H. (1981). *Phys. Fluids* **24**, 1999.
Boozer, A. H. (1982). *Phys. Fluids* **25**, 520.
Braginskii, S. I. (1965). ‘Review of Plasma Physics’, Vol. 1 (Consultants Bureau: New York).
Cordey, J. G., and Hastie, R. J. (1977). *Nucl. Fusion* **17**, 523.
Gentle, K. W. (1981). *Nucl. Tech. Fusion* **1**, 479.
Horton, W. (1989). *Phys. Fluids* B **1**, 524.
Horton, W., Estes, R., Kwak, H., and Choi, D. (1978). *Phys. Fluids* **21**, 1366.
Kadomtsev, B. B. (1965). ‘Plasma Turbulence’ (Academic: New York).
Lewandowski, J. L. V. (1997*a*). *Phys. Plasmas* **4**, 4023.
Lewandowski, J. L. V. (1997*b*). *J. Phys. Soc. Japan* **66**, 3831.
Liewer, P. C. (1985). *Nucl. Fusion* **25**, 543.
Manheimer, W. D., and Lashmore-Davies, C. N. (1989). ‘MHD and Microinstabilities in Confined Plasma’ (Hilger: Bristol).
Romanelli, F., and Zonca, F. (1993). *Phys. Fluids* B **5**, 4081.

Tang, W. M. (1978). *Nucl. Fusion* **18**, 1089.
Tsui, H. Y. W., Wootton, A. J., Bell, J. D., Bengston, R. D., Diebold, D., Harris, J. H., Hershkowitz, N., Hidalgo, C., Ingraham, J. C., Kilpatrick, S. J., Li, G. X., Lin, H., Manos, D. M., Meier, M. A., Miller, G. M., Munson, C. P., Pew, J., Prager, S. C., Ritz, Ch. P., Rudyj, A., Schoenberg, K. F., Sorensen, J., Tanaka, T., Uckan, T., and Weber, P. G. (1993). *Phys. Fluids* B **5**, 24916.
Wagner, F., and Stroth, U. (1993). *Plasma Phys. Controll. Fusion* **35**, 1321.

Manuscript received 10 September 1998, accepted 5 January 1999