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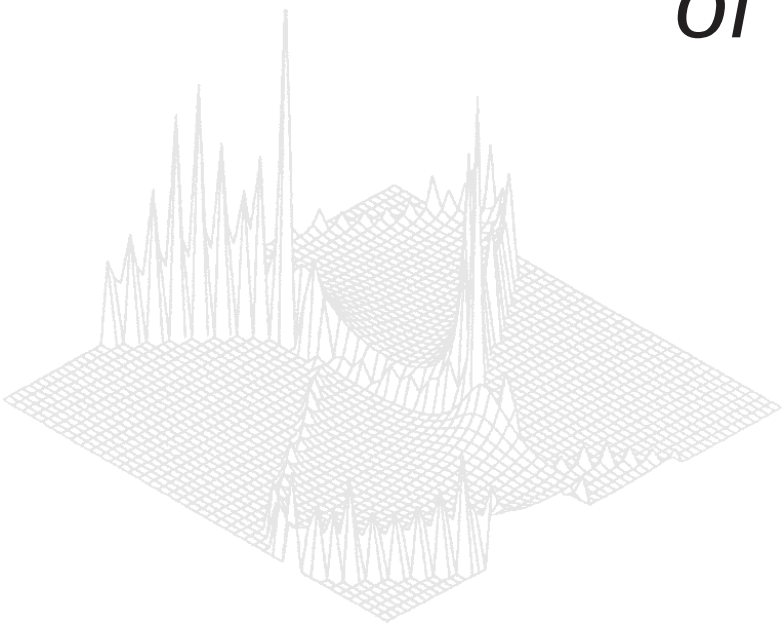

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Position Indeterminacy in Ortho-positronium Decay

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Abstract

Position indeterminacy contributions to the decay width of ortho-positronium (o-Ps) are calculated for lowest order. Contributions improve agreement between theory and the Ann Arbor group measurements, while suggesting a value larger than that of the Tokyo group.

1. Introduction

The possibility of position indeterminacy to explain a possible discrepancy between theory and experiment for the Lamb shift of low-lying hydrogen-like atoms is examined in an accompanying paper (Ruzzene 2000; present issue p. 631). An approximate $(Z\alpha)^6$ term is introduced to the classic Lamb shift, giving position indeterminacy contributions of the same magnitude as the accuracy of current QED theory and experiment.

The position indeterminacy length is given by the relation

$$\delta = \hbar c \Delta E / 2 E_0^{\,2} \,. \tag{1}$$

This expression is obtained by imposing a maximal acceleration (Caianiello 1984) on the dynamics of the uncertainty relations. Since the quantum mechanical particle position cannot then be precisely defined, the momentum operator is a difference operator, i.e.

$$p_i \Psi = -i\hbar \Delta_i \Psi = -i\hbar [\partial/\partial x_i + \delta \partial^2/\partial x_i^{\,2}/2 + \delta^2 \partial^3/\partial x_i^{\,3}/6 + \ldots] \Psi \,, \tag{2}$$

where Δ_i and p_i are the finite and momentum difference operators respectively. Position indeterminacy contributions are then calculated treating the second and higher order terms in standard perturbation theory.

In this work position indeterminacy contributions to the decay width of o-Ps are calculated to lowest order. The experimental situation for o-Ps is interesting, but ambiguous. Measurements by two different groups are currently in disagreement. The Ann Arbor group (Nico *et al.* 1990) gave the two values 7.0514(14) μs^{-1} and 7.0482(16) μs^{-1} for gas and vacuum measurements respectively. A more recent measurement by the Tokyo group gives the result of 7.0398(29) μs^{-1} (Czarnecki *et al.* 1999). Clarification of this experimental situation is reported to be in progress. The current theoretical prediction for the lifetime is 7.039970(10) μs^{-1} (Adkins *et al.* 2000; Kniehl *et al.* 2000). Should the Ann Arbor results be confirmed, agreement arising from higher order QED is expected to be most unlikely.

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For atomic bound states the energy change in the expression for position indeterminacy is calculated as the rms excitation energy (Bethe and Salpeter 1957). It is then of the order of the Rydberg energy. Accordingly, for hydrogen-like bound states the position indeterminacy length per Bohr radius is of order α^3. Contributions of this magnitude are too small for the current precision of the o-Ps decay width.

However, positronium decay at lowest order occurs by electron–positron annihilation at contact (Ore and Powell 1949). For annihilation the energy change in relation (1) would be the positronium rest mass energy at reduced mass. The position indeterminacy length per Bohr radius is then of order α. Contributions of this size would be of a magnitude similar to the current discrepancy.

The non-relativistic model of Ore and Powell will be used in this work. They calculated the decay rate as a product of the annihilation cross section for free electron–positron pairs, times an effective flux of particles at contact (Adkins 1983; Stroscio 1975). The probability of the particles at contact is taken as the positronium Schrödinger wavefunction at the origin.

2. Calculations

The complete decay matrix element takes the form (Alekseev 1959; Stroscio and Holt 1974)

$$A = \int \psi(p)\, M(p)\, dp\,, \tag{3}$$

where $M(p)$ relates to the three-photon annihilation of ortho-positronium and $\psi(p)$ is the positronium wavefunction in momentum space. At lowest order the matrix element reduces to $\phi(0)M(0)$, which is just the Ore and Powell expression. The function $\phi(0)$ is the coordinate space Schrödinger wavefunction at contact, i.e.

$$\phi(0) = \int \phi(\boldsymbol{p})\, d\boldsymbol{p}\,. \tag{4}$$

Position indeterminacy contributions arising from modification of this wavefunction will be obtained in this work.

The Schrödinger equation in coordinate space transforms into an integral equation in momentum space (Salpeter 1951)

$$(p^2 + \gamma^2)\, \phi(\boldsymbol{p}) = -2m \int d^{(3)}p'V\,(\boldsymbol{p} - \boldsymbol{p'})\, \phi(\boldsymbol{p'})\,, \tag{5}$$

where $V(\boldsymbol{p})$ and $\phi(\boldsymbol{p})$ are the p-space potential and wavefunction respectively, and where natural units are used and $\gamma^2 = -2mE$. The mass in this equation is the positronium reduced mass which is half that of the electron. Position indeterminacy is incorporated by replacing p^2 on the left-hand side by P^2, i.e. the momentum difference operator in second order.

The second order momentum difference operator can be written as the following series relation (see the Appendix):

$$P_i^{\,2} = p_i^{\,2}\,(1 + i\delta p_i/\hbar - 7\delta^2 p_i^{\,2}/12\hbar^2 + \ldots)\,, \tag{6}$$

where p_i is the usual momentum operator of standard quantum mechanics, i.e. the zero order term in (2). The positronium reduced rest mass is expressed as $\mu = m_e/2$. The at-

contact energy, being the reduced mass energy for annihilation, is given by $\Delta E = \mu c^2$. It is convenient to express the momentum in units of the Bohr value (i.e. $p_0 = \hbar/2a_0$), which results in terms of $p_0\delta/\hbar$ in equation (6). This term equals $\alpha/2$, where α is the fine structure constant. Relation (6) then becomes a series in the fine structure constant, i.e.

$$P_i^2 = p_0^2\, p_i^2(1 + i\alpha p_i/2 - 7\alpha^2 p_i^2/48 + \ldots)\,, \tag{7}$$

where $i = x, y, z$ and where p_i is redefined in units of the Bohr momentum. The positronium wavefunction, as modified by relation (7), is then obtained from equation (5) by iteration (Salpeter 1951; Bethe and Salpeter 1957). Using the standard ground-state positronium wavefunction as the initial trial function, the resulting modified wavefunction is

$$\phi(p) = N_0/\{(p^2 + 1)[(p^2 + 1) + i\alpha\Sigma p_i^3 - 7\alpha^2\Sigma p_i^4/48 + \ldots]\}\,, \tag{8}$$

where N_0 is the normalisation constant, p^2 relates to the radial momentum per Bohr momentum and summation is over the three coordinate directions. To more readily evaluate the momentum integral a series expansion via the binomial theorem is appropriate, giving

$$\phi(p) = N_0(p^2 + 1)^{-2}(1 - \beta + \beta^2 - \beta^3 + \beta^4 - \ldots)\,, \tag{9}$$

where

$$\beta = (i\alpha\Sigma p_i^3/2 - 7\alpha^2\Sigma p_i^4/48 + \ldots)/(p^2 + 1)\,. \tag{10}$$

This expression reduces to the standard ground-state positronium wavefunction if the fine structure terms are ignored. Position indeterminacy contributions are obtained by evaluating the integral of (5) for the modified wavefunction (9). Only even powers in p_i give nonzero contributions. It is convenient to work in polar coordinates (Bethe and Salpeter 1957).

For the fine structure terms in (9), the leading radial momentum integral terms are of the form

$$\alpha^n \int p^{n-2}\,dp = (\alpha^n p_{\text{cutoff}}^{n-1})/(n-1). \tag{11}$$

Introducing the Bethe relativistic cutoff momentum (Bethe and Salpeter 1957) as μc gives p_{cutoff} to be $1/\alpha$, resulting in terms of order α. This relativistic momentum cutoff is also necessary for ensuring that $|\beta| < 1$ to enable the binomial series expansion of (9). To leading term contributions, the modified lowest order decay rate becomes

$$\Gamma_0^{\text{modified}} = \Gamma_0[1 + (\alpha/\pi)0.19585]\,. \tag{12}$$

Using a lowest order decay rate of 7.2112 μs^{-1} (Caswell *et al.* 1977), position indeterminacy gives an additional correction of 0.00328 μs^{-1}, giving a net theoretical value of 7.04325 μs^{-1}.

Further position indeterminacy contributions will also arise from additional standard quantum-mechanical terms. Higher order contributions to the decay rate are obtained by using the Bethe–Salpeter positronium wavefunction, thereby introducing corrections relativistic in magnitude. Additional position indeterminacy terms, as discussed below, will

then be introduced via the momentum operator in the Bethe–Salpeter equation. Furthermore, the modified ground-state positronium wavefunction (9) is no longer even. Terms that previously gave zero contribution to the standard positronium wavefunction in the complete decay matrix element are then nonzero for the modified function (Stroscio and Holt 1974).

The modified at-contact positronium wavefunction of (9) also gives position indeterminacy contributions to the para-positronium decay at lowest order. Lifetimes for p-Ps are 7989.476(13) μs^{-1} and 7990.9(1.7) μs^{-1} for theory and experiment respectively. Position indeterminacy gives a correction of 3.6 μs^{-1}. This modifies the net theoretical value to 7993.1 μs^{-1}, which is slightly outside experimental uncertainty. However, as with the ortho-positronium, position indeterminacy contributions will also arise from higher order terms. Decay of para-positronium differs from the ortho-positronium by decaying to two rather than three photons. Hence, remaining position indeterminacy contributions will likewise differ. Nevertheless, these initial calculations predict that a discrepancy between QED theory and experiment also exists for para-positronium and will become evident with greater precision in experiment.

The degree of precision in the latest o-Ps calculations suggests that higher order QED contributions are unlikely to resolve the current situation. Clearly no definitive conclusions can be made until the experimental situation is clarified. However, as stated by Adkins *et al.* (2000), the situation is now further complicated by the current disagreement between QED theory and the positronium hyperfine splitting (Hoang *et al.* 2000). Position indeterminacy contributions will also arise with the splitting. Whether such contributions can account for the current disagreement awaits further investigation.

Nevertheless, these difficulties between theory and experiment do invite consideration of non-QED contributions. Recently, an interesting non-QED explanation for the ortho-positronium decay has been suggested (Foot and Gninenko 2000). The discrepancy in the lifetime measurements is explained by oscillations of o-Ps into mirror matter ortho-positronium. Presently, this explanation does not take into account possible additional non-QED contributions. Rather, approximate calculations give possible threshold oscillation rates to the mirror matter that could explain differences in the different experimental measurements.

3. Conclusion

These initial calculations show that position indeterminacy gives lowest order contributions of the same magnitude as the current discrepancy between theory and the Ann Arbor measurements for ortho-positronium. We also predict that greater experimental precision of the para-positronium decay will show a discrepancy with the theory of quantum electrodynamics.

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Appendix

The series relation (6) is obtained by the following reasoning. For simplicity, only one dimension will be considered. Extension to three dimensions is straightforward. The difference and differential first order operators are related by the basic relation (Jordan 1965)

$$\Delta = (e^{\delta\,\partial/\partial x} - 1)/\delta\,. \tag{13}$$

By repeated application, and then expansion of the exponential function as a series, the second order difference operator becomes a series in the differential operators

$$\Delta^2 = \partial^2/\partial x^2 + \delta\partial^3/\partial x^3 + (7/12)\delta^2\partial^4/\partial x^4 + (1/4)\delta^3\partial^5/\partial x^5 + \ldots \tag{14}$$

The momentum difference operator at second order is expressed as

$$p^2 = (-i\hbar\Delta)^2 = -\hbar^2\Delta^2$$

$$= -\hbar^2[\partial^2/\partial x^2 + \delta\partial^3/\partial x^3 + (7/12)\delta^2\partial^4/\partial x^4 + (1/4)\delta^3\partial^5/\partial x^5 + \ldots]\,. \tag{15}$$

Introducing the standard momentum differential operator, i.e. $p^2 = -(\hbar^2\,\partial^2/\partial x^2)$, at lowest order in (13), together with corresponding higher powers for higher order terms, will give equation (6).

Further approximate calculations for positronium, both for the decay and hyperfine splitting, require including position indeterminacy terms, as given by (6), into the Bethe–Salpeter equation. Various perturbation techniques can then be used. Positronium decay contributions can be calculated by following the method of Adkins (1983) in obtaining a Bethe–Salpeter wavefunction perturbatively. For possible examination of the hyperfine splitting, the Bethe–Salpeter equation, including position indeterminacy terms, can be solved to either the equivalent Schrödinger (Caswell and Lepage 1978) or the Dirac

(Lepage 1977) wavefunction. Alternatively, a Foldy–Wouthuysen transformation of the Bethe–Salpeter equation, containing position indeterminacy terms, can be done (Barker and Glover 1955; Pachucki 1997). This approach leads to an effective Hamiltonian. Standard perturbation methods can then be used to obtain contributions arising from the additional position indeterminacy terms. These further calculations, although still approximate, can examine the question raised by Adkins, namely, whether the same non-QED mechanism can explain the possible discrepancy in both the decay and hyperfine splitting.

A complete treatment requires examining the manifest covariance of position indeterminacy. Apart from the mathematical question of replacing differential by difference operators, two conceptual issues arise. Firstly, the question of what the consequence, if any, position indeterminacy has on the time coordinate. A further question arises about a basic assumption of quantum field theory (Weinberg 1995), namely, in field theory the creation and destruction of particles is assumed to occur at a space–time point. Position indeterminacy imposes an indeterminacy length, at least in the space coordinate, to the particle creation–annihilation process.

The question arises as to whether in a complete covariant form the position indeterminacy corrections obtained in these non-relativistic approximate calculations, will cancel, as happens with renormalisation in QED. In QED, infinities occur in integrals over the virtual quanta momentum. These terms are then renormalised into the mass (and charge) of the electron. Infinite terms are found to cancel in summations. The singularity obtained here is of a different type. It arises for the momentum of the positronium two-body system at reduced mass. However, the reduced mass is the ‘experimental’ mass not the ‘mechanical’ mass of QED. Renormalisation in the sense of QED does not apply. Applying a relativistic cutoff momentum in this non-relativistic calculation seems plausible. Initial non-relativistic calculations of the Lamb shift use the same procedure of a relativistic cutoff momentum after renormalisation. Subsequent relativistic calculations, although not using manifestly covariant perturbation, resolved the relativistic cutoff question (Kroll and Lamb 1949).

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