CALCULATION OF ACCURACY OF RESULTS OF GRAPHICAL SQUARE INTERCOMPARISONS

By P. M. GILET* and G. S. WATSON†

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Summary

Graphical squares of two types are used for calculating the results of intercomparisons of standards of length, mass, and other quantities, and the results of calibration of scales. By means of graphical squares, estimates of the true values or " improved " values are obtained from observations. From the differences between the improved and observed values, the residuals, a measure of the accuracy of the intercomparison, may be obtained. In this paper expressions are derived which enable the calculation, from the residuals, of accuracies of observed values, of improved values, and of various quantities derived from the improved values.

I. INTRODUCTION ‡

In the maintenance of standards, much use is made of intercomparisons by which several nominally equal standards are compared in all possible pairs. The results of such comparisons in pairs are used to form a so-called graphical square. The graphical square enables the calculation by simple arithmetic of improved values for the differences between the standards. Such a graphical square is illustrated by Johnson (1923, Fig. 19).

From the residuals obtained by subtracting the observed value for each difference between a pair of standards from the corresponding improved value, it should be possible, of course, to calculate the accuracy both of the observed values and of the improved values. Johnson, however, gives no indication of any method whereby the accuracies can be calculated from the residuals.

One of the uses of the type of square mentioned above is the intercomparison of the main divisions of a standard scale. However, this type of square is not suitable for the calibration of the subdivisions of the main divisions of a scale,

* Division of Metrology, C.S.I.R.O.; present address: Department of Mechanical Engineering, University of Melbourne.

† Department of Statistics, University of Melbourne.

‡ The expressions for the variances given in this paper were originally derived by P. M. Gilet from first principles using very little mathematical statistics. Acknowledgments for assistance in this part of the work are given at the end of this paper. At a later stage G. S. Watson derived the same expressions by using various standard theorems in mathematical statistics, with a consequently great shortening of the derivations. These shorter derivations are those given in this paper together with the further formulae (**2.6**) and (**3.8**) and simpler estimates for the second type of square. For the benefit of those who may prefer the derivations requiring little mathematical statistics, a limited number of copies of the paper in its original form are available from the Division of Metrology, C.S.I.R.O.

and for this work a second type of graphical square is used, resulting from the comparison of each subdivision of one main division with each subdivision of another main division. Such a square of the second type is shown in Figures 20 and 21 of the above-mentioned article (Johnson 1923).* But here again no mention is made of any method for calculating accuracies from the residuals.

The discussion in this paper will be confined mainly to the calibration of scales, circular and linear, but the expressions derived are applicable to intercomparisons of quantities of many kinds. For example, although one set of expressions has been derived by consideration of the calibration of the main divisions of a linear scale, the set of expressions is equally applicable to the results of comparing, say, four 3 in. slip gauges in combination against a known 12 in. end bar, and of intercomparing the four slip gauges in pairs, or of comparing, say, five 200 g. masses against a known kilogram and of intercomparing the five 200 g. masses in pairs.

II. Accuracy of Results from a Square of the First Type

(a) *Procedure for Use of Square*

The method of using a graphical square of the first type is given in detail by Johnson (1923) and for convenience is summarized below. The method is described in the above article for the case of 10 quantities, but in the following the general case is taken where there are n quantities intercompared in all possible pairs. Table 1 represents a partly completed square for n quantities. $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the true values of the quantities or, since we are usually comparing nominally equal quantities, the true departures from their nominal values. For definiteness, the first interpretation will be used below. $e_{12}, e_{13}, \ldots, e_{n-1\,n}$ are the errors in measurement made in the comparisons.

The observed differences between quantities, such as $y_{12}=\alpha_1-\alpha_2+e_{12}$ are entered, as shown, below the diagonal line, and again entered with a change of sign in the appropriate spaces above the diagonal.

The observed differences are added in columns and the sums entered in the M row. Each sum is divided by n and the resulting mean values are entered in the $\bar{M}$ row.

The square is now at the stage shown in Table 1, excluding the last two rows.

The improved value for a difference between any pair of quantities is given by the difference between the corresponding $\bar{M}$ values. It can be shown that, in fact, the graphical square gives a least squares solution for the $\frac{1}{2}n(n-1)$ observational equations. For example, $(\bar{M}_1-\bar{M}_2)$ is the best (least squares) value corresponding to the true difference $(\alpha_1-\alpha_2)$.

The values of these improved estimates, $(\bar{M}_1-\bar{M}_2)$, $(\bar{M}_1-\bar{M}_3)$, etc., are calculated and entered in the appropriate spaces of the square. Each improved estimate is subtracted from the corresponding observed difference and the resulting residual also entered in the appropriate space.

* An example of a square of this type is given in Table 3 of this paper.

These residuals, as stated by Johnson, are a measure of the accuracy of the whole comparison. In the following, expressions are derived enabling accuracies to be calculated from the residuals.

TABLE 1

SQUARE OF THE FIRST TYPE

	α_1	α_2		α_{n-1}	α_n
α_1		$-y_{12}$		$-y_{1\ n-1}$	$-y_{1\ n}$
α_2	y_{12}			$-y_{2\ n-1}$	$-y_{2\ n}$
α_{n-1}	$y_{1\ n-1}$	$y_{2\ n-1}$			$-y_{n-1\ n}$
α_n	$y_{1\ n}$	$y_{2\ n}$		$y_{n-1\ n}$	
M	M_1	M_2		M_{n-1}	M_n
$\overline{M}$	$\overline{M}_1$	$\overline{M}_2$		$\overline{M}_{n-1}$	$\overline{M}_n$
M adjusted for departure of overall length	$\overline{M}_1+\frac{A}{n}$	$\overline{M}_2+\frac{A}{n}$		$\overline{M}_{n-1}+\frac{A}{n}$	$\overline{M}_n+\frac{A}{n}$
Progressive total . .	$P_1=\overline{M}_1+\frac{A}{n}$	$P_2=\overline{M}_1+\overline{M}_2+2\frac{A}{n}$		$P_{n-1}=\overline{M}_1+\ldots+\overline{M}_{n-1}+(n-1)\frac{A}{n}$	$P_n=\overline{M}_1+\ldots+\overline{M}_n+A=A$

Reverting to the procedure for obtaining the results and leaving for the moment the matter of their accuracy, it will be noted that the procedure has been outlined to the stage of obtaining improved estimates of the differences between the quantities concerned. In many applications this is all that is necessary. For example, when several end bars, several masses, or several overall lengths of line standards have been intercompared, the aim is simply to obtain improved estimates of the differences between the quantities. One or more of the quantities is known, and the values for the others can be determined from the known values and the appropriate improved estimates of the differences.

It happens sometimes, however, that the sum of the quantities compared is known, but none of the individual quantities. This would happen, for example, when five unknown 200 g. masses have been compared with a known kilogram and then intercompared in pairs. The sum of the five 200 g. masses would then be known, as would the improved estimates of their differences. Again, in the calibration of the main divisions of a line standard, the divisions are intercompared in pairs and improved estimates obtained for their differences. The overall length of the standard is known, and it is necessary to determine the value of each main division from the known sum of the main divisions (i.e. the overall length) and the improved estimates of the differences.

In such cases the square is extended as indicated in the last two rows of Table 1 where A is the value assumed for the sum of the quantities $\alpha_1, \alpha_2, \ldots, \alpha_n$.

In the example given above relating to five 200 g. masses, the improved value for each mass would be given by the appropriate space in the second last row of Table 1.

In the case of the line standard, the improved value for each main division would be given by the second last row, and the last row would give improved values for all the main intervals of the standard starting from one end.

If the procedure given by Johnson for the intercomparison of the main divisions of a line standard is followed exactly, it will be found that compartments of the square which are diagonally adjacent contain a common error and are not independent. If all the observations are repeated, it is possible to choose observations alternately from the two sets and combine them to form two squares in each of which the observed values are now wholly independent. The results from the two squares are used as a check upon each other.

The analysis by the methods set out below applies to either of the two squares formed as described.

(b) *Statistical Investigation of a Square of the First Type*

Denoting the observed differences between quantities whose true values are α_i and α_j by y_{ij}, it is seen from Table 1 that

$$y_{ij}=\alpha_i-\alpha_j+e_{ij}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2.1)$$

with $i=1, 2, \ldots, n-1$ and $j=i+1, i+2, \ldots, n$. Let it be supposed that all the errors e_{ij} are independent random variables with zero means and a common variance σ_0^2.

If the true value of the sum of all the quantities is known, A say, then the α's must be estimated from the observations subject to the restriction that $\sum_1^n \alpha_i=A$. The best linear unbiased estimates may be obtained by the method of least squares.*

* All the statistical concepts and theorems used below may be found, *inter alia*, in Mood (1950).

The normal equations are found by minimizing

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (y_{ij} - x_i + x_j)^2$$

by choice of $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $\sum_1^n \alpha_i = A$. They are easily seen to be

$$M_j = n\hat{\alpha}_j - \sum_1^n \hat{\alpha}_j \qquad (j=1, \ldots, n), \quad \ldots\ldots\ldots \text{(2.2)}$$

that is,

$$\hat{\alpha}_j = \bar{M}_j + \frac{A}{n}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.3)}$$

The residuals v_{ij} are defined by

$$v_{ij} = y_{ij} - \hat{\alpha}_i + \hat{\alpha}_j = y_{ij} - M_i + M_j, \quad \ldots\ldots\ldots\ldots \text{(2.4)}$$

and an estimate S_0^2 of σ_0^2 can be found by dividing the sum of squares of these residuals by the number of degrees of freedom associated with it. This number is given by the general rule :

degrees of freedom = (number of observations)
— (number of independent constants estimated).

In the present instance, there are

$$\frac{n(n-1)}{2} - (n-1) = \frac{(n-1)(n-2)}{2}$$

degrees of freedom. Thus the estimate of σ_0^2 is

$$S_0^2 = 2\frac{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} v_{ij}^2}{(n-1)(n-2)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(2.5)}$$

By using the equation (**2.4**) it may be verified that

$$\sum_{i=1}^{n-1} \sum_{=i+1}^{n} v_{ij}^2 = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} y_{ij}^2 - \frac{1}{n} \sum_{j=1}^{n} M_j^2. \quad \ldots\ldots \text{(2.6)}$$

If a calculating machine is being used, (**2.6**) leads to the most expeditious method for computing S_0^2.

It is now necessary to find the variances of the estimates of α's and functions of them. In their derivations, account must be taken of the fact that every pair of M_i's has a term in common.
Thus

$$\text{var } (\hat{\alpha}_i - \hat{\alpha}_j) = \frac{2\sigma_0^2}{n} \qquad \left.\begin{array}{l}(i=1, \ldots, n-1)\\(j=1+i, \ldots, n)\end{array}\right\}, \quad . \text{(2.7)}$$

$$\begin{aligned}\text{var } (P_r) &= \text{var } (\hat{\alpha}_1 + \ldots + \hat{\alpha}_r)\\ &= \frac{r(n-r)}{n^2}\sigma_0^2 \qquad (r=1, \ldots, n). \quad \ldots\ldots \text{(2.8)}\end{aligned}$$

In particular, (**2.8**) with $r=1$ gives

$$\text{var } (\hat{\alpha}_i) = \frac{n-1}{n}\sigma_0^2 \qquad (i=1, \ldots, n). \quad \ldots\ldots \text{(2.9)}$$

The case of a circular scale may be given as an example in which the sum of the unknown quantities is known exactly. Here the unknown quantities are the true values of the main divisions of the 360° used for a square of the first type.

For a linear scale or a group of equal masses the sum of the unknown quantities is never known exactly but merely through another measurement. Then A will be subject to error with variance σ_A^2, say. Under these circumstances, the best method of estimation is complicated. The most practical method is to use the above estimates of the α's (**2.3**) and take account of the error in A when finding the variance formulae. Then the procedure and results are exactly as before except (**2.8**). This becomes now

$$\text{var }(P_r)=\frac{1}{n^2}[r(n-r)\sigma_0^2+r^2\sigma_A^2] \qquad (r=1,\ldots, n). \quad \ldots . (\mathbf{2.10})$$

Again, in particular, (**2.10**) gives with $r=1$

$$\text{var }(\hat{\alpha}_i) =\frac{1}{n^2}[(n-1)\sigma_0^2+\sigma_A^2] \qquad (i=1,\ldots, n). \quad \ldots . (\mathbf{2.11})$$

To use (**2.10**) or (**2.11**) an estimate of σ_A^2 must be available from a previous investigation.

Naturally S_0^2 of (**2.5**) and (**2.6**) will be used for σ_0^2 in (**2.7**) to (**2.11**).

Thus, for a linear scale, the best estimates of the main divisions are found from a square as in Table 1 and the estimates of the accuracies of the quantities of interest are found from (**2.5**), (**2.6**), (**2.7**), (**2.10**), and (**2.11**).

For a circular scale the same square is used but (**2.8**) and (**2.9**) replace (**2.10**) and (**2.11**) in accuracy calculations. It will be noted that

$$\text{var }(P_{n-r})=\text{var }(P_r)$$

and that the maximum value of var (P_r) occurs for $r=n/2$ (n even) and $r=(n\pm1)/2$ (n odd). This is illustrated by the broken line curve of Figure 2.

When several nominally equal standards are compared, the last row of the square in Table 1 is omitted since it is devoid of interest.

III. Accuracy of Results from a Square of the Second Type

(*a*) *Description of Square*

This type of square is used when each subdivision of a main division of a scale has been compared with each subdivision of another main division.

Such a square is shown in Figures 20 and 21 of the previously mentioned article (Johnson 1923), and the method is described for the case of two decimetre divisions of a metre scale, each subdivided into 10 centimetre divisions.* In the following the general case is taken of a scale divided and numbered as in Figure 1.

The graphical square of Table 2 refers to the comparison of each of the m subdivisions of the $(P+1)$th main division with each of the m subdivisions of

* In Table 3 is given another example of a square of this type.

the $(Q+1)$th main division. The method of representation of divisions is such that, for example, the $(P+1)$th main division is represented by $P/P+m$, and, for example, the third subdivision of the $(Q+1)$th main division is represented by $Q+2/Q+3$.

TABLE 2

SQUARE OF THE SECOND TYPE

	$P/P+1$	$P+1/P+2$	$\ldots$	$P+m-1/P+m$	S	M	$-M+\frac{D_Q}{m}+\frac{G}{m^2}$
$Q/Q+1$	z_{11}	z_{21}		z_{m1}	R_1	$\bar{R}_1$	$\frac{G}{m^2}+\frac{D_Q}{m}-\bar{R}_1$
$Q+1/Q+2$	z_{12}	z_{22}		z_{m2}	R_2	$\bar{R}_2$	$\frac{G}{m^2}+\frac{D_Q}{m}-\bar{R}_2$
$\vdots$							
$Q+m-1/Q+m$	z_{1m}	z_{2m}		z_{mm}	R_m	$\bar{R}_m$	$\frac{G}{m^2}+\frac{D_Q}{m}-\bar{R}_m$
S	C_1	C_2		C_m	G		
M	$\bar{C}_1$	$\bar{C}_2$		$\bar{C}_m$		$\frac{G}{m}$	
$M+\frac{D_P}{m}-\frac{G}{m^2}$	$\bar{C}_1+\frac{D_P}{m}-\frac{G}{m^2}$			$\bar{C}_m-\frac{D_P}{m}-\frac{G}{m^2}$			
Progressive totals ..	P_1	P_2		P_m			

In connection with Table 2 the true lengths of the subdivisions of $P/P+m$ are represented by $\delta_1, \delta_2, \ldots, \delta_m$ and the true lengths of the subdivisions of $Q/Q+m$ are represented by $\theta_1, \theta_2, \ldots, \theta_m$. A typical entry in the table, z_{ij}, stands for the observed difference between the ith subdivision of $P/P+m$ and the jth subdivision of $Q/Q+m$. If the error in the comparison is e_{ij} then $z_{ij}=\delta_i-\theta_j+e_{ij}$.

The rows and columns marked S and M contain respectively the sums and means of observed differences, the grand total being shown as G. The second last row gives the mean values $\bar{C}_j$ adjusted to give the improved values of the lengths of the subdivisions of $P/P+m$ as derived below. It will be noted that the adjustment consists of the addition of D_P/m (where D_P is the estimate of the main division $P/P+m$ obtained from a square of the first type) and of the subtraction of G/m^2. G/m, being equal to $(\Sigma_i\hat{\delta}-\Sigma_j\hat{\theta})$, is the difference between the main divisions $P/P+m$ and $Q/Q+m$ as obtained from the square of Table 2. It can be shown that the adjustment for obtaining the second last row of Table 2 is identical with that given by Johnson (1923, Fig. 20).

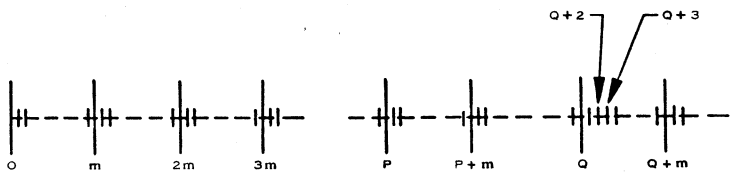

Fig. 1

The last column of Table 2 gives the mean values $\bar{R}_j$ adjusted to give the improved values of the subdivisions of $Q/Q+m$ as derived below. It can again be shown that the adjustment is identical with that given by Johnson. The last row of Table 2 gives progressive totals of the row above, i.e. the improved values for the intervals $P/P+1$, $P/P+2$, . . ., $P/P+m$ of the scale. The corresponding column has been omitted from Table 2.

If the procedure given by Johnson for obtaining the observed differences is followed exactly, the errors e_{11}, e_{12}, . . . will not be independent. As described in Section II, two intercomparisons can be made and alternate entries from the intercomparisons made into two squares such as Table 2. If this is done the errors e_{11}, e_{12}, . . . in each square are independent, the results from one of the squares giving a valuable check on the results of the other. The analysis given below is based on the assumption that the errors are independent.

(b) *Statistical Investigation of Square of Second Type*

Denoting the observed differences of the subdivisions by z_{ij}, the observations in Table 2 may be written as

$$z_{ij}=\delta_i-\theta_j+e_{ij}, \qquad (i,j=1, 2, \ldots, m) \quad \ldots . \quad \mathbf{(3.1)}$$

where the e_{ij}'s are assumed to be independent with zero means and a common variance σ_s^2. As in Section II, the most satisfactory method of estimation is least squares, modified to take account of the estimates of the lengths of the main divisions.

$\sum_1^m\delta_i$ and $\sum_1^m\theta_j$ represent the true lengths of the main divisions which will have been estimated from a square of the first type. Let the appropriate estimates be

$$D_P=\hat{\alpha}_{r+1}, \quad D_Q=\hat{\alpha}_t \qquad (t\neq r+1)$$

in the notation of Section II (*b*). Thus, $\delta_1, \delta_2, \ldots, \delta_m$ and $\theta_1, \theta_2, \ldots, \theta_m$ would ideally be estimated by least squares subject to the restrictions

$$\sum_1^m \delta_i = \hat{\alpha}_{r+1}, \quad \sum_1^m \theta_j = \hat{\alpha}_t.$$

The normal equations, obtained as before, are

$$\left.\begin{aligned} R_j &= \sum_1^m \hat{\delta}_i - m\hat{\theta}_j \qquad (j=1, 2, \ldots, m), \\ C_i &= m\hat{\delta}_i - \sum_1^m \hat{\theta}_j \qquad (i=1, 2, \ldots, m). \end{aligned}\right\} \quad \ldots (3.2)$$

It is evident that only one restriction can be imposed upon the solutions of (**3.2**). Let it be that

$$\sum_1^m \delta_i = \hat{\alpha}_{r+1}. \quad \ldots\ldots (3.3)$$

Then

$$\hat{\theta}_j = \frac{\hat{\alpha}_{r+1}}{m} - \bar{R}_j, \quad \ldots\ldots (3.4)$$

$$\hat{\delta}_i = \bar{C}_i + \frac{\hat{\alpha}_{r+1}}{m} - \frac{G}{m^2}. \quad \ldots\ldots (3.4')$$

If the restriction

$$\sum_1^m \theta_j = \hat{\alpha}_t \quad \ldots\ldots (3.5)$$

is used, the estimates are

$$\hat{\delta}_i = \bar{C}_i + \frac{\hat{\alpha}_t}{m}, \quad \ldots\ldots (3.6)$$

$$\hat{\theta}_j = \frac{G}{m^2} + \frac{\hat{\alpha}_t}{m} - \bar{R}_j. \quad \ldots\ldots (3.6')$$

The estimates used by Johnson are (**3.4′**) and (**3.6′**).

The residuals are given by

$$w_{ij} = z_{ij} - \hat{\delta}_i + \hat{\theta}_j,$$

where $\hat{\delta}_i$ and $\hat{\theta}_j$ are defined by (**3.4**) and (**3.4′**) or (**3.6**) and (**3.6′**). Thus

$$w_{ij} = z_{ij} - \frac{R_i}{m} - \frac{C_j}{m} + \frac{G}{m^2}. \quad \ldots\ldots (3.7)$$

These are the residuals obtained by the method of calculation given by Johnson. It may be shown that

$$\sum_{i=1}^m \sum_{j=1}^m w_{ij}^2 = \sum_{i=1}^m \sum_{j=1}^m z_{ij}^2 - \frac{\Sigma R_i^2}{m} - \frac{\Sigma C_j^2}{m} + \frac{G^2}{m^2}. \quad \ldots\ldots (3.8)$$

This sum of squares carries

$$m^2 - (2m-1) = (m-1)^2$$

degrees of freedom, so that the estimate of σ_s^2, S_s^2, is given by

$$S_s^2 = \frac{\sum_{i=1}^m \sum_{j=1}^m w_{ij}^2}{(m-1)^2}. \quad \ldots\ldots (3.9)$$

When computing (**3.9**) with a calculating machine, use should always be made of (**3.8**).

The variances of the estimates $\hat{\theta}_j$, $\hat{\delta}_i$ from (**3.4**) and (**3.4′**) are found to be

$$\text{var}\ (\hat{\theta}_j)=\frac{\text{var}\ (\hat{\alpha}_{r+1})}{m^2}+\frac{\sigma_s^2}{m}, \quad \ldots\ldots\ldots\ldots \quad (\mathbf{3.10})$$

$$\text{var}\ (\hat{\delta}_i)=\frac{\text{var}\ (\hat{\alpha}_{r+1})}{m^2}+\frac{(m-1)}{m^2}\sigma_s^2. \quad \ldots\ldots\ldots \quad (\mathbf{3.11})$$

When the estimates from (**3.6**) and (**3.6′**) are used, the terms containing σ_s^2 in (**3.10**) and (**3.11**) are interchanged. When the procedure described by Johnson is followed, both estimates have the variance

$$\frac{\text{var}\ (\hat{\alpha}_{r+1})}{m^2}+\frac{(m-1)}{m^2}\sigma_s^2.$$

The variance of the estimate $\hat{\delta}_1+\hat{\delta}_2+\ldots+\hat{\delta}_z=P_z$ of the sum of the lengths of the first z subdivisions is given by

$$\text{var}\ (P_z)=\frac{1}{m^2}[z(m-z)\sigma_s^2+z^2\ \text{var}\ (\hat{\alpha}_{r+1})]. \quad \ldots\ldots\ldots \quad (\mathbf{3.12})$$

IV. Accuracy of the Estimated Values of Intervals of a Scale, Each Interval Starting from Zero Graduation

The full procedure for the calibration of a scale (linear or circular) can now be summarized.

The scale is regarded as being composed of n main divisions $\alpha_1, \alpha_2, \ldots, \alpha_n$, each of which has m subdivisions.

The main divisions are intercompared and the results worked up in a square of the first type. This square gives estimates of $\alpha_1, \alpha_2, \alpha_3, \ldots$ and also of $\alpha_1, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3, \ldots$

Two divisions are chosen (the choice being made on non-statistical grounds) and the m subdivisions of one are compared with the m subdivisions of the other. These divisions are denoted by $P/P+m$ and $Q/Q+m$ so that the subdivisions are denoted by $P/P+1$, $P+1/P+2$, . . ., $P+m-1/P+m$ and $Q/Q+1$, $Q+1/Q+2$, . . ., $Q+m-1/Q+m$. In Section III, the true values of these subdivisions were denoted respectively by $\delta_1, \delta_2, \ldots, \delta_m$ and $\theta_1, \theta_2, \ldots, \theta_m$. Estimates of these subdivisions and their progressive totals were obtained in Section III from a square of the second type.

It remains to obtain estimates of the intervals $O/P+1, O/P+2, \ldots, O/P+m$ and $O/Q+1$, $O/Q+2$, . . ., $O/Q+m$, and expressions for accuracy of these estimates. Consider generally the estimation of $O/P+z$. If $P/P+m$ is the $(r+1)$th main division, then

$$O/P+z=(\alpha_1+\alpha_2+\ldots+\alpha_r)+(\delta_1+\delta_2+\ldots+\delta_z). \quad \ldots \quad (\mathbf{4.1})$$

So that, by (**2.3**) and (**3.4′**), estimate of

$$O/P+z=(\hat{\alpha}_1+\hat{\alpha}_2+\ldots+\hat{\alpha}_r)+(\hat{\delta}_1+\hat{\delta}_2+\ldots+\hat{\delta}_z)$$
$$=\left(\bar{M}_1+\bar{M}_2+\ldots+\bar{M}_r+r\frac{A}{n}\right)+\left(\bar{C}_1+\bar{C}_2+\ldots+\bar{C}_z+\frac{z\hat{\alpha}_{r+1}}{m}-\frac{zG}{m^2}\right). \quad \ldots\ldots (\mathbf{4.2})$$

Thus the estimate of $O/P+z$ is simply the sum of the two appropriate progressive totals in the squares of Tables 1 and 2.

Now

$$\text{var (est. of } O/P+z)=\text{var }(\hat{\alpha}_1+\hat{\alpha}_2+\ldots+\hat{\alpha}_r)+\text{var }(\hat{\delta}_1+\hat{\delta}_2+\ldots+\hat{\delta}_z)$$
$$+2\text{ cov }(\hat{\alpha}_1+\hat{\alpha}_2+\ldots+\hat{\alpha}_r,\ \hat{\delta}_1+\hat{\delta}_2+\ldots+\hat{\delta}_z). \quad \ldots\ldots (\mathbf{4.3})$$

The first two terms of (**4.3**) are given by (**2.10**) and (**3.12**).

To find the last term, it will be noted that

$$\text{cov }(\hat{\alpha}_1+\ldots+\hat{\alpha}_r,\ \hat{\delta}_1+\ldots+\hat{\delta}_z)=\frac{z}{m}\text{ cov }(\hat{\alpha}_1+\ldots+\hat{\alpha}_r,\ \hat{\alpha}_{r+1})$$
$$=\frac{rz}{m}\text{ cov }(\hat{\alpha}_1,\ \hat{\alpha}_{r+1})$$
$$=\frac{rz}{mn^2}(\sigma_A{}^2-\sigma_0{}^2).$$

Combining results,

$$\text{var (est. of } O/P+z)=\frac{1}{n^2}[r(n-r)\sigma_0{}^2+r^2\sigma_A{}^2]+\frac{1}{m^2}[z(m-z)\sigma_s{}^2+\frac{z^2}{n^2}\{(n-1)\sigma_0{}^2+\sigma_A{}^2\}]$$
$$+\frac{2rz}{mn^2}[\sigma_A{}^2-\sigma_0{}^2], \quad \ldots\ldots (\mathbf{4.4})$$

which may be rearranged to give

$$\text{var (est. of } O/P+z)=\frac{1}{m^2n^2}[\sigma_0{}^2\{(rm^2+z^2)(n-r-1)+r(m-z)^2\}$$
$$+\sigma_s{}^2n^2z(m-z)+\sigma_A{}^2(rm+z)^2]. \quad \ldots\ldots (\mathbf{4.5})$$

In the case of a circular scale where A is known exactly, the same result with $\sigma_A{}^2=0$ applies.

V. Summary of Symbols and Expressions relating to Accuracy for Graphical Squares of Both Types and for Combined Results from Squares of Both Types

(*a*) *Square of First Type*

Number of quantities intercompared	$=n$.
Number of comparisons	$=\frac{1}{2}n(n-1)$.*
Residual, i.e. observed difference minus improved estimate	$=v$.
Estimated variance of value used for sum of quantities intercompared	$=S_A{}^2$.

* Note that each of the comparisons must be statistically independent of the remainder.

Estimated variance for observed differences $=S_0^2=\dfrac{2\Sigma v^2}{(n-1)(n-2)}$.*

$$\Sigma v^2=\Sigma y^2-\frac{1}{n}\Sigma M^2.$$

Estimated variance for improved estimates of differences $=\operatorname{var}(\hat{\alpha}_i-\hat{\alpha}_j)=\dfrac{2}{n}S_0^2$.

Estimated variance for improved estimate of each quantity $=\operatorname{var}(\alpha_i)=\dfrac{1}{n^2}[S_A^2+(n-1)S_0^2]$.

Estimated variance for progressive total of improved estimates of quantities up to and including the rth quantity $=\operatorname{var}(P_r)=\dfrac{1}{n^2}[r^2S_A^2+r(n-r)S_0^2]$.

(b) *Square of Second Type*

Number of subdivisions of a main division $=m$.

Number of comparisons $=m^2$.†

Residual, i.e. observed difference minus improved estimate $=w$.

Estimated variance for improved estimate of a division from a square of the first type $=\operatorname{var}(\hat{\alpha}_{r+1})=\dfrac{1}{n^2}[S_A^2+(n-1)S_0^2]$.

Estimated variance for observed differences between subdivisions $=S_s^2=\dfrac{1}{(m-1)^2}\Sigma w^2$.

$$\Sigma w^2=\Sigma z^2-\frac{1}{m}\Sigma R^2-\frac{1}{m}\Sigma C^2+\frac{G^2}{m^2}.$$

Estimated variance for improved estimates of differences between subdivisions $=\operatorname{var}(\hat{\theta}_j-\hat{\delta}_i)=\dfrac{1}{m^2}(2m-1)S_s^2$.

Estimated variance for improved value of each subdivision $=\operatorname{var}(\theta_j)=\operatorname{var}(\delta_i)$

$$=\frac{1}{m^2}\operatorname{var}(\hat{\alpha}_{r+1})+\frac{1}{m^2}(m-1)^2S_s^2.$$

Estimated variance for progressive total of subdivisions up to and including the zth subdivision $=\operatorname{var}(P_z)$

$$=\frac{1}{z^2}[z^2\operatorname{var}(\hat{\alpha}_{r+1})+z(m-z)S_s^2].$$

(c) *Intervals from Zero Graduation*

Estimated variance for value of interval, $O/P+z$, this value being obtained by adding the values for the first r main intervals, to give O/P, and the value for the subinterval $P/P+z$ $=\operatorname{var}(\text{est. of } O/P+z)$

$$=\frac{1}{m^2n^2}[S_0^2\{(rm^2+z^2)(n-r-1)+r(m-z)^2\}+S_s^2n^2z(m-z)+S_A^2(rm+z)^2].$$

* Note that the number of v's used in obtaining Σv^2 is $\frac{1}{2}n(n-1)$, i.e. the number is that of the comparisons made and not the number of compartments in the square which is $n(n-1)$.

† Note that each comparison must be statistically independent of the remainder.

VI. Example of Graphical Square Method applied to the Calibration of a Circular Scale

The graphical square method of calibration was used some time ago in the calibration of the scale of the 40 in. circular dividing engine of the Division of Metrology made by the Société Génevoise de Physique.

The first stage was the intercomparison of the twelve 30° divisions of the scale, using a graphical square of the first type for calculating the results. The second stage was the comparison of each 5° subdivision of one 30° division with each 5° subdivision of another 30° division. This was done six times to cover the six pairs of 30° divisions, the results being calculated using 6×6 squares of the second type. The third stage was the comparison of each 1° subdivision of one 5° division with each 1° subdivision of another 5° division. This was done 36 times, the results being calculated using 5×5 squares of the second type.

For this example, only the first and second stages will be considered, and the square of the first type used for the intercomparison of the 30° divisions is not reproduced, the method being quite simple. It is considered advisable to reproduce one of the squares of the second type.

The square shown in Table 3 is one of the squares used for the second stage of the calibration. The complete square is shown, in the form given by Johnson. Comparing Table 3 with Table 2, it will be seen that the additional items shown in Table 3 are improved estimates (at the top of each compartment), residuals (at the bottom of each compartment), a column corresponding to the last row of Table 2, and a row and a column giving values for the intervals from zero and making use of the values obtained for the intervals 0/30 and 0/210 with the square of the first type. If the value of Σw^2 is to be calculated using (**3.8**), there is no need to include in the square improved estimates of differences or residuals.

The observed differences entered in this square (as for all the other squares used) were obtained alternately from two independent sets of values in order that all the errors of measurement in the observed values should be independent. It was permissible, therefore, to make use of the expressions derived in this paper to calculate the accuracies.

The sum of the squares of the residuals in Table 3 is $2\cdot 0016$ and therefore $\sqrt{\Sigma w^2}=1\cdot 415$ sec.

The standard deviation for the observed differences

$$S_s=\frac{1}{m-1}\sqrt{\Sigma w^2},\qquad \text{with } m=6,$$

$$=0\cdot 28 \text{ sec.}$$

The estimates, $S_s{}^2$ of the variances for the six squares of this stage were pooled, and the pooled value of S_s found to be $0\cdot 32$ sec.

The standard deviation of the observed differences for the square of the first type, S_0, calculated from the residuals of this square, was found to be $0\cdot 33$ sec.

Table 3

EXAMPLE OF A SQUARE OF THE SECOND TYPE

Values in seconds of arc

	30/35	35/40	40/45	45/50	50/55	55/60	S	M	Add $-\frac{1}{6}d_{30/60}+0\cdot17$ $=+0\cdot17$	Successive Addition and Change of Sign	Add Correction for 0/210 $=-0\cdot30$
210/215	—0·27 —0·40 +0·13	—0·17 —0·20 +0·03	—0·41 —0·80 +0·39	—0·31 —0·10 —0·21	—0·12 +0·30 —0·42	—0·52 —0·60 +0·08	—1·80	—0·30	—0·13	(210/215) +0·13	(0/215) —0·17
215/220	+0·23 +0·20 +0·03	+0·33 +0·70 —0·37	+0·09 +0·30 —0·21	+0·19 +0·30 —0·11	+0·38 —0·30 +0·68	—0·02 0·00 —0·02	+1·20	+0·20	+0·37	(210/220) —0·24	(0/220) —0·54
220/225	+0·03 —0·20 +0·23	+0·13 0·00 +0·13	—0·11 —0·10 —0·01	—0·01 +0·30 —0·31	+0·18 +0·30 —0·12	—0·22 —0·30 +0·08	0·00	0·00	+0·17	(210/225) —0·41	(0/225) —0·71
225/230	—0·05 0·00 —0·05	+0·05 —0·30 +0·35	—0·19 +0·10 —0·29	—0·09 —0·30 +0·21	+0·10 +0·40 —0·30	—0·30 —0·40 +0·10	—0·50	—0·08	+0·09	(210/230) —0·50	(0/230) —0·80
230/235	—0·10 —0·20 +0·10	0·00 +0·10 —0·10	—0·24 —0·30 +0·06	—0·14 —0·40 +0·26	+0·05 +0·10 —0·05	—0·35 —0·10 —0·25	—0·80	—0·13	+0·04	(210/235) —0·54	(0/235) —0·84
235/240	—0·32 +0·10 —0·42	—0·22 —0·20 —0·02	—0·46 —0·50 +0·04	—0·36 —0·50 +0·14	—0·17 —0·40 +0·23	—0·57 —0·60 +0·03	—2·10	—0·35	—0·18	(210/240) —0·36	(0/240) —0·66
S	—0·50	+0·10	—1·30	—0·70	+0·40	—2·00	—4·00				
M	—0·08	+0·02	—0·22	—0·12	+0·07	—0·33		—0·66			
Add $\frac{1}{6}d_{210/240}+0\cdot17$ $=0\cdot11$	+0·03	+0·13	—0·11	—0·01	+0·18	—0·22			—0·11		
Successive addition	(30/35) +0·03	(30/40) +0·16	(30/45) +0·05	(30/50) +0·04	(30/55) +0·22	(30/60) 0·00					
Add correction for 0/30=—0·64	(0/35) —0·61	(0/40) —0·48	(0/45) —0·59	(0/50) —0·60	(0/55) —0·42	(0/60) —0·64					

The third basic standard deviation, S_A, that of the value used for the total length of the scale, is in this case equal to zero.

From the above basic standard deviations, the accuracy of all the results of the calibration were computed using the expressions summarized in Section V.

It is of some interest to show how the accuracies vary for the different intervals starting at the zero graduation.

From Section V, the standard deviations for the improved values of the intervals 0/5, 0/10, . . ., 0/355 are given by the expression for var (est. of $O/P+z$) with $S_A=0$, $n=12$, $m=6$,

and $r=1, 2, \ldots, 12$,

$z=1, 2, \ldots, 6$.

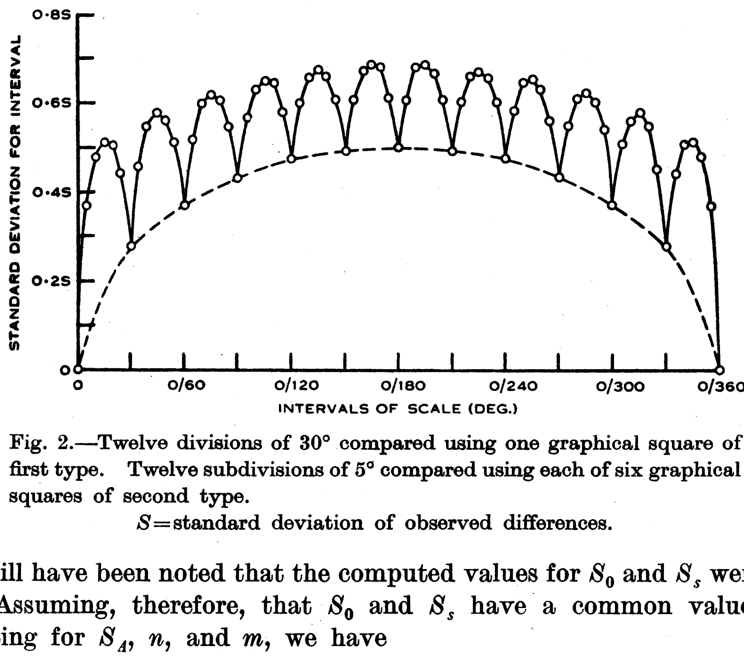

Fig. 2.—Twelve divisions of 30° compared using one graphical square of first type. Twelve subdivisions of 5° compared using each of six graphical squares of second type.

S=standard deviation of observed differences.

It will have been noted that the computed values for S_0 and S_s were closely equal. Assuming, therefore, that S_0 and S_s have a common value S, and substituting for S_A, n, and m, we have

$$S_{0/P+z}=\frac{S}{72}\sqrt{\{432r+864z-12rz-36r^2-133z^2\}}$$

with $r=1, 2, \ldots, 12$,

$z=1, 2, \ldots, 6$.

These values of $S_{0/P+z}$ are shown in the graph of Figure 2. The points joined by the broken curve refer to the intervals obtained solely from the square of the first type.

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