EXPLICIT MATRIX ELEMENTS FOR MULTIPOLE RADIATION

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Summary

Using a simple model, a formula is given for the radiation intensity from a multipole of any order. This formula is extended to include polarization properties of quanta. As an example, a matrix element is calculated for multipole radiation of order four.

I. Introduction

In the theory of angular correlations, it is relatively simple to calculate matrix elements for transitions which involve the emission of heavy particles. Thus, in the case of a nucleus of total angular momentum J and component m, which disintegrates into two particles of spins $\mathbf{S}_1$ and $\mathbf{S}_2$ and relative orbital angular momentum L, the lighter particle proceeding in a direction specified by spherical polar coordinates (θ, φ) in the usual way, the matrix element connecting transitions to a final state of spin $S=|\mathbf{S}_1+\mathbf{S}_2|$ and component $m_S=m_{S1}+m_{S2}$ is simply

$$(Jm \mid H_L \mid Sm_S)=\alpha_{LS} \,.\, (LSJm \mid LSm_Lm_S) \,.\, P_L^{m_L}(\cos \theta)e^{-im_L\varphi}/(2\pi)^{\frac{1}{2}}. \quad \ldots\ldots\ldots\ldots (1)$$

Here $(LSJm \mid LSm_Lm_S)$ is a transformation coefficient for vector addition, in the notation of Condon and Shortley (1935); α_{LS} is a complex quantity taking account of (unknown) orbital and spin dependence of the matrix element. The remaining term is $Y_L^{m_L*}(\theta, \varphi)$, a normalized spherical harmonic of order L and $m_L=m-m_S$, consistent with the definition of an arbitrary phase in $(LSJm \mid LSm_Lm_S)$. (The symbol * denotes complex conjugate.) Contributions from different S and m_S values must be combined incoherently using equation (1).

It is the purpose of this note to show that formulae similar to (1) can be used in the calculation of matrix elements for multipole radiation, assuming as a model that a quantum is a " particle " of intrinsic spin one, and components $+1$, 0, and -1. With this assumption the intensity of a radiation process may be written down immediately, and a simple extension makes it possible to evaluate the polarization of the radiation field. A general formula for radiation intensity of a multipole has been given by Ling and Falkoff (1949) (see also Falkoff, Colladay, and Sells 1952). In their method polarization effects do not appear automatically. Fierz (1949) has calculated multipole matrix elements for most cases of interest, while the straightforward calculation of matrix elements of field vectors has been carried out to multipole order 3 by French (1951),

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using the method of Condon and Shortley (1935). The present model has the advantage of simplicity and of providing a formula completely explicit in its angular dependence.

II. Formulae for Multipole Radiation

Consider a radiative transition of order L from a system described by (J, m) to a lower state (J', m'). Because of the nature of the electromagnetic interaction, we may define the matrix element of the radiation field $\mathbf{E}_L$ (cf. Ling and Falkoff 1949) as

$$\begin{aligned}(Jm \mid \mathbf{E}_L \mid J'm') &= f(J, J')(JLJ'm' \mid JLm\overline{m'-m})(L\overline{m'-m} \mid \mathbf{E}_l \mid 00)\\ &= f^*(J', J)(J'LJm \mid J'Lm'\overline{m-m'})(L\overline{m-m'} \mid \mathbf{E}_L \mid 00),\end{aligned} \quad \ldots\ldots (2)$$

where $f^*(J', J)$ is unspecified but independent of m and m'. In the matrix element $(L\overline{m-m'} \mid \mathbf{E}_L \mid 00)$, the order of complexity is to $\cos^L\theta$, and therefore the model is not inconsistent in supposing that this represents the emission of a particle of spin one and orbital angular momentum L. The intensity of radiation

$$\left|(L\overline{m-m'} \mid \mathbf{E}_L \mid 00)\right|^2 = \left|(LM \mid \mathbf{E}_L \mid 00)\right|^2$$

is thus given as the sum of the squares of three terms. In the notation of equation (1) these are

$$\left.\begin{aligned} a &= (LM \mid H_L \mid 11) \quad = (L1LM \mid L1\overline{M-1}1)\alpha_{L1}Y_L^{M-1*}(\theta, \varphi),\\ b &= (LM \mid H_L \mid 1-1) = (L1LM \mid L1\overline{M+1}-1)\alpha_{L1}Y_L^{M+1*}(\theta, \varphi),\\ c &= (LM \mid H_L \mid 10) \quad = (L1LM \mid L1M0)\alpha_{L1}Y_L^{M*}(\theta, \varphi).\end{aligned}\right\} \quad \ldots\ldots (3)$$

Using the known explicit forms of transformation coefficients given by Condon and Shortley (1935), we find

$$\begin{aligned}\left|(LM \mid \mathbf{E}_L \mid 00)\right|^2 &= |a|^2+|b|^2+|c|^2\\ &= |\alpha_{L1}|^2\left\{\frac{(L+M)(L-M+1)}{2L(L+1)}\,|Y_L^{M-1}(\theta, \varphi)|^2\right.\\ &\quad + \frac{(L-M)(L+M+1)}{2L(L+1)}\,|Y_L^{M+1}(\theta, \varphi)|^2\\ &\quad \left. + \frac{M^2}{L(L+1)}\,|Y_L^M(\theta, \varphi)|^2\right\},\end{aligned} \quad \ldots\ldots (4)$$

as given by Ling and Falkoff (1949).

It might be objected that the model is not consistent with the fact that a quantum can have only two components of angular momentum about its direction of motion. This, however, is not necessarily true for the intrinsic spin of the quantum as defined here. The property of the total angular momentum L, that

$$(LM \mid \mathbf{E}_L \mid 00)_{(\theta=\varphi=0)} = 0, \text{ unless } M = \pm 1,$$

is not violated by equation (4).

In order to complete the method it is necessary to introduce polarization properties of the radiation field. We have found it possible to describe matrix elements simply in terms of dipole field vectors. From a well-known property of spherical harmonics (Margenau and Murphy 1943) one may write

$$|a|^2+|b|^2+|c|^2=\left|\frac{a}{\sqrt{2}}e^{-i\varphi}+\frac{b}{\sqrt{2}}e^{i\varphi}\right|^2+\left|\left(\frac{a}{\sqrt{2}}e^{-i\varphi}-\frac{b}{\sqrt{2}}e^{i\varphi}\right)\cos\theta+c\sin\theta\right|^2. \quad \ldots\ldots (5)$$

It is therefore consistent with equation (4) to write the matrix element for electric multipole radiation as

$$(LM\mid \mathbf{E}_L\mid 00)_{\mathrm{el}}=\mathbf{j}\left(\frac{a}{\sqrt{2}}e^{-i\varphi}+\frac{b}{\sqrt{2}}e^{i\varphi}\right)-i\mathbf{k}\left\{\left(\frac{a}{\sqrt{2}}e^{-i\varphi}-\frac{b}{\sqrt{2}}e^{i\varphi}\right)\cos\theta+c\sin\theta\right\}$$
$$=\frac{a}{\sqrt{2}}(\mathbf{j}-i\mathbf{k}\cos\theta)e^{-i\varphi}+\frac{b}{\sqrt{2}}(\mathbf{j}+i\mathbf{k}\cos\theta)e^{i\varphi}$$
$$+c(-i\varphi_0\sin\theta). \quad \ldots\ldots (6)$$

Here **j** and **k** are unit vectors in the direction in increasing θ and φ respectively. Equation (6) is in agreement with the calculations of French. In it, the coefficients of a, b, and c are field vectors for a magnetic dipole: for magnetic multipole radiation, it would be necessary to replace them by field vectors for an electric dipole. Otherwise, equation (6), in conjunction with equations (2) and (3), gives an explicit form for a radiation matrix element of order L.

This method of separating orbital and intrinsic angular momenta is not unique. For instance, electric multipole radiation has been described (Bethe and Placzek 1937) as the emission of a particle of orbital angular momentum $(L-1)$ with intrinsic electric dipole field vector. The present system, however, has the advantage that the coefficients a, b, and c are sufficient for calculation of intensity (equation (4)) and polarization patterns (equation (6)), independently of one another.

As an example of the method, we shall calculate the matrix element $(42\mid \mathbf{E}_4\mid 00)_{\mathrm{el}}$ for electric multipole radiation of order 4. From equation (3), with $L=4$, $M=2$, we find

$$\left.\begin{aligned} a&=\left(\frac{9}{128}\right)^{\frac{1}{2}}\{3\sin\theta\,(7\cos^3\theta-3\cos\theta)\}e^{-i\varphi},\\ b&=\left(\frac{9}{128}\right)^{\frac{1}{2}}\{-7\sin^3\theta\cos\theta\}e^{-3i\varphi},\\ c&=\left(\frac{9}{128}\right)^{\frac{1}{2}}\{\sqrt{2}\sin^2\theta\,(7\cos^2\theta-1)\}e^{-2i\varphi}.\end{aligned}\right\}$$

Neglecting the common numerical factor $(9/128)^{\frac{1}{2}}$, the coefficients of **j** and $-i\mathbf{k}$ are

$$\left.\begin{aligned} \frac{a}{\sqrt{2}}e^{-i\varphi}+\frac{b}{\sqrt{2}}e^{i\varphi}&=\frac{4}{\sqrt{2}}\sin\theta\cos\theta\,(7\cos^2\theta-4)e^{-2i\varphi},\\ \left(\frac{a}{\sqrt{2}}e^{-i\varphi}-\frac{b}{\sqrt{2}}e^{i\varphi}\right)\cos\theta+c\sin\theta&=\frac{2}{\sqrt{2}}(7\cos^2\theta-1)\sin\theta e^{-2i\varphi},\end{aligned}\right\}$$

and equation (6) yields

$$(42\mid \mathbf{E}_4\mid 00)_{\mathrm{el}}=\frac{1}{\sqrt{2}}\{\mathbf{j}\,.\,4\sin\theta\cos\theta\,(7\cos^2\theta-4)-i\mathbf{k}\,.\,2\sin\theta\,(7\cos^2\theta-1)\}e^{-2i\varphi},$$

whence the intensity pattern is

$$|(42 \mid \mathbf{E}_4 \mid 00)_{\mathrm{el}}|^2 = 2 \sin^2 \theta \, (196 \cos^6 \theta - 175 \cos^4 \theta + 50 \cos^2 \theta + 1),$$

in agreement with that calculated from equation (**4**).

It should be pointed out that the expression $(Jm \mid \mathbf{E}_L \mid J'm')$ introduced in equation (**2**) is only valid for " pure " multipole radiation of order L, defined such that its intensity pattern is the same as that of $(LM \mid \mathbf{E}_L \mid 00)$. This definition does not coincide with that given by Condon and Shortley. Thus, for instance, in the case of octupole radiation, the matrix elements of the quantity **rrr** (French 1951) lead partly to a dipole pattern, which has been suppressed (French, personal communication, 1953). This means that a dipole pattern could appear mixed with an octupole pattern even in the absence of " pure " dipole radiation, and would have the same intensity dependence on quantum energy as the accompanying octupole pattern.

III. References

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