AN INTERPRETATION OF THE FIELD TENSOR IN THE UNIFIED FIELD THEORY

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Summary

It is assumed that the skew symmetric part of the field tensor g_{ik} is a complex, self-dual tensor. This permits the whole set of field equations *for free space* to be derived directly from the theory without the introduction of an electric current density tensor. However, with this assumption it appears impossible for spherically symmetric electric and magnetic fields to exist in free space.

I. The Field Equations

It has been shown (Taylor 1952) how the complete set of Maxwell's electromagnetic equations in free space can be expressed simply by

$$\frac{\partial}{\partial x^k}(\mathrm{F}^{ik})=0,$$

where F^{ik} is the tensor density derived from a complex, self-dual tensor F_{ik}. Hence, if the skew symmetric part of the field tensor g_{ik} is supposed to be of the same type as this F_{ik}, then the system of equations (I) of Einstein (1951, p. 138) gives the complete description of the field for the case of free space, without the insertion of an additional equation defining the current and charge density.

The derivation of Einstein's system (I) was a generalization of the process of obtaining the field equations for empty space in the purely gravitational field. Also, the system (I) reduces to these equations when the skew symmetric part of g_{ik} vanishes. Hence it appears as an advantage of the present interpretation of the skew symmetric part of g_{ik} that the current and charge vector density is zero.

II. The Self-Dual Field Tensor

The condition for self-duality of the skew symmetric part of g_{ik} requires special investigation in the present theory. Write

$$g_{ik}=a_{ik}+\varphi_{ik}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where a_{ik} is the symmetric part and φ_{ik} the skew symmetric part of g_{ik}. In the electromagnetic field theory the restriction on F_{ik} becomes

$$\sqrt{a}a^{ir}a^{ks}F_{rs}=\tfrac{1}{2}\varepsilon^{iklm}F_{lm}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

In the present theory, however, tensor densities may be formed using the determinant g as well as a, and suffixes may be raised by a^{ik}, g^{ik}, or $g^{\underline{ik}}$ (the symmetric

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part of g^{ik}). The various possible combinations give many equations restricting the φ_{ik}, all of which could be accepted as generalizations of (2). However, it can be shown by direct calculation that in the case of the spherically symmetric field described in Section III, each of these combinations gives an inconsistent set of equations, except the three

$$\sqrt{a}a^{ir}a^{ks}\varphi_{rs}=\tfrac{1}{2}\varepsilon^{iklm}\varphi_{lm}, \quad \ldots\ldots\ldots\ldots \quad (3)$$

$$\sqrt{g}g^{ir}g^{ks}\varphi_{rs}=\tfrac{1}{2}\varepsilon^{iklm}\varphi_{lm}, \quad \ldots\ldots\ldots\ldots \quad (4)$$

$$\sqrt{g}g^{ri}g^{sk}\varphi_{rs}=\tfrac{1}{2}\varepsilon^{iklm}\varphi_{lm}. \quad \ldots\ldots\ldots\ldots \quad (5)$$

The first of these involves only three distinct equations and obviously does not impose too rigid a restriction even should an arbitrary field be under discussion. The equations (4) and (5) are not linear in the φ_{ik} as they stand, and their investigation in the arbitrary field would be much more complicated.

It is possible to show that (3) is a solution to (4) and (5), however. This can be done using formulae given by Kurşunoğlu (1952) adjusted to suit the present notation. These are

$$g=a(1+\Omega+\Lambda^2), \quad \ldots\ldots\ldots\ldots \quad (6)$$

where

$$\Omega=\tfrac{1}{2}\varphi_{ik}\varphi^{ik}, \quad \ldots\ldots\ldots\ldots \quad (7)$$

and

$$\Lambda=\tfrac{1}{4}f^{ik}\varphi_{ik}, \quad \ldots\ldots\ldots\ldots \quad (8)$$

with

$$f^{ik}=\varepsilon^{iklm}\varphi_{lm}/2\sqrt{a}. \quad \ldots\ldots\ldots\ldots \quad (9)$$

Also

$$g^{\underline{ik}}=a^{ik}-g^{\underset{\vee}{il}}\varphi_{\cdot l}^{k}, \quad \ldots\ldots\ldots\ldots \quad (10)$$

and

$$g^{\underset{\vee}{ik}}=(\varphi^{ik}+\Lambda f^{ik})/(1+\Omega+\Lambda^2). \quad \ldots\ldots\ldots\ldots \quad (11)$$

There is also the identity

$$\delta_k^{\,i}\Lambda=f^{il}\varphi_{kl}. \quad \ldots\ldots\ldots\ldots \quad (12)$$

All indices (except those of g^{ik} which has its usual definition) are raised by a^{ik} in the formulae (6)–(12).

Since it is assumed that (3) holds, (9) reduces to

$$f^{ik}=\varphi^{ik}. \quad \ldots\ldots\ldots\ldots \quad (13)$$

From the equations (7) and (8) we then have $\Lambda=\tfrac{1}{2}\Omega$, and so (11) simplifies to

$$g^{\underset{\vee}{ik}}=\varphi^{ik}/(1+\tfrac{1}{2}\Omega). \quad \ldots\ldots\ldots\ldots \quad (14)$$

(10) and (11) then give

$$g^{ik}=g^{\underline{ik}}+g^{\underset{\vee}{ik}}=a^{ik}+(\varphi^{ik}-\varphi^{il}\varphi_{\cdot l}^{k})/(1+\tfrac{1}{2}\Omega).$$

Since (12) reduces to

$$\tfrac{1}{2}\delta_k^{\,i}\Omega=\varphi^{il}\varphi_{kl}, \quad \ldots\ldots\ldots\ldots \quad (15)$$

this becomes

$$g^{ik}=(a^{ik}+\varphi^{ik})/(1+\tfrac{1}{2}\Omega). \quad \ldots\ldots\ldots\ldots \quad (16)$$

Consider (4). Since (6) is now $g=a(1+\frac{1}{2}\Omega)^2$, the left-hand side is

$$\sqrt{a}(a^{ir}+\varphi^{ir})(a^{ks}+\varphi^{ks})\varphi_{rs}/(1+\tfrac{1}{2}\Omega),$$

that is,

$$\sqrt{a}(\varphi^{ik}+\varphi^{ir}\varphi_r{}^k+\varphi^{ks}\varphi^i{}_s+\varphi^{ir}\varphi^{ks}\varphi_{rs})/(1+\tfrac{1}{2}\Omega),$$

that is,

$$\sqrt{a}(\varphi^{ik}+\varphi^{ir}\varphi^{ks}\varphi_{rs})/(1+\tfrac{1}{2}\Omega).$$

Using (15) this expression simplifies to $\sqrt{a}\varphi^{ik}$. By (3), this is equal to $\frac{1}{2}\varepsilon^{iklm}\varphi_{lm}$, which is also the right-hand side of (4). Hence (4) is satisfied by (3).

It can be shown at once also that if (3) holds then (5) is satisfied. For, the left-hand side of (5) is

$$\sqrt{a}(a^{ir}-\varphi^{ir})(a^{ks}-\varphi^{ks})\varphi_{rs}/(1+\tfrac{1}{2}\Omega),$$

that is,

$$\sqrt{a}(\varphi^{ik}+\varphi^{ir}\varphi^{ks}\varphi_{rs})/(1+\tfrac{1}{2}\Omega),$$

as before.

III. The Spherically Symmetric Field

Papapetrou (1948) has found that the general static spherically symmetric g_{ik} can be written in the form

$$g_{ik}=\left.\begin{matrix} -\alpha & 0 & 0 & w \\ 0 & -\beta & r^2v\sin\theta & 0 \\ 0 & -r^2v\sin\theta & -\beta\sin^2\theta & 0 \\ -w & 0 & 0 & \gamma \end{matrix}\right\} . \qquad \ldots\ldots (17)$$

It is necessary to determine now the restriction imposed on these components by the condition that the skew symmetrical part is self dual. In this case it is easily seen by straightforward calculation that (4) and (5) have just one solution, that given by (3), namely,

$$w=i\sqrt{\frac{\alpha\gamma}{\beta^2}}r^2v=i\sqrt{\frac{\alpha\gamma}{\beta^2}}f, \qquad \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

using $f=r^2v$ (Bonnor 1951, 1952).

Without this restriction the form (17) would be sufficiently general to permit the description of the spherically symmetric field in a dielectric and magnetic medium (Taylor 1953). However, for the reasons given in Section I, the present theory does not include such media.

The terms w and f are complex. The electric field components are now not distinguished from the magnetic by position in the array (17). The separation is made by resolving the w and f into real and imaginary parts, there being freedom of choice, until an electric current vector is defined, as to which part corresponds to the electric and which to the magnetic field.

Since w and f cannot now be made to vanish separately, it is necessary to use the completely general static spherically symmetric solution of Einstein's (1951) system of equations (I) given by Bonnor (1952). In the following analysis the references will be to this paper.

Substituting for γ from (**18**) into Bonnor's equation (**2.21**),

$$(f^2+\beta^2)w^2/\alpha=-cf^2. \qquad (19)$$

According to Bonnor's equation (**2.20**) the expression on the left-hand side of (**19**) is also equal to l^2. Hence

$$f=\pm il/\sqrt{c}=\text{a constant}. \qquad (20)$$

From (**20**) and (**2.2**) of Bonnor

$$B=f\beta'/(\beta^2+f^2),$$

and so Bonnor's equations (**2.24**) and (**2.25**) become, using this and (**20**),

$$\beta''-(\alpha'/2\alpha)\beta'-2\alpha(\beta^2-f^2)/(\beta^2+f^2)=0, \qquad (21)$$

and

$$(\beta')^2-4\alpha\beta=0. \qquad (22)$$

Eliminating α from (**21**) and (**22**),

$$\beta''-(\beta''/\beta'-\beta'/2\beta)\beta'-\beta'^2(\beta^2-f^2)/[2\beta(\beta^2+f^2)]=0.$$

This simplifies to

$$\beta'=0,$$

which, together with (**20**), leads to inconsistencies in the field equations. For example, R_{22}, given by Bonnor's equations (**2.8**) and (**2.2**), cannot be made to vanish.

The present interpretation of $g_{\underset{\vee}{ik}}$ would imply the impossibility of the existence of static spherically symmetric electromagnetic fields when the field equations are Einstein's system (I).

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V. References

Bonnor, W. B. (1951).—Static spherically symmetric solutions in Einstein's unified field theory. *Proc. Roy. Soc.* A **209** : 353-68.

Bonnor, W. B. (1952).—The general static spherically symmetric solution in Einstein's unified field theory. *Proc. Roy. Soc.* A **210** : 427-34.

Einstein, A. (1951).—"The Meaning of Relativity." 5th Ed. Appendix II. (Methuen: London.)

Kurşunoğlu, B. (1952).—Gravitation and electrodynamics. *Phys. Rev.* **88** : 1369.

Papapetrou, A. (1948).—Static spherically symmetric solutions in the unitary field theory. *Proc. R. Irish Acad.* A **52** : 69-86.

Taylor, N. W. (1952).—A simplified form of the relativistic electromagnetic equations. *Aust. J. Sci. Res.* A **5** : 423-9.

Taylor, N. W. (1953).—The relativistic electromagnetic equations in a material medium. *Aust. J. Phys.* **6** : 1-9.