ON A THEORY DUE TO I. FÉNYES

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Summary

Fényes (1952) has attempted to base quantum mechanics on a theory of Markoff processes in the configuration space of any system considered.

It is shown here that for certain systems the allowable stationary-state solutions of the basic equations of this theory are more numerous than the stationary states predicted by quantum mechanics through the Schrödinger equation, and that it is not possible to identify the probability amplitude defined in this theory with the Schrödinger wave function. It is pointed out that the " total stochastic velocity " of a particle in this theory cannot equal the linear momentum divided by the mass, that Fényes' uncertainty relation is not equivalent to the Heisenberg relation, and that the total-stochastic-velocity operator has properties different from those of the quantum-mechanical momentum operator. It is shown that Fényes' proof that the addition of further parameters to a set of statistical parameters cannot lead to a set affording a causal description of the motion of a system is unsatisfactory, but that this conclusion may reasonably be drawn from his theory, by making a consistent auxiliary assumption.

I. INTRODUCTION

Fényes (1952) has attempted to describe the motion of a general microscopic system as a Markoff process in a configuration space, which is implied to be that of the Cartesian position coordinates of the particles constituting the system. Then quantum mechanics becomes a statistical theory. Only the briefest outline can be given here of Fényes' lengthy exposition.

The theory is based upon an analytical treatment of stochastic processes due to Kolmogoroff (1931, 1933). Fényes begins with a set of parameters (position coordinates) and probabliity and transition probability densities w and v which are functions of these parameters and the time. He takes these densities to be appropriately normalized, and interrelates them through an integral equation which defines a Markoff process. An integral relation then follows between the transition probability densities at three distinct instants of time. Fényes assumes the existence of certain limits in order to derive two differential equations from the original integral relations. One equation leads to a continuity or conservation equation, the Fokker equation. Fényes defines a " total stochastic velocity " and corresponding operators, and deduces an " uncertainty relation ". He then further restricts the class of systems considered and defines a probability amplitude ψ. When expressed in terms of ψ, the Fokker equation is formally identical with the equation of " conservation of

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probability" in quantum mechanics (see e.g. Tolman (1938)). Fényes modifies his original total stochastic velocity definition to define "the velocity corresponding to ψ", which he claims is the same as the quantum-mechanical velocity.

After a consideration of formal analogies and further discussion, Fényes decides that the mathematical apparatus of quantum mechanics is identical with that of his treatment of Markoff processes, and that quantum-mechanical processes are special Markoff processes. This conclusion leads Fényes to suppose that further degrees of freedom, hidden parameters, must be assigned to the electron. Fényes dismisses von Neumann's (1932) objections to the existence of hidden parameters in quantum mechanics by stating that von Neumann proposes to reach a causal description in terms of ψ plus further parameters, and arguing that, since ψ characterizes the state of the system from the statistical standpoint, it cannot be considered as a causal state parameter.

Fényes then attempts to demonstrate that his set of statistical configuration coordinates cannot be completed by the addition of further parameters to a set of coordinates in terms of which a causal description of the motion may be given. He also tries to show that the state of a system for which the "scattertensor" does not vanish cannot be characterized by a phase-space distribution function of the kind considered in statistical mechanics.

Finally, Fényes assumes a Lagrangian function and by using a variation principle deduces the Fokker equation and the Schrödinger equation for a certain class of systems.

This paper is an attempt to show that Fényes' theory is unsatisfactory. Many points of the theory which seem open to debate will not be discussed here because they are part of long verbal arguments which would require considerable analysis. Some further discussion may be found in the writer's thesis (Nicholson 1953). The theory has also been criticized by Weizel (1953).

II. Fényes' Markoff Relation and Quantum Mechanics

The Lagrangian function used by Fényes appears to be of arbitrary form. Fényes has not shown that the resulting Schrödinger equation and its physically admissable solutions are equivalent to his original equations and their allowable solutions.

We shall now show that quantum mechanics cannot be based solely on Fényes' original integral relations which define a Markoff process, but that further restrictions are necessary on the class of solutions of the Markoff equations before these solutions can be taken to describe quantum-mechanical motions.

Let us consider a quantum-mechanical system for which the Schrödinger equation has only a discrete set of energy eigenvalues E_i and associated physically-admissible normalized linearly-independent eigenfunctions ψ_i. We suppose that no continuous ranges of eigenvalues exist and that none of the energy eigenstates is degenerate. One such system is the one-dimensional linear harmonic oscillator which has a denumerably infinite set of discrete energy eigenstates. Let ψ_1, ψ_2 be any two of the energy eigenfunctions. Then $\psi_1^*\psi_1$, $\psi_2^*\psi_2$ are time independent and are interpreted as probability densities. If

Fényes' theory is applicable then the motion of the system can be represented as a Markoff process defined by the relations

$$w(y,t)=\int_{-\infty}^{\infty} v(x,s;\, y,t)w(x,s)\mathrm{d}x, \qquad (1)$$

$$\int_{-\infty}^{\infty} w(y,t)\mathrm{d}y=1, \qquad (2)$$

$$w(y,t)=\psi^*(y,t)\psi(y,t), \qquad (3)$$

with obvious meanings. Fényes supposes that any non-negative solution of (1) and (2) represents a possible mode of motion of the system. If Fényes' theory is applicable we must suppose that the time-independent distributions

$$w_1(y)=\psi_1^*\psi_1, \quad w_2(y)=\psi_2^*\psi_2$$

are solutions of (1) and (2). But it is then clear that any linear combination

$$w(y)=kw_1(y)+(1-k)w_2(y), \quad k \text{ constant, } 0\leqslant k\leqslant 1,$$

also satisfies (1) and (2) and is non-negative and so must represent a possible stationary state of the system.

Since k can assume a continuous infinity of values in the range (0,1), Fényes' theory predicts an infinite set of power continuum of stationary states of the system. But the system considered has only a discrete set of energy eigenvalues, so that the set of stationary states is at most denumerably infinite. Hence Fényes' equations permit an infinitely greater number of stationary states than are allowed by quantum mechanics, for certain systems. Such a continuum of states can be constructed from each pair of discrete states.

Hence any theory, like that of Fényes, based on the equations (1), (2), and (3) cannot be equivalent to quantum mechanics, which is based on a linear equation for the probability amplitude ψ. This conclusion is also suggested by a discussion of the principle of superposition of states given by Feynman (1948). The solutions w of (1) and (2) would need to satisfy some further equation involving w non-linearly before the resulting theory could be equivalent to quantum mechanics.

In a representation of quantum mechanics as a form of statistical mechanics by Moyal (1949), the state of the system is described by a phase-space distribution function $F(p,q;t)$ which satisfies an equation of Markoff form, namely,

$$F(p,q;t)=\int\int K(p,q \mid p_0,q_0;t-t_0)F(p_0,q_0;t_0)\mathrm{d}p_0\mathrm{d}q_0.$$

As for Fényes' theory, we can see that for certain systems the possible normalized time-independent solutions of such an equation, if they exist at all, are more numerous than the stationary states of the systems as given by quantum mechanics, so that quantum mechanics cannot be based solely on equations of this form. However, Moyal defines his F directly in terms of the Schrödinger amplitude ψ, and, although F satisfies the equation above, this equation does not define the class of admissible F in his theory.

III. Generality of Fényes' Theory

The Schrödinger equation deduced by Fényes applies only to a certain class of systems, in which the interactions between the particles and the external field can be expressed completely in terms of scalar potential functions. The Schrödinger equation given by Fényes is not applicable, for instance, to a particle moving in an external electromagnetic field.

Fényes' form of the Schrödinger equation is in position-coordinate language. If his basic variables were declared to be, say, linear momenta instead of position coordinates, then the Schrödinger equation would be in momentum language, but the class of known physical systems to which it would then apply would be very small: for the Hamiltonian would have to involve the position coordinates q_i only in the form of the term $\sum_i q_i^2$. The harmonic oscillator is a member of this small class. Further, if Fényes' basic variables were linear momenta, then the various " velocities " and the " uncertainty relation " which occur in his theory would have no familiar interpretation.

It appears to be difficult to extend Fényes' method to yield the wave equation in coordinate language for more general systems, or to deduce a satisfactory eigenvalue-eigenfunction equation in other languages, or for other dynamical variables.

IV. Fényes' ψ and the Probability Amplitude of Quantum Mechanics

We now show that it is impossible to identify Fényes' ψ with the probability amplitude ψ of quantum mechanics.

Consider two energy eigenstates ψ_1, ψ_2 of a suitable quantum-mechanical system, which eigenstates belong to the discrete unequal eigenvalues E_1, E_2. Then we can write

$$\psi_1(q,t)=u_1(q)e^{-2\pi i E_1 t/h}, \quad \psi_2(q,t)=u_2(q)e^{-2\pi i E_2 t/h}.$$

In the quantum-mechanical case, Fényes' definition of ψ becomes

$$\psi=\alpha e^{2\pi(\sigma+i\Sigma)/h},$$

where α can be taken as a real positive constant. Equating the probability densities in the two theories we obtain

$$\psi_1^*\psi_1=u_1^*u_1=\alpha_1^2 e^{-4\pi\sigma_1/h}$$

and similarly for ψ_2, so that

$$-\frac{2\pi\sigma_1}{h}=\ln\frac{|u_1|}{\alpha_1}, \quad -\frac{2\pi\sigma_2}{h}=\ln\frac{|u_2|}{\alpha_2}.$$

Allowing for an arbitrary phase factor, we equate the probability amplitudes from the two theories to obtain

$$\psi_1=\alpha_1 e^{\ln(|u_1|/\alpha_1)}e^{-2\pi i\Sigma_1/h}=|u_1|e^{-2\pi i\Sigma_1/h}=u_1 e^{-2\pi i\eta_1/h}e^{-2\pi i E_1 t/h},$$

and similarly for ψ_2, where η_1 is a constant (cf. Witmer and Vinti 1935).

We can write

$$u_1 = |u_1| e^{-i\delta_1(q)},$$

where $\delta_1(q)$ is a real function of q. Then

$$\Sigma_1 = \delta_1 + E_1 t + n_1 h + \eta_1, \quad n_1 \text{ an integer.}$$

Similarly

$$\Sigma_2 = \delta_2 + E_2 t + n_2 h + \eta_2.$$

But by Fényes' definitions

$$\Sigma = s + \sigma,$$

where $s \equiv s(q,t)$ is independent of the state of the system and depends only upon its nature. Therefore

$$\sigma_1 + s = \delta_1 + E_1 t + n_1 h + \eta_1,$$

$$\sigma_2 + s = \delta_2 + E_2 t + n_2 h + \eta_2,$$

and

$$\sigma_1 - \sigma_2 - \delta_1 + \delta_2 = (E_1 - E_2)t + h(n_1 - n_2) + (\eta_1 - \eta_2).$$

But this is impossible, as the right side is a non-zero function of t (since $E_1 \neq E_2$) but not of q, whereas the left side is a function of q only. Hence we cannot equate the probability amplitudes ψ in the two theories, and Fényes' theory is not a possible representation of quantum mechanics.

V. Total Stochastic Velocity and the Uncertainty Relation

We next discuss Fényes' total stochastic velocity of components c_i, and his uncertainty relation. If $w(y,t)$ is Fényes' space-time distribution function then

$$c_i = a_i - \frac{1}{w} \sum_k \frac{\partial}{\partial y_k}(b_{ik} w), \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

where $a_i(y,t)$ and $b_{ik}(y,t)$ depend upon the nature of the system but not upon its state. The important thing here is that c_i is a precise function of position and time.

Fényes defines the means $\bar{c}_i, \bar{y}_i$ and the variances $\Delta c_i, \Delta y_i$ of the stochastic velocity components c_i and the configuration coordinates y_i with respect to w over all configuration space, and deduces a lower bound for the product $\Delta y_i \Delta c_i$. A similar relation has been given by Fürth (1933) for the case of one-dimensional Brownian motion of a particle. Fényes attempts to identify this uncertainty relation with the Heisenberg uncertainty relation of quantum mechanics. To do this for a particle it is necessary to identify c_i with p_i/m where m is the particle mass and p_i is the quantum-mechanical linear momentum component corresponding to the coordinate y_i. But c_i is a function of the configuration coordinates y_i as well as the time, whereas the quantum-mechanical momentum distribution is a function of the p_i and time only, for any state of the system. Any dependence of p_i on the configuration coordinate y_i would amount to a violation of the Heisenberg uncertainty principle. Hence it is impossible to identify c_i with p_i/m or Δc_i with $\Delta p_i/m$. The "uncertainties" Δc_i and Δp_i have different meanings.

This basic difference is reflected in Fényes' definition of a stochastic velocity operator $\mathbf{c}_i$ which satisfies

$$\mathbf{c}_i w = c_i w$$

for any admissible state w of a system, that is, every possible state of the system is an eigenstate of c_i. In quantum mechanics only a subset of all admissible states ψ of a system are eigenstates of p_i, that is, satisfy

$$\mathbf{p}_i \psi = p_i \psi.$$

Later in his paper Fényes modifies his velocity definition to define a velocity of components c'_i which, he states, agree with the quantum-mechanical velocity components. But again c'_i is a precise function of position and time, and thus cannot be identified with p_i/m. Further c'_i is in general complex, whereas p_i is always real.

VI. Causal and Statistical Description

Some of the mathematical arguments in Fényes' paper appear doubtful. We shall only consider one such argument here, namely, Fényes' proof that the addition of further parameters $y_{n+1}, \ldots, y_N$ to the basic parameters $y_1, \ldots, y_n$ cannot cause the scattertensor b_{ik} to vanish in the completed parameter system if it does not vanish in the original system. This proposition is probably correct though the argument seems wrong. From this proposition Fényes concludes that a causal description cannot be obtained from a statistical description by the addition of further parameters.

In the original parameter system the transition probability density is denoted by $v(y_1, \ldots, y_n, t, z_1, \ldots, z_n, t+\Delta)$ while in the completed set it is $V(y_1, \ldots, y_N, t, z_1, \ldots, z_N, t+\Delta)$.

The scattertensors in the two systems are

$$\left.\begin{aligned} b_{ik}(y_1, \ldots, y_n, t) &= \lim_{\Delta\to 0} \frac{1}{2\Delta} \int (z_i - y_i)(z_k - y_k) v \, dz_1 \ldots dz_n, \\ \beta_{ik}(y_1, \ldots, y_N, t) &= \lim_{\Delta\to 0} \frac{1}{2\Delta} \int (z_i - y_i)(z_k - y_k) V \, dz_1 \ldots dz_N. \end{aligned}\right\} \quad \ldots (5)$$

Fényes states that it is obvious that

$$v = \int V \, dz_{n+1} \ldots dz_N. \quad \ldots\ldots\ldots\ldots (6)$$

But V is a function of $y_{n+1}, \ldots, y_N$ while v is not. There is a class of functions V for which $\int V dz_{n+1} \ldots dz_N$ is independent of $y_{n+1}, \ldots, y_N$, but Fényes has not shown that the functions V considered in his paper belong to this class.

Fényes' conclusion can be drawn, however, on the basis of a plausible assumption. It seems clear from the meaning of a probability density that

$$w(y_1, \ldots, y_n, t) = \int W(y_1, \ldots, y_N, t) dy_{n+1} \ldots dy_N,$$

where w and W are the probability densities in the two systems. Then equation (1) applied in the two systems yields

$$\int w(y_1, \ldots, y_n, t) v(y_1, \ldots, y_n, t, z_1, \ldots, z_n, t+\Delta) \mathrm{d}y_1 \ldots \mathrm{d}y_n$$

$$= \int\int W(y_1, \ldots, y_N, t) V(y_1, \ldots, y_N, t, z_1, \ldots, z_N, t+\Delta) \mathrm{d}y_1 \ldots \mathrm{d}y_N \, \mathrm{d}z_{n+1} \ldots \mathrm{d}z_N,$$

which suggests that

$$wv = \int\int WV \mathrm{d}y_{n+1} \ldots \mathrm{d}y_N \, \mathrm{d}z_{n+1} \ldots \mathrm{d}z_N \quad \ldots\ldots\ldots \quad (7)$$

is true, instead of (6).

In Fényes' theory the probability of a transition from the point $(y_1, \ldots, y_n)$ at t to the region $\mathrm{d}z_1 \ldots \mathrm{d}z_n$ about $(z_1, \ldots, z_n)$ at $t+\Delta$ is

$$v \mathrm{d}z_1 \ldots \mathrm{d}z_n,$$

and similarly for V. Classically, the probability of transition from $(y_1, \ldots, y_N)$ at t to $\mathrm{d}z_1 \ldots \mathrm{d}z_n$ about $(z_1, \ldots, z_n)$ at $t+\Delta$ is

$$\mathrm{d}z_1 \ldots \mathrm{d}z_n \int V \mathrm{d}z_{n+1} \ldots \mathrm{d}z_N,$$

and the probability of transition from $(y_1, \ldots, y_n)$ at t to $\mathrm{d}z_1 \ldots \mathrm{d}z_n$ about $(z_1, \ldots, z_n)$ must then be

$$\mathrm{d}z_1 \ldots \mathrm{d}z_n \int \Big\{ \text{Relative probability of } (y_{n+1}, \ldots, y_N) \text{ at } t$$

$$\times \int V \mathrm{d}z_{n+1} \ldots \mathrm{d}z_N \Big\} \mathrm{d}y_{n+1} \ldots \mathrm{d}y_N$$

$$= \mathrm{d}z_1 \ldots \mathrm{d}z_n \int \left\{ \frac{W(y_1, \ldots, y_N, t)}{\int W \mathrm{d}y_{n+1} \ldots \mathrm{d}y_N} \cdot \int V \mathrm{d}z_{n+1} \ldots \mathrm{d}z_N \right\} \mathrm{d}y_{n+1} \ldots \mathrm{d}y_N$$

$$= \frac{\mathrm{d}z_1 \ldots \mathrm{d}z_n}{w(y_1, \ldots, y_n, t)} \int\int WV \mathrm{d}y_{n+1} \ldots \mathrm{d}y_N \, \mathrm{d}z_{n+1} \ldots \mathrm{d}z_N.$$

But this transition probability is $v \mathrm{d}z_1 \ldots \mathrm{d}z_n$. Hence (7) is true for a classical theory, and we assume it to hold true in Fényes' theory as it is consistent with Fényes' relations between v and w, which are of a classical kind.

We now consider the integral

$$I = \frac{1}{2\Delta} \int (z_i - y_i)(z_k - y_k) V(y, t, z, t+\Delta) W(y, t) \mathrm{d}y_{n+1} \ldots \mathrm{d}y_N \mathrm{d}z_1 \ldots \mathrm{d}z_N,$$

for $i \leqslant n$, $k \leqslant n$.

Clearly,

$$I=\frac{1}{2\Delta}\int(z_i-y_i)(z_k-y_k)\left\{\int VW\mathrm{d}y_{n+1}\ldots\mathrm{d}y_N\mathrm{d}z_{n+1}\ldots\mathrm{d}z_N\right\}\mathrm{d}z_1\ldots\mathrm{d}z_n$$

$$=\frac{1}{2\Delta}\int(z_i-y_i)(z_k-y_k)w(y,t)v(y,t,z,t+\Delta)\mathrm{d}z_1\ldots\mathrm{d}z_n, \text{ by equation (7)},$$

$$=w(y,t)\frac{1}{2\Delta}\int(z_i-y_i)(z_k-y_k)v(y,t,z,t+\Delta)\mathrm{d}z_1\ldots\mathrm{d}z_n.$$

But also

$$I=\int W(y,t)\left\{\frac{1}{2\Delta}\int(z_i-y_i)(z_k-y_k)V\mathrm{d}z_1\ldots\mathrm{d}z_N\right\}\mathrm{d}y_{n+1}\ldots\mathrm{d}y_N.$$

We equate these expressions for I, and let $\Delta\rightarrow 0$. Then the definitions (5) yield, for $i,k\leqslant n$,

$$w(y_1,\ldots,y_n,t)b_{ik}(y_1,\ldots,y_n,t)=\int W(y_1,\ldots,y_N,t)\beta_{ik}(y_1,\ldots,y_N,t)\mathrm{d}y_{n+1}\ldots\mathrm{d}y_N.$$

Hence, if, for $i,k\leqslant n$,

$$b_{ik}\neq 0,$$

then

$$\beta_{ik}\neq 0.$$

This is Fényes' conclusion, and shows that the addition of further coordinates $y_{n+1},\ldots,y_N$ cannot cause the scattertensor to vanish in the completed system if it does not vanish in the original system. Fényes also concludes that, for $i,k\leqslant n$, $\beta_{ik}=b_{ik}$, but this does not follow from the present argument, and is doubtful since β_{ik} is in general a function of $y_{n+1},\ldots,y_N$ while b_{ik} is not.

Fényes next proves that a system can only be assigned a phase-space density function $f(u,y,t)$ and be treated by the methods of statistical mechanics if the scattertensor b_{ik} vanishes identically. Some objections to this proof are stated in the writer's thesis (Nicholson 1953).

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