CONVECTION FROM A LARGE HORIZONTAL SURFACE

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Summary

A theory is put forward for convection from a large horizontal heated surface in a semi-infinite medium, by buoyant elements which are subject to continuous mixing with the environment by turbulence on a smaller scale. It is assumed, with support from similarity arguments, that the (potential) temperature θ at sufficient height z above the surface obeys the form $-(g/\theta)(\partial\theta/\partial z)=Cz^{-\delta}$, where δ and C are positive constants. It is then shown : (i) that δ must in practice be close to 4/3 and equal to it under steady conditions, except in layers where radiational heating is large, where δ will be smaller ; (ii) that the rate of heat loss varies as $C^{3/2}$; and (iii) that the r.m.s. temperature fluctuations are proportional to $Cz^{-1/3}$. Experimental results from the surface layers of atmosphere support these predictions quite well.

The principal results are first suggested for free convection by dimensional and similarity arguments. They receive independent confirmation from the mechanistic theory, which extends into conditions when forced convection is present but not dominant. The theory also provides information about the multiplying constants in the above relationships, though it does not so far lead to a prediction of their exact values. The multiplying constants depend, *inter alia*, on the mass ratio between the ascending and descending air, and this remains constant through the layer of constant heat flux. The behaviour of the ascending elements and that of the descending air are shown to be quite differently governed.

I. Introduction

The problem of free convection is commonly approached on empirical lines working on a foundation of dimensional analysis. Accounts of the work appear in textbooks by Fishenden and Saunders (1932), Bosworth (1952), and others, and in the meteorological literature by Sutton (1953). The formulation is in terms of three dimensionless bulk parameters, the Nusselt, Grashof, and Prandtl numbers, and formulae for heat transfer by free convection are in general suitably expressed as a functional relation between these three numbers. Experiments on convection from heated wires and pipes and vertical planes have led to the common result that the heat loss is proportional to the 5/4 power of the temperature difference between the heated body and the surrounding medium, the index changing to 4/3 when the motion becomes sufficiently turbulent. Other experiments (Chandra 1938 ; de Graaf and van der Held 1953) relate to convection between parallel planes. In both these types of study, the formation of large buoyant elements is inhibited by the scale and geometry of the experiment.

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Special considerations must apply to convection from large (theoretically infinite) horizontal surfaces, in a medium of large vertical extent, which is the case to be considered here. The temperature difference $\Delta\theta$ between body and surroundings is no longer clearly defined since it will continue to increase, though at a continuously decreasing rate, as the height z above the surface increases. Secondly, the heated body ceases to have an identifiable linear dimension d, which is required for the specification of both the Nusselt and Grashof numbers. With bodies of finite size this length d enters largely as a factor determining the size and structure of the significant heat-carrying elements. For an unlimited heated body and medium, the size of these elements becomes a function of z and the problem becomes one-dimensional, the statistical properties of the flow and temperature fields being dependent on z but not on any horizontal coordinate. It is evident that in some respects z must replace d as one of the basic independent variables of the problem and that the Grashof number, which contains both d and $\Delta\theta$ as constituents, must lose its character as a uniquely defined bulk parameter. It is possible to employ a formally similar number with z replacing d and $\Delta\theta$ recognized as being dependent on z, but a number so defined can only be of local significance; similar considerations apply to the Nusselt number and with these changes of character it seems desirable to approach the problem afresh.

When the heated surface is large enough, it may be supposed that the main buoyant elements at sufficient height are so large that ranges of smaller-scale motions are interposed between them and the molecular scale; the mixing between these elements and their environment will itself be a turbulent type of process. This has been recognized in an earlier treatment by G. I. Taylor, described by Sutton (1953). Very close to the ground there is no scope for the full development of the sub-elemental turbulence, so that the assumption above can only be valid above a certain layer of thickness Δ; consideration will be restricted to $z>\Delta$. In conformity, $\Delta\theta$ must relate to the temperature difference between the level under consideration and that at $z=\Delta$ rather than at the surface itself.

The problem will be subjected to dimensional analysis in Section II in order to indicate the form of relations to be expected. Following the type of postulate commonly made in modern theories of turbulence, it will here be assumed that the mixing of the elements and the consequences of that mixing are not directly dependent on the molecular constants (thermal conductivity and viscosity) of the medium in which the convection takes place. In the remaining sections a more mechanistic theory will be formulated and checked against experimental data. The concern here is pre-eminently with the meteorological problem of convection from the heated ground or water surface but the treatment should permit of considerably wider application.

II. Dimensional Analysis

Given the geometry of a problem in free convection, it is known (e.g. Bosworth, loc. cit.) that six independent quantities are required to specify the physical condition, and the task is to express any dependent quantity, notably the rate of heat loss, in terms of these six. From purely dimensional arguments the solution is not determinate but admits of two degrees of freedom.

In the present problem, the assumption concerning the scale of convective elements effectively eliminates two of the independent variables, and the form of the solution becomes determinate on dimensional grounds alone. It is taken that the heat flux F through the level z will in general depend on z and on the (potential) temperature difference $\Delta\theta$ between the levels z and Δ, on the specific heat per unit volume, c, and on gravity, which exerts its influence through the effect of thermal expansion and so for a perfect gas appears as g/T, where T is the absolute temperature. Putting then

$$F \propto z^{\alpha}(\Delta\theta)^{\beta}\left(\frac{g}{T}\right)^{\gamma}c^{\delta},$$

and equating dimensions the required relation is derived as

$$F \propto c\left(\frac{gz}{T}\right)^{1/2}(\Delta\theta)^{3/2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)^*$$

This law contrasts with the experimental $(\Delta\theta)^{5/4}$ and (turbulent) $(\Delta\theta)^{4/3}$ laws for small heated bodies.

It has been implied that the assignment of a value $\Delta\theta$ at a single specified height z is sufficient to specify the temperature state; were this not so, the result (**1**) should contain an additional factor

$$f\left(\frac{z}{\Delta\theta}\,\frac{\partial\theta}{\partial z},\ z\frac{\partial^2\theta}{\partial z^2}\bigg/\frac{\partial\theta}{\partial z},\ \ldots\right),$$

where f is an arbitrary function of the non-dimensional characteristics of the temperature profile which appear within the bracket. Now it has been shown more generally by Batchelor (1953) that the condition for similarity of two flow patterns of the type considered here is that their Richardson numbers Ri shall be equal. The dimensionless profile characteristics are therefore functions solely of Ri. In the special case of free convection $Ri = -\infty$, the characteristics must be constants and the supposition that the profile is specified by a single value $\Delta\theta$ at a stated height z is confirmed. As a corollary, since, for example, $z(\partial^2\theta/\partial z^2)/(\partial\theta/\partial z)$ is constant with height, the temperature profile will take the form

$$\frac{\partial\theta}{\partial z} \propto z^{-\delta},$$

where δ is a constant, a form which has been anticipated in the literature (Sutton 1953).

* Using the Grashof and Nusselt numbers in the modified sense discussed above, (**1**) may be written

$$(Nu)^2(Gr)^{-1}(Pr)^{-2} = \text{const.}$$

This represents the only relation between these three numbers which involves neither of the molecular coefficients.

Returning to equation (**1**), in the special case where the heat flux is constant with height and provided that, as in the meteorological problem, the absolute temperature varies only within narrow limits, it follows that

$$\Delta\theta \propto z^{-1/3},$$

or

$$\frac{\partial\theta}{\partial z} \propto z^{-4/3}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Thus in the steady state the form of the temperature profile is determinate from dimensional considerations.

From the similarity argument it is equally valid to carry through the dimensional analysis in terms of the local gradient $\partial\theta/\partial z$ rather than $\Delta\theta$, and the result corresponding to (**1**) is then

$$F \propto c\left(\frac{g}{T}\right)^{1/2} z^2 \left(-\frac{\partial\theta}{\partial z}\right)^{3/2}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

from which of course (**2**) again follows as the condition for constant flux. In meteorological work it is conventional further to define the *eddy-conductivity* $K(z)$ by

$$F = -cK\frac{\partial\theta}{\partial z},$$

whence from (**3**)

$$K \propto z^2 \sqrt{-\frac{g}{T}\left(\frac{\partial T}{\partial z}+\Gamma\right)}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

with Γ denoting the adiabatic lapse rate. This relation is the same as that derived by Taylor and by Sutton (1948) with L, defined by these authors as the length over which the convection currents preserve their identity, replaced by z outside the radix. It seems clear from dimensional considerations and the similarity result that L must be proportional to z, since no other length can be constructed from the independent variables of the problem. In the remainder of the paper a detailed mechanistic theory will be put forward to confirm this proportionality, to provide a physical derivation of the results obtained in this section, and a basis for discussion of the considerations which govern the constants of proportionality therein.

III. Mechanistic Theory

(*a*) *Basis of the Theory*

An entirely independent line of approach may be taken by considering the detailed mechanics of the individual element of fluid which is the agent of the heat transfer. This element is to be regarded not as the finest-grained " particle " of the fluid medium, but as an aggregate of particles which have a measure of buoyancy and of vertical motion in common ; the latter properties of the element will accordingly be subject to a continuous mixing with those of the environment by turbulence on a scale which is smaller than that of the element itself. This sub-elemental turbulence may take its origin in part from horizontal wind shear (a mean horizontal flow is allowed in the present treatment provided it is not so strong that forced convection becomes dominant) but mainly as degraded motions from the buoyant motions themselves. The strength of the turbulence, being controlled by the *aggregate* of larger motions, will be a field quantity rather than one which pertains to the individual element.

On this basis it has been shown elsewhere (Priestley 1953) that the equations for the rate of change of temperature T and vertical velocity w of a buoyant element of constant size may be taken as

$$\left.\begin{aligned} \dot{w} &= \frac{g}{T_e}T' - k_1 w, \qquad (5a) \\ \dot{T} &= -w\Gamma - k_2 T', \qquad (5b) \end{aligned}\right\}$$

where T' denotes temperature excess of the element over the environmental temperature T_e and dots denote time derivatives following the mean motion of the particles composing the element at a given instant. k_1 and k_2 are the mixing rates for momentum and sensible heat respectively between the element and its surroundings, proportional to the corresponding sub-elemental turbulent interchange coefficients divided by R^2, where R characterizes the size of the element. k_1 and k_2 are to be regarded as decreasing with increasing size, but as independent of the other properties of the individual element, and as constants following a given element of constant size. These concepts have been discussed more fully in the earlier paper (loc. cit.). It will also be taken that $k_1 = k_2$; this step is made purely for simplification, and the analysis does not depend on it in any fundamental way. A single mixing rate k is accordingly used to identify the size of the particular element under consideration.

The equations (**5**) may then be differentiated following the element and T' eliminated therefrom, which leads (loc. cit.) to a single equation of motion

$$\ddot{w} + 2k\dot{w} + \left[\frac{g}{T_e}\left(\frac{\partial T_e}{\partial z} + \Gamma\right) + k^2\right] w = 0, \qquad (6)$$

provided that T'/T_e is small, which is always the case under natural conditions. The temperature of the element may then be obtained from (**5a**) in the form

$$T' = \frac{T_e}{g}(\dot{w} + kw). \qquad (5a)$$

It is required to solve these equations for a form of environmental temperature profile which is typical of conditions above a heated surface, and thence to formulate the heat flux by recombining the different solutions for elements of different sizes moving through the reference level to which the heat flux relates. The profile of temperature or of potential temperature θ will be taken to be of the form

$$\frac{g}{\theta}\frac{\partial \theta}{\partial z} = \frac{g}{T_e}\left(\frac{\partial T_e}{\partial z} + \Gamma\right) = -Cz^{-\delta}, \qquad (7)$$

where δ and C are constants characterizing its shape and the general magnitude of the gradient respectively. Since the absolute θ in the denominator varies relatively slowly with height, this is in effect equivalent to the general form deduced in Section II, but to preserve independence of the two treatments (7) will here be regarded as an assumed form with sufficient flexibility to cover all practical cases to which the theory will be applied. C is clearly positive, but no theory has previously provided a value of δ. From the dimensional analysis a value of 4/3 is suggested as of special significance and this is confirmed in the following, but for the present δ will be allowed to take any positive value.

(*b*) *Solution of the Equations*

Equation (6) may now be written

$$\dddot{z}=Cz^{-\delta}\dot{z}-k^2\dot{z}-2k\ddot{z}. \qquad (8)$$

We shall consider here only the ascending elements; the principal results are modified only by a numerical multiplier when the descending air is allowed for (Section VI (*a*)). The effects of any externally produced irregularities will be disregarded. That is to say, an individual element will be treated as starting from rest at some level z_0 with a temperature equal to that of the environment at that level. The consequences of the removal of this last assumption are discussed in Section VI (*b*). From (5a) it follows that $\ddot{z}=0$ at z_0, whence (8) can be integrated to

$$\ddot{z}=\dot{w}=\frac{C}{1-\delta}(z^{1-\delta}-z_0{}^{1-\delta})-k^2(z-z_0)-2k\dot{z}. \qquad (9)$$

This is now integrated with respect to z from the starting level z_0 to the reference level z_1 at which the heat flux is required, and we obtain after some rearrangement

$$\tfrac{1}{2}w^2=Cz_1{}^{2-\delta}\left\{\frac{1-\left(\dfrac{z_0}{z_1}\right)^{1-\delta}}{1-\delta}-\frac{1-\left(\dfrac{z_0}{z_1}\right)^{2-\delta}}{2-\delta}\right\}-\tfrac{1}{2}k^2z_1^2\left(1-\frac{z_0}{z_1}\right)^2-2k\int_{z_0}^{z_1}w\,\mathrm{d}z. \qquad (10)$$

This is the equation* for w, regarded as a function of z_1 and dependent also on C, δ, k, and z_0, in a form which lends itself to numerical solution by successive approximation when the values of the independent quantities are specified.

It is possible, without solving explicitly, to infer the form of dependence of w on certain of these independent variables. For this purpose we shall define a critical mixing rate k_c by

$$k_c{}^2=Cz_1{}^{-\delta}. \qquad (11)$$

* The equation is not valid for $\delta=1$ or $\delta=2$, when logarithmic terms would appear; the alternative forms are not set out here, but they have been used whenever applicable in all calculations whose results are quoted hereafter.

This rate has local physical significance in that elements for which $k<k_c$, that is, which are larger than the critical size to which k_c refers, are absolutely buoyant or unstable at the level z_1 (Priestley, loc. cit.). The term containing k^2 in (**10**) may now be expressed as

$$-\tfrac{1}{2}Cz_1^{2-\delta}\left(\frac{k}{k_c}\right)^2\left(1-\frac{z_0}{z_1}\right)^2,$$

and k in the last term as $(k/k_c)C^{1/2}z_1^{-\delta/2}$. It may then be seen on close scrutiny that w will take the functional form

$$w=C^{\frac{1}{2}}z_1^{\frac{2-\delta}{2}}n_1, \qquad (\mathbf{12})$$

where n_1 is a dimensionless number which depends only on δ and the values of z_0/z_1 and k/k_c, and can be calculated from assigned values of these ratios. The formal demonstration that there is a solution of this form is obtained by substitution from (**12**) in (**10**) which then may be reduced to an equation for n_1 in the form

$$\tfrac{1}{2}n_1^2=\frac{1-\left(\dfrac{z_0}{z_1}\right)^{1-\delta}}{1-\delta}-\frac{1-\left(\dfrac{z_0}{z_1}\right)^{2-\delta}}{2-\delta}-\tfrac{1}{2}\left(\frac{k}{k_c}\right)^2\left(1-\frac{z_0}{z_1}\right)^2-2\frac{k}{k_c}\int_{z=z_0}^{z=z_1}n_1\left(\frac{z}{z_1}\right)^{\frac{2-\delta}{2}}\mathrm{d}\left(\frac{z}{z_1}\right), \qquad (\mathbf{13})$$

which shows clearly the quantities on which n_1 must depend. The uniqueness of the solution may be established from (9), for, if w_1 and w_2 are two solutions,

$$\dot{w}_1-\dot{w}_2=-2k(w_1-w_2),$$

or

$$w_1-w_2=\text{const. } \mathrm{e}^{-2kt},$$

and, since both satisfy $w=0$ at $t=0$, $w_1=w_2$.

Turning now to the temperature excess of the element, it is seen from (**5a**) and (9) that

$$T'=\frac{T_e}{g}\left\{Cz_1^{1-\delta}\left(\frac{1-\left(\dfrac{z_0}{z_1}\right)^{1-\delta}}{1-\delta}\right)-k^2z_1\left(1-\frac{z_0}{z_1}\right)-kw\right\}, \qquad (\mathbf{14})$$

which can be calculated after w has been obtained from (**10**). From similar reasoning to the above,

$$T'=\frac{T_e}{g}Cz_1^{1-\delta}n_2, \qquad (\mathbf{15})$$

whence

$$wT'=\frac{T_e}{g}C^{3/2}z_1^{\frac{4-3\delta}{2}}n_3, \qquad (\mathbf{16})$$

where n_2 and n_3, like n_1, are numbers dependent only on δ and on z_0/z_1 and k/k_c.

(c) *Formulation of the Heat Flux and other Field Statistics*

To formulate the heat flux F equation (**16**) must be averaged over all particles crossing the reference plane z_1 at a given instant. The elements to which the particles belong have a variety of sizes, characterized by k, and of starting levels z_0, so that a double averaging process is involved. We introduce a probability function $f(k, z_1, C, \delta)$ to represent the probability that a particle at z_1 will, for a given C and δ, belong to an element of size k; the probability distribution is in effect a frequency distribution for k weighted according to element size. The four arguments of f permit of only two dimensionless combinations, δ and $kC^{-1/2}z_1{}^{\delta/2}$ or k/k_c, so that the probability distribution can be represented by $f(k/k_c, \delta)$, where

$$\int_0^\infty f\left(\frac{k}{k_c}, \delta\right) \mathrm{d}\left(\frac{k}{k_c}\right) = 1.$$

The averaging of (**16**), for a given value of δ, is now carried out first over all admissible values of the starting level z_0 for a fixed value of k; this step will be indicated by a bar (a bar will also be used to denote averages taken over all particles, but there should be no confusion). The average is then taken over all values of k, and the result is seen to be

$$\frac{F}{\rho c_p} = \overline{wT'} = \frac{T_e}{g} C^{3/2} z_1{}^{\frac{4-3\delta}{2}} N_3. \qquad \text{(17)}$$

N_3 is another number, which is given in terms of n_3 by

$$N_3 = \int_0^\infty f\left(\frac{k}{k_c}, \delta\right) \overline{n_3\left(\frac{k}{k_c}, \delta, \frac{z_0}{z_1}\right)} \mathrm{d}\left(\frac{k}{k_c}\right).$$

To understand the meaning of the term " admissible value " it is necessary to go more closely into the details of the motion and temperature behaviour. This involves numerical calculation in the equations set out above; some of the computations are presented in another context in Section V, but here it will be sufficient to state the main features and illustrate them by a diagram (Fig. 1).

Consider, first, elements all of a given size; that is, fix the value of k, which will be the critical value (equation (**11**)) for some level P on the profile. For an element at rest at P the tendency to mix will just compensate for the instability of the lapse rate, and the element will just not move. Elements at rest above P are in a less unstable environment and will not move spontaneously. At Q, a short distance below P, the instability at first outweighs the mixing and an element here will move upwards, reaching the level of P with finite velocity and temperature excess; from here onwards the coefficient of w in (**6**) has changed sign and becomes increasingly positive, the mixing will outweigh the decreasing environmental instability, and the element will be brought to rest at some higher level Q'. The final approach to rest is of the exponential type and $\dot{w}$ and hence T' tend to zero simultaneously, the element being completely merged into its new surroundings. An element from a lower level R will reach the level of P with a larger w and T' than that from Q and so will penetrate above Q', to R';

and so on, SS', TT', etc., representing paths of elements starting from successively lower levels.

Now consider all elements as they cross the reference level z_1. If $k<k_c$, P will lie above z_1, and an element of this size reaching z_1 may have started at any level below z_1. For these values of k, therefore, the averaging of n_3 in the integral for N_3 must extend from $z_0=z_1$ down to $z_0=\Delta$. If, however, $k>k_c$, P will lie below z_1 as in the figure, and it is evident that elements reaching z_1 must have started from some level considerably below z_1. In Figure 1, an

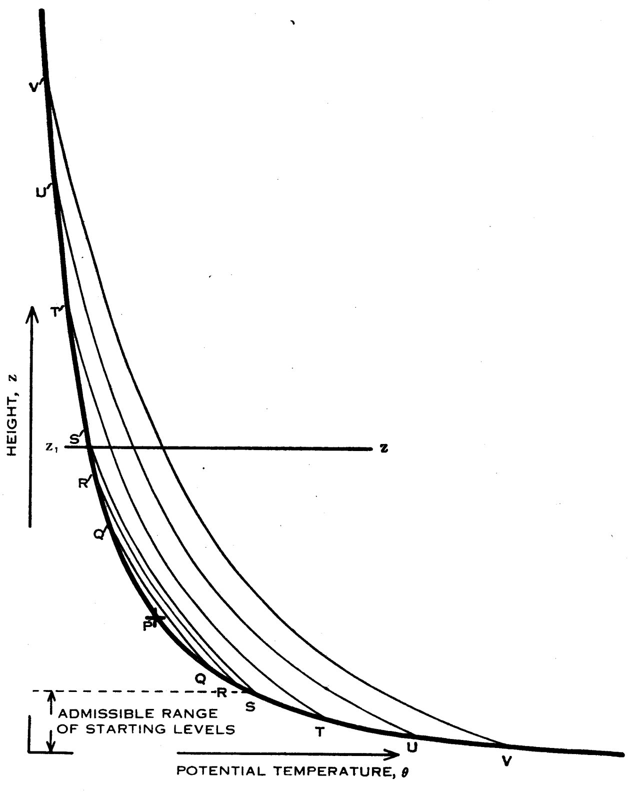

Fig. 1.—Schematic representation of paths of elements. These are represented as thin lines against the thick line for the mean profile of the environment.

element starting from between z_1 and S would fail to reach z_1. In this case the averaging of n_3 over z_0 must extend only over the " admissible range " from the level S down to $z_0=\Delta$. The level S will depend on k, and the value of z_0 appropriate to it may be derived most readily from (9) or (14). It is given by

$$\left(\frac{k}{k_c}\right)^2=\frac{1-\left(\dfrac{z_0}{z_1}\right)^{1-\delta}}{(1-\delta)\left(1-\dfrac{z_0}{z_1}\right)},$$

which shows that z_0/z_1, at the top of the admissible range, is determined by δ and k/k_c only.

It has now been shown that the upper limit of the range of values of z_0, over which n_3 must be averaged, is either equal to z_1 (for $k<k_c$) or (for $k>k_c$) *to a given submultiple of* z_1 which depends only on k/k_c and δ. The lower limit is $z_0=\Delta$, and this is taken to be effectively equivalent to $z_0=0$, assuming convergence of the integral as $z_0 \to 0$ which will be established later in the paper (Section V). It now follows from the result established above that $\overline{n_3}$ will be a function of k/k_c and δ only ; that is, *it will not otherwise contain* z_1. It then follows further that N_3, which is essentially a weighted average of $\overline{n_3}$ with respect to k/k_c, is independent of z_1.

It has thereby been proved that N_3, though dependent on δ and the form of the unknown function f, is independent of z_1 and of C ; and hence that the indices of z_1 and C appearing in (**17**) correctly represent the functional dependence of heat flux on height and on lapse rate.

In a similar way, corresponding expressions may be derived for other statistical representations of the temperature and velocity distributions at the level z_1. Those to which subsequent appeal will be made are

$$\overline{w}=C^{\frac{1}{2}}z_1^{\frac{2-\delta}{2}}N_1, \quad \text{(18)}$$

$$\overline{T'}=\frac{T_e}{g}Cz_1^{1-\delta}N_2, \quad \text{(19)}$$

$$\sigma_w=\sqrt{\overline{w^2}}=C^{\frac{1}{2}}z_1^{\frac{2-\delta}{2}}N_4, \quad \text{(20)}$$

$$\sigma_T=\sqrt{\overline{T'^2}}=\frac{T_e}{g}Cz_1^{1-\delta}N_5, \quad \text{(21)}$$

where in general

$$N_i=\int_0^\infty f\left(\frac{k}{k_c},\,\delta\right)\overline{n_i}\,\mathrm{d}\left(\frac{k}{k_c}\right), \quad \text{(22)}$$

and the derivation of n_4 and n_5 from (**10**) and (**14**) is along similar lines to that of n_1, n_2, and n_3.

N_1 to N_5 have the common property that they depend on δ and on the form of f, but not on z_1 and C. The foregoing analysis has thus separated into two parts the problem of describing the fields of temperature and motion. The first part relates to the shape of the temperature profile, that is, the value of δ, and to those relationships which depend on it alone, namely the form of F, σ_T, etc., as functions of height and of lapse rate. This part will be treated in Sections III (*e*) and IV, and the second, relating to the absolute values of the multiplying factors N_1 to N_5 which depend on f as well as on δ, in Section V.

In passing, it has been established from equations (**10**) and (**14**) that z_1 is itself the length scale which characterizes all aspects of the temperature and velocity fields at the level z_1 ; in particular that the admissible range of initial levels is proportional to z_1. It follows that L, the convective length envisaged by Taylor and Sutton, is proportional to z_1, and that the form of the relation (**4**) proposed here can be reconciled with theirs. Up to this point there is no conflict between the two treatments, but a relation introduced by Sutton (1948) at a later stage, that $L \propto z^{1 \cdot 35}$, is not in harmony with the theory offered here.

(*d*) *Forced Convection and Radiation*

The relations (**17**)–(**21**) are derived on the basis that the statistics of the temperature and vertical velocity distribution are determined from the operation of free convection alone. In nature, both radiation and forced convection exert complicating influences. Whereas the effects of the former lie clearly outside the bounds of the present treatment, this is not entirely so in the case of forced convection ; rather in the atmospheric problem is it idle to attempt too fine a distinction between free and forced convection, since they occur together and their effects are inseparable. The extent to which the vertical movement in an unstably stratified boundary layer exceeds that in a neutral layer at the same wind strength would appear to be a consequence solely of the processes which are here under consideration. For this reason it might be expected that the predictions of the theory would have approximate validity when a wind is blowing, provided it is not too strong or the lapse rate too weak.

In short, the results should extend to finite values of the Richardson number, but at the present stage it is probable that experiment will be more successful than theory in indicating the limits of validity. The relations (**18**) and (**20**) will evidently cease to hold when the buoyant motions no longer outweigh those deriving from wind shear but it may well be that at this stage the temperature fluctuations, and hence the heat flux, will still be dominated by the former. Thus the range of applicability of the other predictions may extend considerably beyond that of (**18**) and (**20**).

(*e*) *Specification of the Index* δ

From (**17**) it follows immediately that F is constant with height when $\delta = 4/3$, and that it decreases and increases with height when δ is greater and less than 4/3 respectively. Thus, in the absence of radiation influences and strong forced convection, $\delta = 4/3$ will characterize the steady state, $\delta > 4/3$ a condition in which the air is warming, and $\delta < 4/3$ a condition in which it is cooling.

In practice, in the meteorological case at any rate, an upwards flux of any considerable magnitude will only vary with height very slowly. For a modest F of about 10 mW/cm^2, a 5 per cent. change in flux between 1 and 2 m would represent heating or cooling at 15 °C/hr, a rate which is never approached ; yet such changes would be represented by $\delta = 1 \cdot 29$ and $\delta = 1 \cdot 38$. The conclusion is that δ will in practice be constrained to a value quite close to 4/3. This value will accordingly be adopted as a definite prediction to be tested, together with the consequences which follow from it in equations (**17**)–(**21**), against actual measurements of the profile and of F, σ_T, and σ_w.

An exception to the value 4/3 may be expected from the bottom layer (below about 1 m), which normally under lapse conditions is strongly warmed by radiation (Robinson 1950). To maintain the quasi-steady state in this layer the effects of radiation and convection must approximately compensate, and a δ lower than 4/3 is therefore to be expected.

It is of interest to remark that an index of 4/3 and approximately constant flux imply that the eddy-conductivity K is proportional to $z_1^{4/3}$, which resembles theoretical and empirical results in other problems of turbulent transfer over a very wide range of scales.

IV. Experimental Data

(a) *The Mean Temperature Profile*

The first prediction to be tested is the temperature profile law, $\delta=4/3$. For a few clear days in summer at Leafield, England, Sutton (1948) finds $\delta=1\cdot75$ but an analysis by Deacon (1948) of a far larger number of observations from the same site yields values ranging from $1\cdot23$ to $1\cdot53$ at different ranges of wind speed. Deacon (unpublished data) has analysed data from other sites with the results shown in Table 1.

Table 1

ANALYSIS OF TEMPERATURE PROFILES BY DEACON

Location	Height Range (m)	Reference for Original Data	Conditions	δ
Porton, England	1·2–17	Deacon (1953)	256 clear days (May, June, July) 1923–43, 1100–1400 hr	1·35
Ismailia	1·1–46	Flower (1937)	All days Apr.–Sept. 1932, 1400–1500 hr	1·43
Manor, Texas	2–25	Gerhardt *et al.* (1948), Gerhardt (1949)	Fair or fine days,* July–Oct. 1948, 1100–1300 hr	1·25
Edithvale, Victoria	1–30	Unpublished	Fine days, Nov. 1951–Mar. 1952, 1200–1600 hr	1·25

* With wind at 12 m less than 4·5 m/sec.

There is some scatter, but much of this may result from attributing too fine an accuracy to the experimental measurements. Extraction of a value of δ from a set of measured temperatures involves essentially the evaluation of second differences between temperatures which are subject to errors of calibration, retention of calibration, standardization of shielding and of aspiration; such errors may be to a considerable extent systematic over the period of observation. The point may be illustrated by Table 2 in which are given two calculated profiles for $\delta=1\cdot5$ and $1\cdot3$. These two profiles, which represent a strong lapse rate, differ nowhere by more than 0·11 °F, a measure of accuracy which can hardly be claimed for much of the original data.

A separate analysis has been made of some published data on strong lapse profiles in the layers mainly below 1 or 2 m where radiational heating becomes large (Robinson, loc. cit.) and the present theory would predict a value of δ less than 4/3. The results are shown in Table 3. Save in the case of the Porton data, the number of levels of observation were more than the three required to provide a value of δ, and the value given was obtained by least squares applied to the logarithms of the heights and measured temperatures.

The data from Poona are of particular interest because they provide a simultaneous and detailed profile both above and below the level where radiation effects become important. Analysis of the profile in two overlapping regions indicates a considerable difference in δ.

TABLE 2

CALCULATED TEMPERATURE PROFILES FOR $\delta=1\cdot5$ AND $1\cdot3$

Height (m)	2	4	8	16	32
Temperature (°F), $\delta=1\cdot5$ . .	73·45	72·92	72·16	71·06	69·41
,, (°F), $\delta=1\cdot3$ . .	73·55	72·89	72·05	70·96	69·51

The individual mean profiles summarized in Tables 1 and 3 are shown diagrammatically* in Figure 2. A striking feature is that all the gradients from the stations in the lower latitudes (Ismailia, Manor, Edithvale, and Poona) lie approximately on *the same* profile above 1 m, and a single dashed line of slope $\delta=4/3$ has been drawn by eye to illustrate the closeness of fit of the prediction.

TABLE 3

PROFILES AT LOW LEVELS

Location	Height Range (cm)	Reference	Conditions	δ
Porton, England	2·5–120	Best (1935, Table VI)	58 clear occasions in June, 1100–1300 G.M.T.	1·22
Cambridge, England	25–200	Pasquill (1949)	15 lapse profiles	1·05
Kew, England	5–202	Rider & Robinson (1951)	15 lapse profiles	1·40
Poona, India	1–180	Ramdas (1953)	April, hr of max. temp.	0·93
,, ,,	90–750	,, ,,	,, ,, ,,	1·29

The full lines I, II, and III represent the calculated regression lines for the Poona, Cambridge, and Kew data respectively, the first being shown in two parts relating to the two overlapping ranges.

On the whole the theoretical predictions, that δ shall be 4/3 above about 1 m and less below, fit the facts within the expected tolerance.

* Where the original data are given as temperatures at, or differences between, fixed levels, the slope of the chord has been plotted against the geometric mean height. This is strictly legitimate only for $\delta=1$, but the error involved in other cases is very small.

(b) Heat Flux and Temperature and Velocity Fluctuations

With substitution of the value $\delta=4/3$ in (**17**)–(**21**) the variations of the flow and temperature characteristics with height, as well as with lapse rate, become determinate.

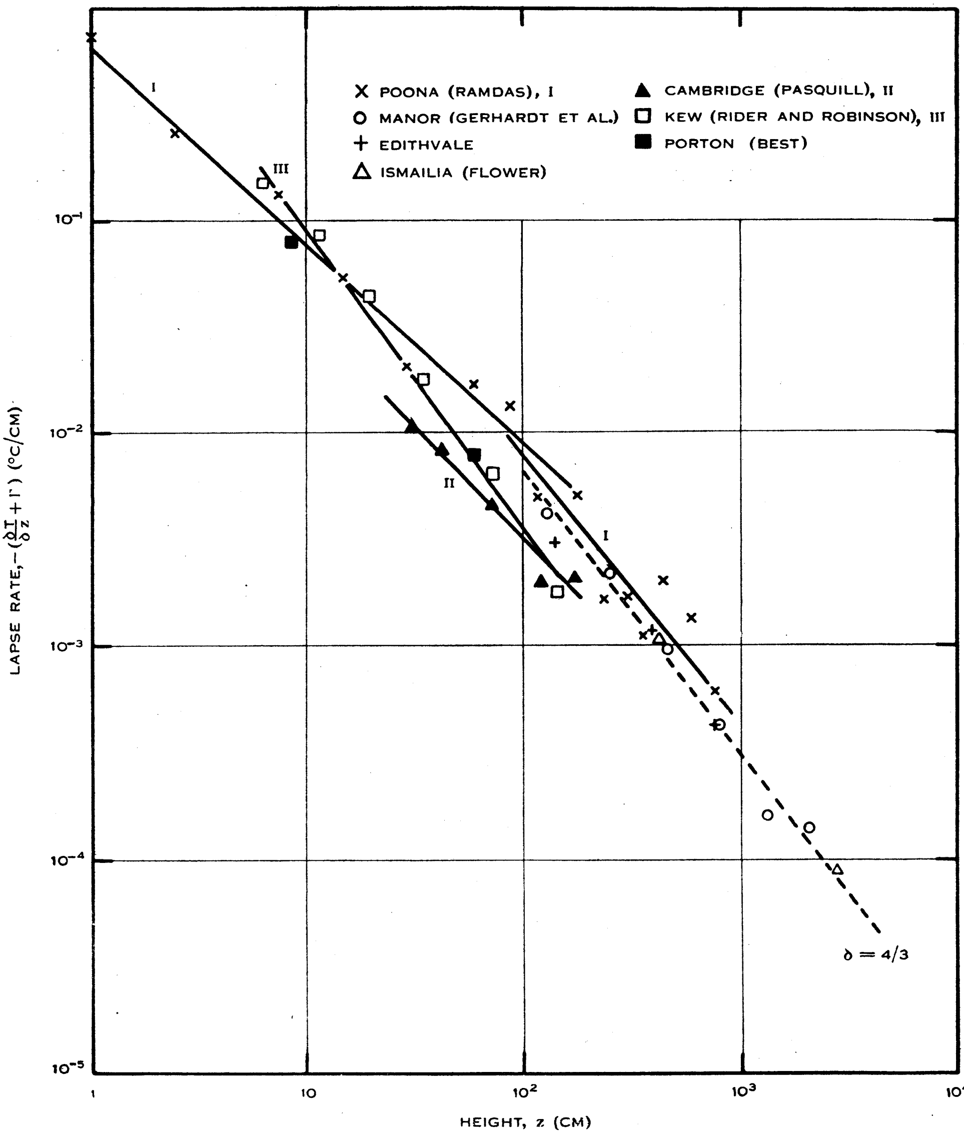

Fig. 2.—Lapse profiles from 1 cm to 100 m.

The published literature does not contain the information on which a satisfactory test of these theoretical predictions can be made. However, a group of workers in the Section of Meteorological Physics, C.S.I.R.O., has made measurements of the quantities involved at Edithvale, Vic., using the instruments

TABLE 4

EXPERIMENTAL DATA FROM EDITHVALE ON 3.iii.1952

Run No.	1	2	3	4	5	6	7	8	9	10	11	12
Wind velocity at 2 m (m/sec)	4·1	4·2	4·5	3·9	4·5	4·9	4·6	5·1	4·4	4·6	4·4	4·4
Potential temperature gradient $-\partial\theta/\partial z$ at 1·5 m (°C/m)	0·83	0·71	0·71	0·59	0·55	0·58	0·51	0·50	0·24	0·22	0·18	0·13
Height of exposure of fine structure apparatus (m)	1·92	29	1·92	29	1·92	29	1·92	29	1·92	29	1·92	29
Heat flux F (mW/cm²)	19·9	17·0	20·1	23·4	14·4	37·7	10·9	4·8	5·4	9·9	2·8	0·4
σ_T (°C)	1·00	0·38	0·88	0·32	0·70	0·47	0·56	0·18	0·27	0·14	0·20	0·11
σ_w (cm/sec)	39	70	41	92	36	91	39	58	39	84	37	55
Negative Richardson number at 1·5 m	0·14	0·11	0·11	0·12	0·10	0·07	0·06	0·04	0·04	0·02	0·02	0·02

and techniques developed by Swinbank (1950). Although at the time of writing much of the data still awaits analysis, sufficient is available to allow preliminary tests of the predictions.

In the experiments the apparatus designed by Swinbank for the measurement, among other things, of heat flux was exposed for 5-min periods usually, though not always, alternately at two different heights above ground level. From the records obtained, the values of F, σ_T, and σ_w have been extracted. Auxiliary observations included the wind and temperature profile ; for purposes of the present tests these have been reduced by an objective method to temperature gradient and Richardson number at a standard level (1·5 m).

In Table 4 are presented the results of a sequence of measurements made on a single day in which a large range of lapse rates was experienced, conditions otherwise remaining steady.

To analyse these results, the logarithms of the measured quantities F, σ_T, and σ_w have been taken and the regression line

$$\log \lambda + \mu \log C + \nu \log z$$

fitted to them, the values of μ and ν and of their standard errors being calculated by the method of least squares. The regressions are given in Table 5, standard errors being shown in brackets and the theoretical relations set alongside.

TABLE 5

TEST OF THEORETICAL PREDICTIONS

Theoretical Relation	Empirical Result
$F \propto C^{3/2}$, independent of z	$F \propto C^{1\cdot73(\pm0\cdot32)} z^{0\cdot00(\pm0\cdot14)}$
$\sigma_T \propto C z^{-1/3}$	$\sigma_T \propto C^{0\cdot88(\pm0\cdot10)} z^{-0\cdot26(\pm0\cdot04)}$
$\sigma_w \propto C^{1/2} z^{1/3}$	$\sigma_w \propto C^{0\cdot02(\pm0\cdot06)} z^{0\cdot24(\pm0\cdot03)}$

Only the σ_w relation departs radically from its predicted form, and this was expected since a wind of 4–5 m/sec blew steadily throughout, while the theoretical σ_w relates only to those motions attributed to buoyancy. The results lend support to the speculations of Section III (*d*) that in fair winds the buoyant *motions* may constitute only a minor part of the vertical motion, but that it is this minor part which is correlated with the temperature fluctuations and so dominates the heat flux.

The agreement for F and σ_T justifies a more severe test by using less homogeneous material. All the complete sets of measurements so far available from other occasions are set out in Table 6, the only criterion for acceptance being that $|Ri|$ should be not less than 0·02.

The regression relations, from the whole set of 30 runs, are

$$F \propto C^{1\cdot68(\pm0\cdot20)} z_1^{-0\cdot04(\pm0\cdot08)},$$

$$\sigma_T \propto C^{0\cdot92(\pm0\cdot09)} z_1^{-0\cdot25(\pm0\cdot04)}.$$

TABLE 6

EXPERIMENTAL DATA FROM OCCASIONS OTHER THAN 3.iii.52

Date	21.i.52					30.i.52			18.iii.52			4.iii.52		20.ii.53			25.ii.52	
Run No. (for identification)	1	2	3	4	5	1	2	3	3	5	6	47	48	4	5	9	7	38
Wind at 2 m (m/sec)	5·24	5·12	5·54	5·26	5·31	3·10	3·41	3·12	1·86	1·90	1·91	4·39	4·10	1·56	1·87	2·01	4·45	5·08
$-\partial\theta/\partial z$ at 1·5 m (°C/m)	0·71	0·72	0·65	0·63	0·60	0·63	0·59	0·57	0·24	0·20	0·25	0·25	0·22	0·16	0·38	0·42	0·51	0·33
Height z (m) . .	1·5	29	1·5	29	1·5	1·5	29	1·5	1·5	1·5	23	1·92	29	1·5	1·5	1·5	1·85	1·85
F (mW/cm²) . .	18·7	21·0	22·3	17·1	23·3	20·4	6·4	21·1	4·4	5·3	2·2	7·0	5·1	0·5	2·6	6·8	16·3	11·3
σ_T (°C)	0·80	0·435	0·71	0·345	0·885	0·87	0·485	0·81	0·395	0·470	0·187	0·345	0·126	0·135	0·208	0·526	0·82	0·40
$-Ri$ at 1·5 m . .	0·04	0·05	0·04	0·03	0·03	0·18	0·11	0·12	0·24	0·11	0·24	0·04	0·06	0·20	0·22	0·33	0·02	0·02

The agreement is on the whole a little better than in Table 5, and the uncertainty of the index of C in F is much reduced. It is desirable that more measurements be made, sufficient to be analysed in ranges of the Richardson number, before the goodness of fit be finally judged.

Measurements of σ_T from 7 to 75 m at Leafield by Johnson and Heywood (1938) have been analysed by Sutton (1948) who finds $\sigma_T \propto z^{-0\cdot 4}$, but this result must carry less weight owing to the much lower sensitivity of the instruments used in the early measurements.

It is usual to regard F as determined by some suitable combination of the characteristics of the wind and temperature profiles, and to this end the data of Tables 4 and 6 may be analysed in another way, by calculating the partial correlations of F with each of the three characteristics C, Ri, and u (wind speed). The results are

$$r_{FC,uRi} = +0\cdot 63(\pm 0\cdot 11 \text{ S.E.}),$$

$$r_{Fu,CRi} = +0\cdot 11(\pm 0\cdot 18),$$

$$r_{FRi,Cu} = -0\cdot 02(\pm 0\cdot 18).$$

The total correlation between F and C was $+0\cdot 77$. For two quantities between which the true relation is non-linear, this is a very high coefficient. And the general thesis of this paper, that in the more unstable situations the heat flux is dominated by its dependence on lapse rate alone, appears to be established under the conditions of these experiments.

V. The Constants of Proportionality

(a) *Upper Limits to the Constants*

It remains to examine in greater detail the constants of proportionality, N_1 to N_5, which occur in equations (**17**)–(**21**). Although the exact values of these numbers have been shown to depend on the form of the function f, and therefore must remain unknown from theoretical considerations until this function is provided, useful inferences can be made about their relative magnitudes and it will be shown that their orders of magnitude also are determinable from the theory at its present stage.

In continuing the theoretical treatment of Sections III (b) and III (c) we shall retain the freedom there allowed to the value of δ, delaying the identification $\delta = 4/3$ as late as possible.

An element starting from z_0 will have a velocity and temperature excess at z_1 smaller than if there were no mixing ($k=0$). We shall denote the values attained on the latter basis by $\overset{*}{w}$ and $\overset{*}{T}'$, and refer to them as the *free* velocity and temperature excess. The free values, when substituted in the expressions for F, σ_T, etc. provide an upper limit to the values of these quantities. The device is equivalent to the supposition of a special form for the function $f(k/k_c, \delta)$, assigning f the value 1 for $k=0$ and $f=0$ for all other k. We shall evaluate this upper limit and, by comparing it with measured values, obtain a quantitative assessment of the importance of the mixing process.

From (10) and (14), then

$$\tfrac{1}{2}\overset{*}{w}^2=Cz_1^{2-\delta}\left\{\frac{1-\left(\frac{z_0}{z_1}\right)^{1-\delta}}{1-\delta}-\frac{1-\left(\frac{z_0}{z_1}\right)^{2-\delta}}{2-\delta}\right\}, \quad \ldots\ldots \quad (23)$$

$$\overset{*}{T}'=\frac{T_e}{g}Cz_1^{1-\delta}\left\{\frac{1-\left(\frac{z_0}{z_1}\right)^{1-\delta}}{1-\delta}\right\}, \quad \ldots\ldots \quad (24)$$

and the substitution of these values leads to relations identical with (17)–(21) with the numbers N_1 to N_5 replaced by integrals I_1 to I_5 given by

$$I_1(\delta)=\frac{z_1}{z_1-\Delta}\int_{\Delta/z_1}^{1}\sqrt{2}\left\{\frac{1-\theta^{1-\delta}}{1-\delta}-\frac{1-\theta^{2-\delta}}{2-\delta}\right\}^{\frac{1}{2}}d\theta,$$

$$I_3(\delta)=\frac{z_1}{z_1-\Delta}\int_{\Delta/z_1}^{1}\sqrt{2}\left\{\frac{1-\theta^{1-\delta}}{1-\delta}-\frac{1-\theta^{2-\delta}}{2-\delta}\right\}^{\frac{1}{2}}\left\{\frac{1-\theta^{1-\delta}}{1-\delta}\right\}d\theta,$$

and I_2, I_4, and I_5 in similar form. θ has been written for z_0/z_1 (z_1 fixed while z_0 varies) and Δ is defined in Section I. For $z_1 \gg \Delta$, so long as the integrals converge as the lower limit tends to zero, we may in practice take it as zero and drop the factor $z_1/(z_1-\Delta)$ outside the integral sign.

These integrals have been evaluated, the curve for $I_3(\delta)$ being shown in Figure 3 and values of I_1 to I_5 at discrete values of δ in Table 7.

TABLE 7

UPPER LIMITS TO VALUES OF N_1 TO N_5

δ	0·7	0·9	1·1	4/3	1·5	5/3
I_1	0·67	0·74	0·81	0·94	1·02	
I_2	0·77	0·91	1·11	1·5	2·0	
I_3	0·82	1·22	1·91	4·0	8·4	∞
I_4	0·875	0·93	1·07	1·34	1·63	
I_5	0·98	1·23	1·67	3·0	∞	

None of the integrals diverge for $\delta=4/3$, the case of greatest physical interest. This establishes the required convergence property for $\overline{n_i}$, which was referred to in Section III (*c*), for $k=0$. For $k>0$, the convergence holds *a fortiori*.

Reverting now and hereafter to the special case $\delta=4/3$, it has been shown that upper limits to the statistical quantities are provided by the theory as

$$\overset{*}{w}=0\cdot94C^{1/2}z_1^{1/3}, \quad \ldots\ldots \quad (25)$$

$$\overset{*}{T}'=\frac{3}{2}\frac{T_e}{g}Cz_1^{-1/3}, \quad \ldots\ldots \quad (26)$$

$$\overset{*}{F}=4\rho c_p\frac{T_e}{g}C^{3/2}, \quad \ldots\ldots \quad (27)$$

$$\overset{*}{\sigma}_w=1\cdot34C^{1/2}z_1^{1/3}, \quad \ldots\ldots \quad (28)$$

$$\overset{*}{\sigma}_T=\frac{3T_e}{g}Cz_1^{-1/3}. \quad \ldots\ldots \quad (29)$$

(*b*) *Correction Factors to the Upper Limits*

Comparing (**27**) and (**29**) with the data in Tables 4 and 6, we find

$$\text{Geometric mean ratio } \frac{\overset{*}{F}}{F(\text{observed})} = 4\cdot 7,$$

$$\text{Geometric mean ratio } \frac{\overset{*}{\sigma}_T}{\sigma_T(\text{observed})} = 3\cdot 1.$$

That the upper limits of F and σ_T are not divorced by an order of magnitude from the measured values is an achievement, considering that the estimate in

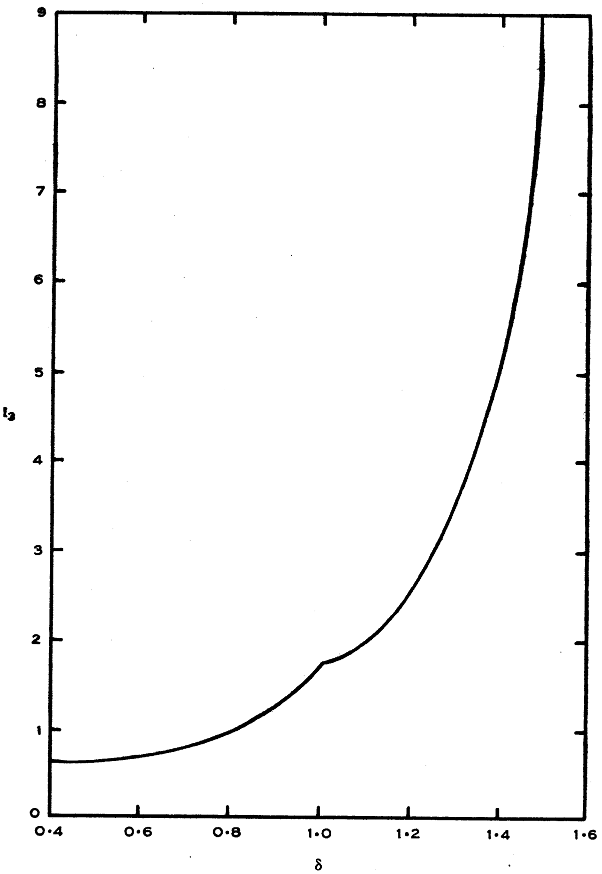

Fig. 3.—Values of flux integral (I_3) with no mixing.

effect invokes the measured lapse rate only. Further examination of the " free " motion device is suggested, in order to see how much can be stated about the correction factors to $\overset{*}{F}$ etc., when mixing is allowed for.

Following (**12**) we may write, for $\delta = 4/3$,

$$w = C^{1/2} z_1^{1/3} n_1\left(\frac{k}{k_c}, \frac{z_0}{z_1}\right), \qquad \ldots\ldots\ldots\ldots\ldots \quad (\mathbf{30})$$

and similarly

$$\overset{*}{w}=C^{1/2}z_1{}^{1/3}\overset{*}{n}_1\left(\frac{z_0}{z_1}\right), \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{31})$$

so that

$$w/\overset{*}{w}=n_1/\overset{*}{n}_1,$$

which will be referred to as the mixing factor for w; numbers $\overset{*}{n}_2$ to $\overset{*}{n}_5$ and corresponding mixing factors may be similarly defined. All the mixing factors may be calculated by substitution from (**30**) etc. in (**10**) and (**14**), and from (**31**) etc. in (**23**) and (**24**).

The calculation has been carried out for z_0/z_1 from 1/2 to 1/200, over the full significant range of k/k_c, for w from equation (**10**) and for T' from (**14**). The range of z_0 chosen was such as to contribute three-quarters of the total value of the flux integral I_3, most of the remaining quarter coming from $z_0<z_1/200$, to which the calculation may be extended as required. Figures 4 and 5 show isopleths of the factors $n_1/\overset{*}{n}_1$ and $n_2/\overset{*}{n}_2$, drawn on diagrams with z_0/z_1 as abscissa and k/k_c as ordinate. Table 8 gives in detail the contributions to $\overset{*}{F}$ from various ranges of z_0/z_1 (or θ), indicating the potential relative importance of the different possible levels of origin.

TABLE 8

CONTRIBUTION TO $\overset{*}{F}$ OR I_3 FROM VARIOUS RANGES OF $\theta=z_0/z_1$

z_0/z_1 from ..	0	0·002	0·005	0·01	0·05	0·1	0·2	0·5
to	0·002	0·005	0·01	0·05	0·1	0·2	0·5	1
Percentage of integral	14·5	7	7·5	29	15	12	12	3

These calculations complete the information as far as the theory at present allows, enabling the multiplying constants in (**17**)–(**21**) to be immediately worked out once the single remaining unknown, the function $f(k/k_c)$, is assigned. An approximate simplification is possible by use of the features of Figures 4 and 5 that the mixing factors are found to depend on k/k_c much more strongly than on z_0/z_1. That is to say, the variation of n_i with z_0/z_1 is largely already taken up in $\overset{*}{n}_i$, which allows as a working approximation

$$\frac{n_i\left(\dfrac{k}{k_c},\dfrac{z_0}{z_1}\right)}{\overset{*}{n}_i\left(\dfrac{z_0}{z_1}\right)}\simeq\frac{n_i\left(\dfrac{k}{k_c},\dfrac{1}{25}\right)}{\overset{*}{n}_i\left(\dfrac{1}{25}\right)},$$

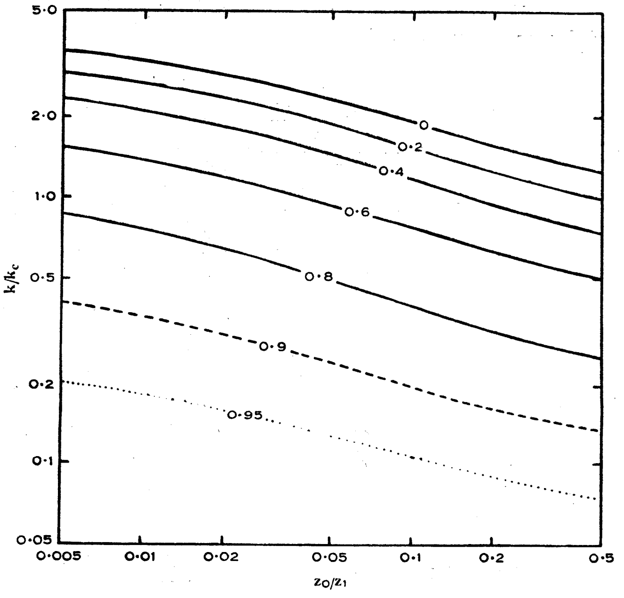

Fig. 4.—Mixing factor for velocity ($w/\overset{*}{w}=n_1/\overset{*}{n}_1$).

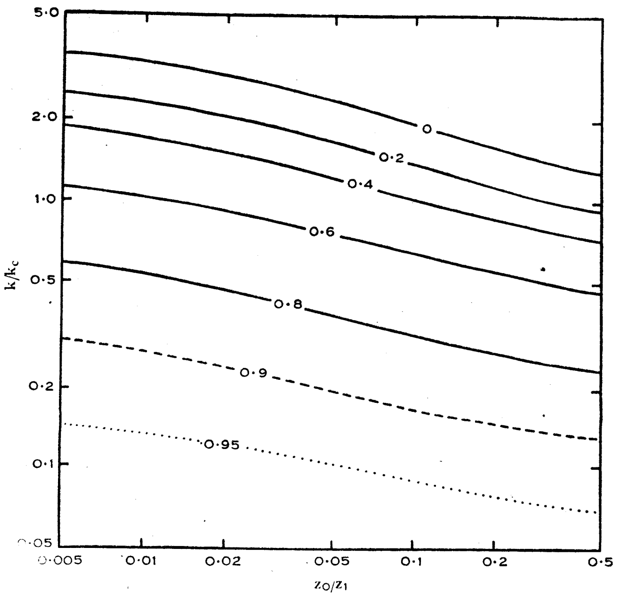

Fig. 5.—Mixing factor for temperature ($T'/\overset{*}{T'}=n_2/\overset{*}{n}_2$).

the value 1/25 being taken as it is roughly the median for the integral I_3. This means that the numerical results (**25**)–(**29**) are approximately corrected for mixing by multiplication by the factor

$$\int_0^\infty f\left(\frac{k}{k_c}\right)\frac{n_i\left(\frac{k}{k_c},\frac{1}{25}\right)}{\overset{*}{n}_i\left(\frac{1}{25}\right)}\mathrm{d}\left(\frac{k}{k_c}\right). \quad \ldots\ldots\ldots\ldots\ldots \quad (\mathbf{32})$$

It is seen from Figures 4 and 5 that the mixing factor for w always exceeds the mixing factor for T'. It follows that the mixing factors for σ_T^2, F, and σ_w^2 are in ascending order of magnitude.

As predicted, the mixing factors for F and σ_T show no considerable variation with C and z_1 in the data of Tables 4 and 6, and their mean values of $1/4\cdot7$ and $1/3\cdot1$ indicate, on reference to (**32**) and Figures 4 and 5, that the dominant part of the $f(k/k_c)$ distribution is at k/k_c between about 1 and 2. The dominant mixing rate at z_1 is derived accordingly from (**11**) and (**7**) as

$$k=1 \text{ to } 2\sqrt{Cz_1^{-4/3}}=1 \text{ to } 2\sqrt{\frac{g}{T}\left|\frac{\partial T}{\partial z}+\Gamma\right|}. \quad \ldots\ldots \quad (\mathbf{33})$$

This expression provides a yardstick for the time of response of sensing elements and recorders necessary for accurate measurement of heat flux and temperature variations.

VI. Extensions of the Theory

(*a*) *Allowance for Descending Air*

Throughout the development of the theory, consideration has been confined to ascending elements. It has been shown in Section III (*c*) that for an element, starting from rest, to penetrate a finite distance it must be absolutely buoyant ($k<k_c$) at its level of origin ; and it will then continue upwards until, through mixing, it is finally brought to rest at some level above that at which it ceases to be absolutely buoyant.

Descending elements, though subject to the same mixing equations as the ascending elements, encounter the differing environmental conditions in the reverse order, and this profoundly alters their behaviour. An element descending with a temperature deficit, if absolutely buoyant in the negative sense at one level, will be so at all lower levels. Under mixing and buoyancy alone, the velocity and temperature deficit would increase indefinitely. Near the heated boundary of a semi-infinite medium, therefore, the factors limiting the motion of the ascending and descending air are not the same ; practical evidence comes from the work of Ramdas and colleagues (Ramdas 1953) who have observed a striking difference in character between the motion of the ascending and the descending air. Whereas the ascending elements are restrained by mixing, the principal constraint on the descending air must be exercised by some other influence which is, almost certainly, the presence of the boundary itself.

In order to describe in detail the behaviour of the descending air it would be necessary to formulate the effects of this influence, but for present purposes this can be avoided provided that the influence may be assumed to be independent of the size of the element on which it operates. The descending air may then, as a reasonable working approximation, be treated as a single mass, of uniform velocity and temperature at any given level. Using suffixes a and d for ascending and descending air, the condition for continuity of mass is then

$$-pw_d = q\overline{w}_a, \qquad (p+q=1)$$

where p, q denote the respective mass weighting fractions. Since T' is defined as difference from the average temperature of the *whole* environment, there is also the condition

$$-pT'_d = q\overline{T'_a}.$$

The overall average of wT' is then

$$q\overline{(wT')}_a + pw_dT'_d = q\overline{(wT')}_a + \frac{q^2}{p}\overline{w}_a\overline{T'_a}$$
$$= \frac{T_e}{g}C^{3/2}z_1^{\frac{4-3\delta}{2}}\left(qN_3 + \frac{q^2}{p}N_1N_2\right),$$

while the overall average of T'^2 is

$$q\overline{T'^2_a} + pT'^2_d = q\overline{T'^2_a} + \frac{q^2}{p}\left(\overline{T'_a}\right)^2$$
$$= \left(\frac{T_e}{g}Cz_1^{1-\delta}\right)^2\left(qN_5^2 + \frac{q^2}{p}N_2^2\right),$$

and similarly for w'^2.

Whereas the numbers N_i have been shown to depend on δ only, it may appear *a priori* that p and q may depend also on the other non-dimensional combination of the basic variables, that is, on $Cz_1^{1-\delta}/g$. But writing the expression for the heat flux as

$$F \propto z_1^{\frac{4-3\delta}{2}}\left\{\frac{1}{p}N_1N_2 + (N_3 - 2N_1N_2) + p(N_1N_2 - N_3)\right\}$$

it is seen that terms in p, 1, and $1/p$ occur in F so that no relation of the form

$$p \propto \left(\frac{Cz_1^{1-\delta}}{g}\right)^{\alpha}$$

can give F independent of z_1, other than $\alpha = 0$. The condition that the heat flux shall be constant with height therefore requires both that $\delta = 4/3$ and that p and q shall be independent of z_1; the ratio of the ascending and descending masses remains constant through the layer of constant F. The relations (**17**), (**20**), and (**21**) giving F, σ_T, and σ_w in terms of C and z_1 then remain unaffected apart from a modification of the numerical factors.

(*b*) *The Effect of Superimposed Temperature Fluctuations*

In the development of the theory in Section III (*b*), all elements have been treated as starting from rest at the local temperature of the environment. The superimposition of temperature fluctuations at the level of origin will in general modify the heat flux and other statistics of the temperature field (Priestley and Swinbank 1947). The effect has been established within the framework of the classical mixing length theory and requires re-examination on the basis of the present theory in which, unlike the former, the influence of elemental buoyancy is taken into account.

Let T_0' denote the representative value for the temperature anomaly at the starting level. This gives rise to an additional term gT_0'/T_e on the right of (**9**), and hence to

$$\frac{gT_0'}{T_e}(z_1-z_0) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{34})$$

on the right of (**10**). To proceed further it is necessary to know the manner in which T_0' depends on lapse rate C and on starting height z_0.

Progress can be made if, as seems reasonable on intuitive grounds, it be assumed that T_0' will be proportional to the temperature fluctuations otherwise present at the level z_0, that is, from (**21**)

$$T_0' \propto \frac{T_e}{g} C z_0^{1-\delta}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{35})$$

and the additional term (**34**) on the right of (**10**) then becomes proportional to

$$C z_1^{1-\delta}(z_1-z_0). \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{36})$$

It may be seen from the modified form of (**10**) or (**13**) that the addition of (**36**) does not affect the previous conclusion that the solution of (**10**) is of the functional form

$$w = C^{\frac{1}{2}} z_1^{\frac{2-\delta}{2}} n_1,$$

although n_1 will not have the same numerical values as when the term (**36**) is omitted. It also follows that the modification to (**14**) consists simply of the addition of T_0' to the right-hand side whence, invoking (**35**) again, (**15**) and hence (**21**) remain unaffected in functional form. It is thereby shown that the hypothesis (**35**) is internally consistent.

On this basis, therefore, the main arguments of the paper extend immediately to embrace the effects of imposed temperature fluctuations, and the temperature profile law and relations derived therefrom would remain unaltered in functional form. If the imposed fluctuations are considerable and obey some form other than (**35**), the mathematics become intractable and no general solution has been found. Solutions have, however, been found for the special case when T_0' and the lapse rate (either super- or sub-adiabatic) are constant with height; these present features of novel interest which will be described in a separate paper.

VII. Acknowledgments

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