THE MULTIPLE SCATTERING OF ELECTRONS AND POSITRONS

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Summary

The angular distribution of the single scattering of 33, 121, and 1065 keV electrons at small angles in gold is calculated and compared with the distributions given by the Born approximation and by the WKB method as used by Molière. The single scattering distribution for 1065 keV electrons is integrated numerically to give mean square angles of multiple scattering, and these are compared with the values given by the various multiple scattering theories. The results are discussed in conjunction with the experimental data for gold and other elements. The discrepancy between theory and the recent experiments with beryllium is shown not to be explained by the use of the Hartree instead of the Thomas-Fermi field. The difference of the root mean square angle for electrons and for positrons is estimated for gold and argon, and its value for argon—the only element for which this difference has been measured—is much less than the observed value.

I. Introduction

The angular distribution of the scattering of fast electrons and positrons at large angles is now fairly well understood (Bartlett and Watson 1940, Massey 1942, McKinley and Feshbach 1948). The calculations just referred to were carried out for an unscreened Coulomb field, and this is justified, since for large angles of scattering the effect of screening is small for electrons with energy as low as 33 keV for even a heavy element like gold (Mohr and Tassie 1954).

For small angles of scattering, however, screening of the nucleus by the atomic electrons is important. For heavy elements and relativistic energies the Born approximation is not likely to give very accurate results. The WKB method of determining phase shifts has been used together with a method of summing the series of partial waves which should give fairly good results (Molière 1947), but certain approximations have been made and it is of interest to assess the accuracy of the distributions obtained. This may be done by carrying out a detailed phase shift analysis, but particular care has to be taken with the summation of the series of partial waves, for it is extremely slowly convergent at small angles.

Experiments on small angle scattering at high energies almost inevitably involve multiple scattering, any theory of which must be based to some extent on the form of the single scattering distribution. While the multiple scattering distribution does not depend critically on the form of the single scattering distribution, it involves a " screening angle " whose value depends on the form of the atomic field. Some of the multiple scattering theories employ an

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exponentially screened field, others a Thomas-Fermi field, and neither field will lie as close to the true field as the Hartree field. The exact single scattering distribution for gold is therefore used to compute mean square angles, and these are compared with the values given by multiple scattering theories. For this reason only those multiple scattering theories are discussed which give as first approximation a Gaussian distribution whose mean square angle is given by simple analytic formulae.

Finally the difference in the intensity of single scattering of electrons and positrons due to spin—though small at small angles—could give rise to a detectable difference in the width of the respective multiple scattering distributions. There is experimental evidence for such a difference, and its magnitude is therefore investigated theoretically.

II. Analytic Theories of Multiple Scattering

The differential cross section for single scattering may be written in the form

$$I(\theta)=(Z^2\varepsilon^4 \operatorname{cosec}^4 \tfrac{1}{2}\theta/4\gamma^2m^2v^4)R, \quad \ldots\ldots\ldots\ldots (1)$$

where Z is the atomic number of the scattering atom, v is the velocity of the electrons,

$$\gamma=(1-\beta^2)^{-\frac{1}{2}} \quad \text{with} \quad \beta=v/c,$$

and R is the ratio of the scattering to the relativistic Rutherford scattering.

The chance of occurrence of a single deviation through an angle between θ and $\theta+d\theta$ is given by

$$P(\theta)=2\pi NtI(\theta)\sin\theta\, d\theta \quad \ldots\ldots\ldots\ldots\ldots (2)$$
$$=QRd\theta/\theta^3 \text{ for small } \theta,$$

where

$$Q=8\pi NtZ^2\varepsilon^4/\gamma^2m^2v^4, \quad \ldots\ldots\ldots\ldots\ldots (3)$$

N is the number of atoms per c.c. of scattering foil, and t is the foil thickness.

For sufficiently large t there will be a large number of collisions, resulting in an approximately Gaussian distribution with mean square angle given by

$$\theta^2_{\text{r.m.s.}}\equiv\overline{\theta^2}=\int_0^{\theta_{\max.}}\theta^2P(\theta)d\theta. \quad \ldots\ldots\ldots\ldots\ldots (4)$$

It is well known that this integral does not converge to a limit as $\theta_{\max.}$ is increased, but $\theta_{\max.}$ must be appreciably smaller than the root mean square angle if the assumption of a Gaussian distribution is to be justified. An arbitrary "cut-off" to the integral is therefore adopted.

In Williams's (1938) theory of multiple scattering $\theta_{\max.}$ is so chosen that the chance of a deviation through an angle greater than $\theta_{\max.}$ is unity, and therefore we have

$$\theta^2_{\max.}=\tfrac{1}{2}Q. \quad \ldots\ldots\ldots\ldots\ldots (5)$$

It is also assumed that Coulomb scattering ($R=1$) holds down to an angle $\theta_{\text{min.}}$, below which screening is supposed suddenly to reduce the scattering to zero, so that (4) gives

Williams: $$\theta^2_{\text{r.m.s.}}=Q \ln (\theta_{\text{max.}}/\theta_{\text{min.}}). \quad \ldots\ldots\ldots\ldots\ldots (6)$$

$\theta_{\text{min.}}$ is chosen to make the mean square angle for Coulomb scattering with a lower cut-off $\theta_{\text{min.}}$ the same as that for scattering by the Thomas-Fermi field, that is, one takes

$$\int_{\theta_{\text{min.}}}^{\theta_{\text{max.}}} \theta^2(1/\theta^3)\mathrm{d}\theta = \int_0^{\theta_{\text{max.}}} \theta^2(R/\theta^3)\mathrm{d}\theta,$$

where the value taken for R is that given by calculating the single scattering by the Thomas-Fermi field using Born's approximation. This finally gives

$$\theta_{\text{min.}}=1/(65\cdot 3\beta\gamma Z^{-1/3}). \quad \ldots\ldots\ldots\ldots\ldots (7)$$

The somewhat more accurate theory of Goudsmit and Saunderson (1940*a*, 1940*b*) gives, for scattering by the Thomas-Fermi field, an approximately Gaussian distribution with a width given by (Mott and Massey 1949)

Goudsmit and Saunderson: $$\theta^2_{\text{r.m.s.}}=Q \ln (0\cdot 64\ \theta_{\text{max.}}/\theta_{\text{min.}}). \quad \ldots\ldots (8)$$

It may be noted that in both theories $\theta_{\text{r.m.s.}}$ varies as $t^{\frac{1}{2}} \ln t+\text{const.}$ for a given Z.

The theory of Molière (1948) is based on a value of the screening angle $\theta_{\text{min.}}$ obtained through a calculation of the single scattering by the Thomas-Fermi field using the more accurate WKB method, the value being given by

$$\theta_{\text{min.}}=(1\cdot 13+3\cdot 76Z^2/137^2\beta^2)^{\frac{1}{2}}\theta_0, \quad \ldots\ldots\ldots\ldots (9)$$

where

$$\theta_0=1/ak=1/(0\cdot 885a_0Z^{-1/3}k)$$
$$=1/(121\beta\gamma Z^{-1/3}),$$

a being the Thomas-Fermi radius of the atom. The theory then gives

Molière: $$\theta^2_{\text{r.m.s.}}=QB/4, \quad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

where

$$B-\ln B=b$$
$$=2 \ln (\theta_{\text{max.}}/1\cdot 08\ \theta_{\text{min.}}),$$

with $\theta_{\text{max.}}$ as previously defined in (5). The appearance of $B/4$ instead of $b/2$ in (10) implies taking a value for the upper limit of the integral (4) which differs from $\theta_{\text{max.}}$ and corresponds to the width of the multiple scattering peak (Bethe 1953).

The theories all contain an addition to the Gaussian distribution which gives the transition to the " single scattering tail " at large angles, and this will slightly alter the width of the multiple scattering peak. We shall for simplicity compare the values of $\theta_{\text{r.m.s.}}$ given by the above formulae and those obtained directly by numerical integration in (4), using single scattering distributions calculated in different ways. Molière's theory gives a multiple scattering

distribution narrower than the Gaussian, which occurs as the first term in his distribution ; and the total distribution may be fitted—up to an angle where the intensity is 1/e of the maximum—by a slightly narrower Gaussian corresponding to a slightly reduced value of B (Hanson *et al.* 1951).

III. Calculation of the Exact Single Scattering Distribution for Gold

The differential cross section $I(\theta)$ is given by

$$I(\theta)=|f|^2+|g|^2,$$

with

$$2ikf(\theta)=\Sigma\{(l+1)(e^{2i\eta_l}-1)+l(e^{2i\eta_{-l-1}}-1)\}P_l(\cos\theta), \quad .. \quad (11)$$

and $g(\theta)$—essentially a spin term—negligible at small angles compared with $f(\theta)$; where $k=2\pi\gamma mv/h$, and the η are phase shifts of the various order waves which have been calculated accurately for gold (Mohr and Tassie 1954).

The series (11) is very slowly convergent at small angles, and therefore two methods were used for summing it in order to check the accuracy of the results.

Method (*a*).—The series was summed to the first 30 or 40 terms, and the summation over higher values of l was replaced by an integral, the higher P_l being calculated with high accuracy from the relation

$$P_l(\cos\theta)\simeq(\theta/\sin\theta)^{\frac{1}{2}}J_0((l+\tfrac{1}{2})\theta). \quad \ldots\ldots\ldots \quad (12)$$

The higher order phases are given with sufficient accuracy by the formula

$$\eta_l\simeq\eta_{-l-1}\simeq(\gamma Z_1/ka_0)K_0(\lambda_1(l+\tfrac{1}{2})/k)+(\gamma Z_2/ka_0)K_0(\lambda_2(l+\tfrac{1}{2})/k), \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

where the Hartree field of the atom is fitted as closely as possible by the expression

$$(Z_1e^{-\lambda_1 r}+Z_2e^{-\lambda_2 r})\varepsilon^2/r \quad \text{with } Z_1+Z_2=Z. \quad \ldots\ldots \quad (14)$$

For gold the values adopted were $Z_1=20$, $Z_2=59$, $\lambda_1=1\cdot3/a_0$, $\lambda_2=6/a_0$. It was found necessary to take into account values of l up to 500.

The integrand being oscillatory, the values of the integral between successive zeros of the integrand were obtained by graphical integration. These values constitute a slowly convergent series with alternating signs, the last few terms of which were treated by the Euler transformation (Rosser 1951) in order to speed the convergence.

Method (*b*).—As a check on the accuracy of the previous method, the following procedure was used in some cases, though it was found in general to be less accurate.

For large l we may take for K_0 in (13) the first term in the asymptotic formula, namely,

$$K_0(x)\sim(\pi/2x)^{\frac{1}{2}}e^{-x}.$$

Let us denote the resulting expression for η_l by $\eta_{l1}+\eta_{l2}$. One may fit $\sin 2\eta_l$ by the expression

$$2\eta_{l1}(1-\alpha_1 l^{-\frac{1}{2}})+2\eta_{l2}(1-\alpha_2 l^{-\frac{1}{2}})$$

and $1-\cos 2\eta_l$ by

$$2\eta_{l1}^2(1-\beta_1 l^{-\frac{1}{2}})+2\eta_{l2}^2(1-\beta_2 l^{-\frac{1}{2}})+4\eta_{l1}\eta_{l2}(1-\beta_3 l^{-\frac{1}{2}}),$$

where α_1, α_2, β_1, β_2, and β_3 are arbitrarily adjusted constants, the fit being good down to relatively small values of l. The discrepancy at the lower values of l is allowed for in a separate numerical calculation. Substituting these expressions for $\sin 2\eta_l$ and $1-\cos 2\eta_l$ in (**11**), replacing the series by an integral, and using (**12**) with $l+\frac{1}{2}$ replaced by l, we require to evaluate the integral $\int_0^\infty l^p e^{-ql} J_0(l\theta)\mathrm{d}l$ for $p=\frac{1}{2}$, 0, and $-\frac{1}{2}$. The value of this integral is given by a hypergeometric series (Watson 1948) which is readily evaluated numerically.

The results of this calculation are shown by the heavy curves marked E_1 and E_2 in Figure 1. Curve E_2 was obtained using the two-term field (**14**), and E_1 using a one-term field ($Z_1=79$, $Z_2=0$, $\lambda_1=3/a_0$) which does not fit the Hartree field for gold so well but which was used as a check on the sensitivity of the results to the precise form of the field. Curve E_H, for 1065 keV electrons, was obtained by using method (*a*) with the phases calculated accurately for the Hartree field of mercury by Gunnersen (1952).

The undulations in the curves occur near those angles at which either the real or the imaginary part of $f(\theta)$ passes through a zero, and appear to have a real existence. Recent calculations by Hoerni and Ibers (1953) for 40 keV electrons in uranium are found to give a curve of R *v.* θ which has undulations very similar to those in curves E_1 and E_2 for 33 keV electrons.

IV. Comparison of Single Scattering Distributions for Gold

For comparison, Figure 1 shows curves obtained using the Born approximation, according to which one has

$$f(\theta)=(8\pi^2 m/h^2)\int_0^\infty V\,(\sin Kr/Kr)r^2\mathrm{d}r, \quad \ldots\ldots\ldots \quad (15)$$

where

$$K=2k\sin\tfrac{1}{2}\theta.$$

We insert the form of the potential energy V for the one-term field

$$V=(\gamma Z\varepsilon^2 e^{-\lambda r}/r)+(Z^2\varepsilon^4 e^{-2\lambda r}/2mc^2 r^2),$$

where the second term is a relativistic spin term which contributes appreciably to the scattering only at large angles and has little effect at the angles considered here. Spin terms which have an even smaller effect have been omitted from V. An elementary integration then leads to the result

$$R=\{(K^2/\lambda^2+K^2)+\beta(Z/137)\,\mathrm{artan}\,(K/2\lambda)\sin\tfrac{1}{2}\theta\}^2. \quad \ldots\,(16)$$

The required modification to the formula for the two-term field is obvious.

The curves so obtained for the one- and two-term fields are labelled B_1 and B_2 respectively in Figure 1. As the second term in brackets in (**16**) is

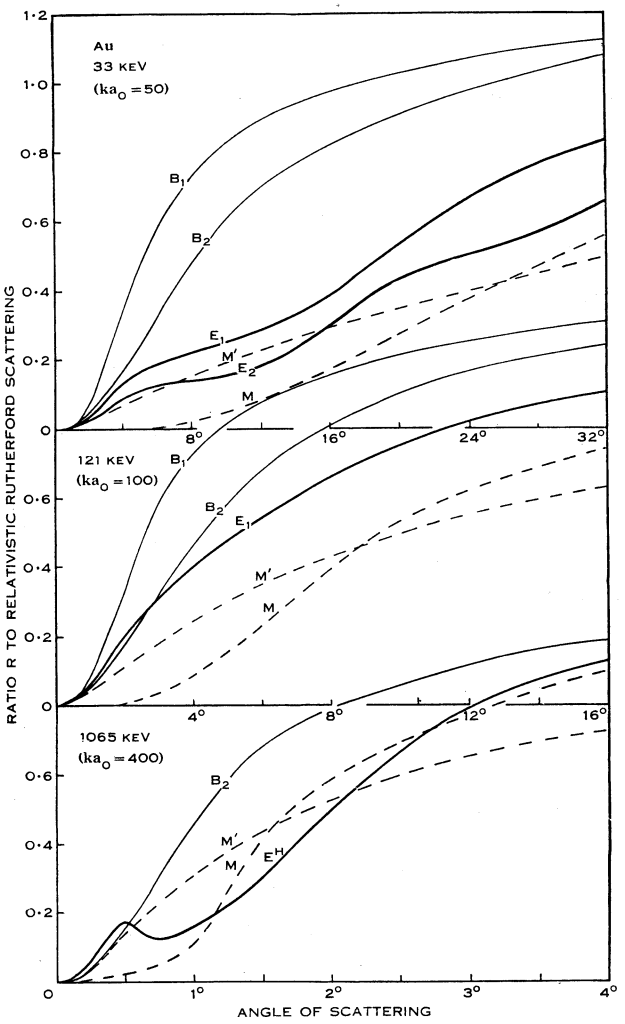

Fig. 1.—Angular distribution of single scattering of electrons in gold. Curves B_1 and B_2 were calculated using the Born approximation with the one-term and two-term potential fields respectively, together with the spin correction term in Z^2. Curve M was calculated from the formulae (**9.1**) and (**9.3**) given by Molière for application to multiple scattering, and M' obtained using Molière's more accurate formula (**8.6**). Curves E_1, E_2, and E_H were calculated using the exact theory in the present paper with the one-term and two-term fields and the Hartree field of mercury respectively.

almost negligible compared with the first term at the angles with which we are concerned, the value of R is practically a function of $k \sin \frac{1}{2}\theta$ only, and this suggests that the sets of curves obtained on the different approximations be

plotted on a scale of θ which is inversely proportional to the value of k. When this is done the Born curves at the different energies are almost identical.

Molière obtains a single scattering distribution, using the WKB method together with a three-term representation of the Thomas-Fermi field, for the limiting cases of $Z/137\beta \to 0$ (Born approximation) and $Z/137\beta \to \infty$ (classical approximation). Then, with the aid of an asymptotic formula he produces his empirical interpolation formula (**8.6**), which gives R for all values of $Z/137\beta$. This leads to the curves labelled M' in Figure 1. For application to the theory of multiple scattering he characterizes the single scattering distribution by a single parameter, the " screening angle " $\theta_{\text{min.}}$—given by his equation (**9.3**) (our equation (**9**)), which implies a single scattering distribution given, somewhat less accurately than by his formula (**8.6**), by his formula (**9.1**), namely,

$$R = \theta^4/(\theta_{\text{min.}}^2 + \theta^2)^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (17)$$

From this formula are derived the curves marked M. This formula, while more accurate than the first term of (**15**), has the same form and significance, since $K = k\theta$ for small angles, and $\lambda \sim k\theta_{\text{min.}}$ by a simple application of the uncertainty principle. $\theta_{\text{min.}}$ is the minimum detectable deflection for a field of range $1/\lambda$, and hence for a particle of angular momentum $kh/2\pi\lambda$.

The Born approximation (**15**) is based on the assumption that the phase shifts are all small compared with unity, whereas in Molière's application of the WKB method this assumption is not made. For a heavy element like gold a large number of the phases are of the order of unity. One therefore finds that the Molière curves M and M' lie closer to the " exact " curves E than do the Born curves B. The supposedly more accurate Molière curve M', however, falls further and further below the true values of R as the angle increases. This is not surprising, since Molière's value of R approaches 1 always from below, whereas the true value of R becomes greater than 1 for electrons as θ increases (Bartlett and Watson 1940). Furthermore the Molière theory gives the same angular distribution for positrons as for electrons, whereas the value of R for positrons is, at the larger angles where screening is less important, less than that for electrons (Massey 1942). The curve M is a better fit than the curve M' except at the very lowest angles, but it is given by a less accurate formula, so that the closeness of fit is to some degree fortuitous.

As the energy increases the various approximations should improve in accuracy, and one sees that the various theoretical curves for a given energy do approach each other more and more closely with increasing energy.

Finally, the degree of sensitivity of the scattering to the form of the atomic field is shown by the difference between the curves E_1 and E_2, and between B_1 and B_2.

V. Mean Square Angle of Multiple Scattering in Gold

The values of $\theta_{\text{r.m.s.}}$ for 1065 keV electrons predicted by the multiple scattering theories discussed in Section II (formulae (**6**), (**8**), (**10**)) are compared in Figure 2 (curves W, G & S, M_1) with the values obtained by direct numerical integration of the exact single scattering distribution, using (**4**) (curve E). Also

included in the figure are values obtained by numerical integration of the single scattering distributions given by the Born approximation with the two-term field (curve B), and by the Molière theory (curve M_2). In the latter case the curve M in Figure 1 (given by (**17**)) is used, as it is this—and not the curve M'—which is given by the use of the screening angle $\theta_{\text{min.}}$ involved in the multiple scattering theory.

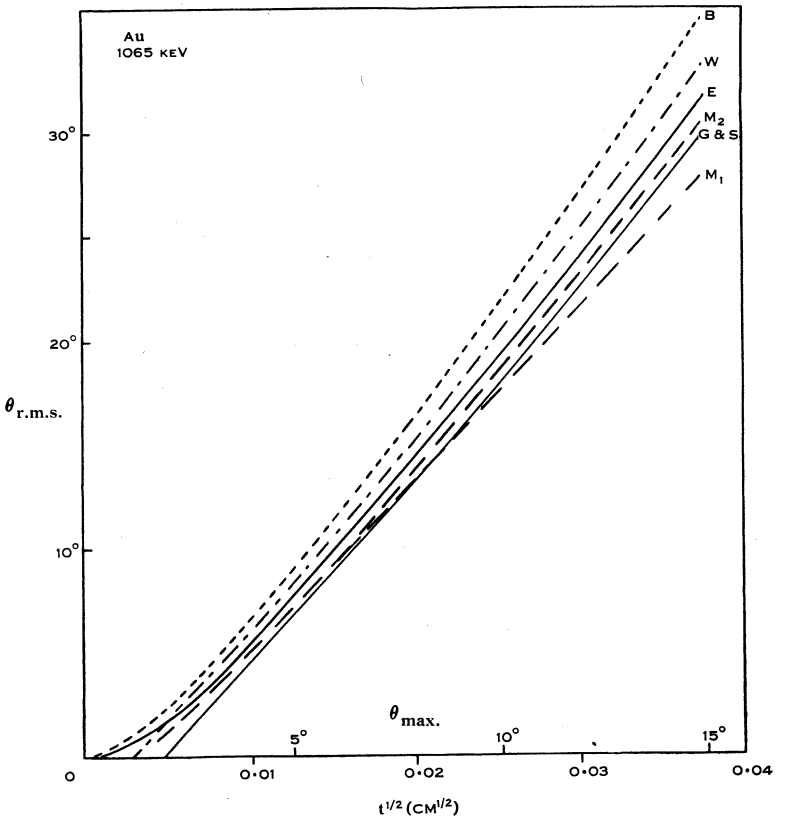

Fig. 2.—Multiple scattering of 1065 keV electrons in gold. Curves W, $G\,\&\,S$, and M_1 were obtained directly from the formulae for $\theta_{\text{r.m.s.}}$ given by Williams, Goudsmit and Saunderson, and Molière respectively (formulae (**6**), (**8**), and (**10**) of the present paper). Curves E, B, and M_2 were obtained by numerical integration of the single scattering distribution given by exact calculation, by the Born approximation with two-term field, and by Molière's formula (**9.1**) respectively (formulae (**11**), (**16**), and (**17**) of the present paper).

The Born and the Williams values of $\theta_{\text{r.m.s.}}$ lie above, and the Molière values below, the exact values. This is to be expected since the Born and Molière single scattering distributions lie respectively above and mostly below the exact distribution. Comparison of formulae (**6**) and (**8**) shows that the Goudsmit and Saunderson values of $\theta_{\text{r.m.s.}}$ lie below the Williams values by an absolute amount which increases with the thickness t, but by a fractional amount which decreases with increasing t.

The Molière curve M_2 lies closest to the exact curve as is to be expected, since it is based on the direct use of the single scattering distribution which lies closest to the exact distribution. The curve M_1 is based virtually on the use of the same single scattering distribution, and therefore coincides with curve M_2 over a fair range. The increasing separation of the curves M_1 and M_2 at layer thicknesses arises partly from the fact that a different value of $\theta_{max.}$ is involved in the two cases. Thus one finds that at $t^{\frac{1}{2}}=0\cdot025$, a change of only 1° in $\theta_{max.}$ in the upper limit of the integral (**4**) alters $\theta_{r.m.s.}$ by 10 per cent. The agreement is good, however, up to $\theta_{r.m.s.}=20°$, by which time the actual path length is appreciably greater than the thickness t, and energy loss is appreciable, so that the simple formulae no longer apply. The arbitrariness in the choice of $\theta_{max.}$ cannot be avoided since some scatters will always occur at still larger angles, and in fact the distribution is not accurately Gaussian. The Molière theory gives the form of the complete distribution, with which comparison should be made in any experimental investigation.

For 121 keV electrons the various single scattering distributions in Figure 1 lie further apart than for 1065 keV electrons, and—as one would therefore expect—the values of $\theta_{r.m.s.}$ on the various theories show larger percentage differences amongst themselves than at 1065 keV. Also the exact curve for $\theta_{r.m.s.}$ is found to lie close to the Williams curve and well above the Molière curve. No practical significance, however, can be attached to results calculated for energies as low as 121 keV, since the thicknesses of foil for which values of $\theta_{r.m.s.}$ less than 30° occur are so small that too few collisions would occur for multiple scattering theory to be applicable.

The experiments of Oleson, Chao, and Crane (1941) at 6 MeV, and those of Kulchitsky and Latyshev (1942) at 2·25 MeV give, for heavy elements, values of $\theta_{r.m.s.}$ less than the Williams values by 10–15 per cent. Using (**6**) and (**10**) one finds that at these energies and for the values of $\theta_{r.m.s.}$ concerned, the Molière value of $\theta_{r.m.s.}$ is less than the Williams value by about the same percentage. This result indicates that, at these higher energies and for heavy elements, there is fairly good agreement with the Molière theory. This conclusion is also supported by the more recent experiments of Hanson *et al.* (1951) on the scattering of 15·7 MeV electrons in gold.

VI. Multiple Scattering in Light and Intermediate Elements

Let us now consider light and intermediate elements. Firstly there are the fairly recent experiments on beryllium, carried out by Hanson *et al.* (1951) with 15·7 MeV electrons. The observed values of $\theta_{r.m.s.}$ are less than Molière's by 3–7 per cent., and the discrepancy is thought to be possibly due to the use of the Fermi field, which would not be very accurate for as light an atom as beryllium.

The sensitivity of the scattering to the field was put to the test, not by the lengthy and tedious phase shift analysis, but by the approximate methods which should be fairly accurate for light elements at this high energy. In any case the methods should give accurately the difference in the scattering due to a small difference in the field.

As a first step the single scattering distribution was calculated in the following ways :

(*a*) using the Born approximation result (**16**) modified for a two-term field fitted to the Hartree field for beryllium,

(*b*) using the Born approximation result (**16**) modified for a three-term field fitted to the Thomas-Fermi field,

(*c*) using Molière's formula ((**17**) above), based on the WKB method and the Thomas-Fermi field.

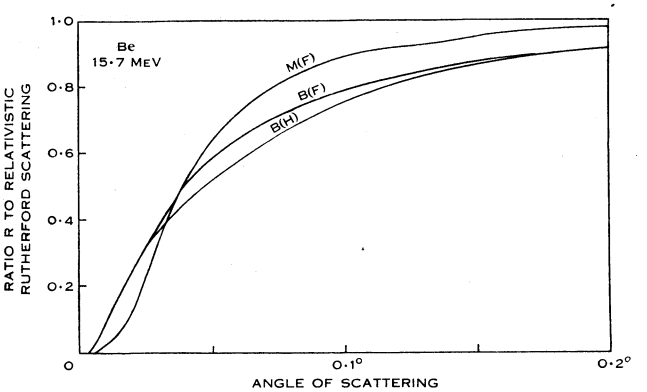

Fig. 3.—Angular distribution of single scattering of 15·7 MeV electrons in beryllium. Curves $B(F)$ and $B(H)$ were calculated using the Born approximation with the Fermi and Hartree fields respectively, and curve $M(F)$ using Molière's formula based on the WKB method and the Fermi field.

The three curves so obtained are shown in Figure 3. The differences between them are much less than the difference between the curves marked B_2 and M in Figure 1 for 1065 keV electrons in gold. One therefore expects fairly small differences in the values of $\theta_{\text{r.m.s.}}$ obtained from them ; the more so since the effect of the Molière curve lying below the Born curves at the smallest angles is largely offset by its lying above at the larger angles.

These single scattering curves were then integrated numerically, using (**4**), to give values of $\theta_{\text{r.m.s.}}$ for the two thicknesses of foil used in the experiment. The results are given in Table 1.

Using in place of the Fermi field the more accurate Hartree field thus makes a difference in $\theta_{\text{r.m.s.}}$ of only $\frac{1}{2}$ per cent. with the Born approximation, and about the same difference should occur with the WKB method as used in the Molière theory. The discrepancy of 3–7 per cent. between theory and experiment can therefore hardly be attributed to the use of the less accurate Fermi field.

Experiments on carbon with 3–11 MeV electrons by Oleson, Chao, and Crane (1941) give values of $\theta_{\text{r.m.s.}}$ less than the Williams values by 10–15 per cent. About the same percentage difference is found between the Williams and Molière values, on substitution in (**6**) and (**10**). The results for carbon are therefore in fair agreement with the Molière theory.

Values obtained for elements of intermediate Z by Oleson, Chao, and Crane (1941) and by Kulchitsky and Latyshev (1942) are in fair agreement with the Goudsmit and Saunderson values, and substitution in (**8**) and (**10**) shows these to be higher by several per cent. than the Molière values. Taking into account the higher terms of the Molière multiple scattering distribution narrows it and merely increases the discrepancy. Theory thus lacks agreement with experiment over the whole range of Z.

TABLE 1

VALUES OF $\theta_{r.m.s.}$ FOR 15·7 MEV ELECTRONS IN BERYLLIUM, OBTAINED BY NUMERICAL INTEGRATION OF THE SINGLE SCATTERING DISTRIBUTION

Thickness (mg/cm²)	Born Approxn. with Fermi Field	Born Approxn. with Hartree Field	WKB (Molière) with Fermi Field
257	2·18°	2·17°	2·14°
495	3·02°	3·01°	2·97°

The Molière theory is based on a more accurate treatment of the single scattering problem than the other theories, and might therefore be expected to give more nearly correct results for multiple scattering. Any discrepancies would be most likely for heavy elements, as suggested by the calculations in this paper; but the experiments indicate better agreement with Molière for heavy than for intermediate elements. Further experiments thus seem desirable.

VII. Difference of Mean Square Angle for Electrons and Positrons

There is a difference—which increases with Z—in the single scattering of electrons and positrons at large angles, due to electron spin; though the difference tends to zero as the angle tends to zero. One may therefore expect a difference in the value of $\theta_{r.m.s.}$ for electrons and positrons, though it is not obvious how large it will be. None of the multiple scattering theories gives a difference, and it must therefore be obtained by numerically evaluating the integral (**4**) for $\theta_{r.m.s.}$ using separate single scattering distributions for electrons and positrons.

(i) *Gold.*—Using the phase shifts given by Gunnersen (1952) for 1070 keV positrons in mercury, the single scattering distribution was calculated and found to diverge appreciably from the corresponding curve for electrons as the angle increased from 5 to 10°. Then, using (**4**), it was found that, for foil thicknesses giving values of $\theta_{r.m.s.}$ less than 2°, the difference between $\theta_{r.m.s.}$ for electrons and positrons was less than 2 per cent.; as $\theta_{r.m.s.}$ increased to 10° the difference increased to about 6 per cent., where it remained for further increase in $\theta_{r.m.s.}$.

Calculations were also carried out at 121 keV, making approximate estimates of the phases for positrons. It was found that for thicknesses giving a $\theta_{r.m.s.}$ greater than 10°, the difference between $\theta_{r.m.s.}$ for electrons and positrons was from 2–3 per cent.

The experimental results of McDonell (1953) on the scattering of 1 MeV electrons and positrons are consistent with a difference in $\theta_{r.m.s.}$, but its magnitude cannot be obtained accurately, since multiple scattering was involved only as a correction to single scattering observations at large angles.

(ii) *Argon.*—Estimates were made for argon, since Groetzinger, Humphrey, and Ribe (1951) report a difference of about 10 per cent. in the value of $\theta_{r.m.s.}$ for electrons and positrons over the range 0·3–2 MeV, the corresponding values of $\theta_{r.m.s.}$ varying from 10° to 3°.

For argon, $\alpha \equiv Z/137$ is sufficiently small to allow one to use for small angles the approximate single scattering formula (McKinley and Feshbach 1948)

$$R = 1 - \beta^2 \sin^2 \tfrac{1}{2}\theta + \pi\alpha\beta \sin \tfrac{1}{2}\theta \,(1 - \sin \tfrac{1}{2}\theta), \quad \ldots\ldots\ldots \quad (18)$$

with α positive for electrons and negative for positrons.

For an energy of 1 MeV, which is near the middle of the experimental range, the observed value of $4\frac{1}{2}°$ for $\theta_{r.m.s.}$ is calculated from (5), (6), and (7) to correspond to $\theta_{max.} = 3°$. From (18) the difference in R for electrons and positrons for an angle of 3° is 2 per cent. A similar calculation at the high and the low energy ends of the experimental range gives differences of 1 and 3 per cent. respectively in R. The value of the integrand in (4) at the upper limit of the integral differs for electrons and positrons by the percentages mentioned, and the differences will be less over the lower part of the range of integration, and hence less for the value of $\theta_{r.m.s.}$.

Furthermore the formula (18) is for scattering by a Coulomb field, and at the angles in question screening will reduce the difference in R, and hence in $\theta_{r.m.s.}$, for electrons and positrons. The difference can hardly exceed 2 per cent. in any part of the experimental range of energies, but the observed difference is 10 per cent. Further experiments with other elements are clearly desirable in order to clarify the situation.

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IX. References

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