ON THE OPTICAL PROPERTIES OF COMPONENTS FOR BIREFRINGENT FILTERS

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Summary

Expressions are derived for the stray light in a birefringent filter caused by polarizers producing incomplete polarization and by inexact half-wave plates. It is shown that, with suitable orientation of retardation plates, less stray light is obtained with incomplete than with complete polarizers. The asymmetry of the field and the stray light arising from incorrect orientation of the crystal axes are also discussed.

I. Introduction

Birefringent filters were first developed by Lyot (1933, 1944) and Öhman (1938), the theory being extended later by Billings (1947) and Evans (1949), while Billings, Sage, and Draisin (1951) and Dunn (1951) discussed some of the practical aspects of their construction.

A simple element of such a filter consists of a plane parallel birefringent plate cut with an axis in the surface and mounted between polarizers whose axes bisect those of the crystal. If the plate is of thickness e and birefringence $\varepsilon-\omega$, ε and ω being the refractive indices, the phase retardation $\varkappa$ introduced between the ordinary and extraordinary rays of wavelength λ is given by $\varkappa=(\varepsilon-\omega)e/\lambda$, and the transmittance of the element by $\tau=\cos^2\pi\varkappa$ or $\sin^2\pi\varkappa$ respectively for parallel or crossed polarizers. Since $\varkappa$ is a function of λ, the element has a sequence of transmission bands throughout the spectrum.

A simple birefringent filter consists of a series of such elements, each having twice the retardation of the preceding, so that the filter transmits narrow, widely spaced bands (the " principal maxima "). The field, which is limited by the variation in the wavelengths of the principal maxima away from the axis, can be extended in various alternative ways suggested by Lyot. In the one most frequently used (known as Lyot type 1), the retardation plate is split into halves which are separated, with their axes crossed, by an appropriately oriented half-wave plate.

Imperfect components introduce various defects into filter performance. Polarizers which do not produce complete polarization change the fraction of residual light outside the principal maxima, that is, the " stray light ". Faulty

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half-wave plates in Lyot type 1 elements increase the stray light. Retardation plates of incorrect thickness or imperfect optical quality result in changes of wavelength of the principal maxima. Incorrect orientation of the components with respect to the polarizers increases the stray light, while, if the crystal axes of the retardation plates do not lie in their surfaces, the field is modified. We discuss these matters in some detail.

II. Incomplete Polarizers

(a) *General*

Although calcite prism polarizers have high transmission and produce perfect polarization, at least over a field of several degrees, their disadvantages of high cost, bulk, and scarcity have meant that most birefringent filters have been constructed with " Polaroid " polarizers. These latter, however, all transmit a small amount of light vibrating in a direction at right angles to that of the main polarized beam, and so affect the performance of a filter in which they are incorporated. In this section we shall investigate the effects resulting from this incomplete polarization of " Polaroid ".

We shall first digress to describe an experiment whose result is of significance in the discussion.

(b) *An Experiment on Retardation in " Polaroid "*

The behaviour of incomplete polarizers in birefringent filters depends in part on any retardation introduced by the polarizer. Let a beam of polarized light be incident on a polarizer, then the beam may be resolved into two components respectively parallel and perpendicular to the polarizer axis and transmitted with unequal absorptions. If there is no phase difference on transmission, the two beams will recombine to give plane polarized light while, if there is a phase difference, elliptically polarized light will result. This effect has been tested on a variety of " Polaroid " films by viewing a sodium lamp through a Thomson prism, a following " Polaroid " sheet being nearly but not quite crossed with it; the yellow sodium glow may then be seen. On rotating a second Thomson prism to examine the light leaving the " Polaroid ", extinction occurs at an appropriate orientation, showing that there is no significant phase difference between the transmitted rays. Through some of the older types of " Polaroid " the red filament of the lamp may also be seen, and this disappears at a different orientation of the second Thomson prism, confirming that with such " Polaroids " the ratio of the transmissions respectively parallel and perpendicular to the polarizer axis varies with wavelength.

(c) *Transmission through a Birefringent Filter*

Let unpolarized light be incident on a birefringent element the first polarizer of which has its axis in direction r (Fig. 1) bisecting the axes x and y of the retardation plate. Neglecting losses by reflection at the various surfaces, the vibrations leaving the polarizer may, in virtue of the above, be represented by $r=T_1^{\frac{1}{2}}a \sin \psi$, $s=T_2^{\frac{1}{2}}a \sin \psi$, where ψ is the phase, a is the amplitude of the incident vibration, and T_1 and T_2 the transmittances of the polarizer for light vibrating respectively along the r and s directions. Considering firstly only the r component incident on the retardation plate, we find that it results in light emerging from the plate with r and s components given by

$$\left.\begin{aligned} r&=T_1^{\frac{1}{2}}a\cos\pi\varkappa\sin(\psi-\pi\varkappa),\\ s&=\pm T_1^{\frac{1}{2}}a\sin\pi\varkappa\cos(\psi-\pi\varkappa),\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (\mathbf{2.1})$$

the sign of s depending on the plane of vibration of the ordinary and extraordinary rays in the plate. Similarly, the s component from the first polarizer gives rise to vibrations leaving the retardation plate

$$r=\pm T_2^{\frac{1}{2}}a\sin\pi\varkappa\cos(\psi-\pi\varkappa),$$

$$s=T_2^{\frac{1}{2}}a\cos\pi\varkappa\sin(\psi-\pi\varkappa).$$

On passing through the following polarizer, the r and s amplitudes are multiplied by $T_1^{\frac{1}{2}}$ and $T_2^{\frac{1}{2}}$ respectively if the polarizers are parallel.

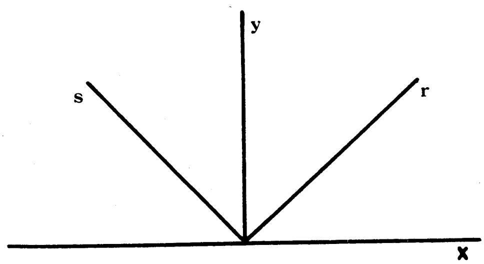

Fig. 1

The beam (**2.1**) is now incident on a second plate of retardation $2\varkappa$, thickness $2e$. The incident r beam behaves as before, its amplitude on striking the retardation plate being $T_1 a\cos\pi\varkappa$. The incident s ray, given by (**2.1**), results in a pair of transmitted components leaving the retardation plate

$$\left.\begin{aligned} r&=\mp T_1^{\frac{1}{2}}T_2^{\frac{1}{2}}a\sin\pi\varkappa\sin 2\pi\varkappa\sin(\psi-\pi\varkappa\mp 2\pi\varkappa),\\ s&=T_1^{\frac{1}{2}}T_2^{\frac{1}{2}}a\sin\pi\varkappa\cos 2\pi\varkappa\cos(\psi-\pi\varkappa\mp 2\pi\varkappa),\end{aligned}\right\} \quad \ldots (\mathbf{2.2})$$

the signs depending on whether the optic axis of the second plate is parallel or at right angles to that of the first plate.

The vibration transmitted through a complete filter is the resultant of the component vibrations which traverse the filter in all possible ways. However, as $T_1 \gg T_2$, terms involving high powers of T_2 may be neglected ; these correspond to rays which have passed through several polarizers in the s direction.

Suppose first that all retardation plates are arranged with their optic axes parallel, all polarizers also being parallel. Omitting the phase, the ray passing through all polarizers in the r direction contributes

$$aT_1^{(N+1)/2}\cos\pi\varkappa\cos 2\pi\varkappa\ldots\cos 2^{N-1}\pi\varkappa$$

to the emergent vibration, N being the number of elements in the filter. Those making only one transition $r \rightarrow s \rightarrow r$ contribute

$$-aT_1^{N/2}T_2^{\frac{1}{2}} \sum_{l=0}^{N-2} (\cos \pi\varkappa \ldots \cos 2^{l-1}\pi\varkappa \sin 2^l \pi\varkappa \sin 2^{l+1}\pi\varkappa \cos 2^{l+2}\pi\varkappa \ldots \cos 2^{N-1}\pi\varkappa). \qquad \textbf{(2.3)}$$

Those incident and transmitted in the r direction, but traversing two polarizers in the s direction, turn out to be negligible and are omitted. The sum of the above two contributions is denoted as A_1.

The r contribution due to the incident s component, of random phase with respect to A_1, is

$$A_2 = aT_1^{N/2}T_2^{\frac{1}{2}} \sin \pi\varkappa \cos 2\pi\varkappa \ldots \cos 2^{N-1}\pi\varkappa.$$

The s disturbance due to the ray which has passed through only one polarizer (the last) in the s direction, is

$$A_3 = aT_1^{N/2}T_2^{\frac{1}{2}} \cos \pi\varkappa \cos 2\pi\varkappa \ldots \cos 2^{N-2}\pi\varkappa \sin 2^{N-1}\pi\varkappa.$$

The transmitted flux is then given by

$$F_\varkappa = A_1^2 + A_2^2 + A_3^2. \qquad \textbf{(2.4)}$$

(*d*) *The Integrated Flux*

For an incident equi-energy spectrum, the transmitted flux is assessed by integrating between two successive principal maxima, with the result

$$F_T = \frac{T_1^{N+1}a^2}{2^N}\left[1 + \frac{T_2}{T_1}(N+1)\right], \qquad \textbf{(2.5)}$$

powers of T_2 higher than the first being neglected.

(*e*) *The Flux in a Principal Maximum Band and the Stray Light*

With an ideal filter the transmittance falls from unity at the principal maxima, $\varkappa = n$, an integer, to zero where $\varkappa = n \pm 2^{-N}$, between which lies a principal maximum band. The flux in a principal maximum band, F_M, is found by integrating from $\varkappa = n - 2^{-N}$ to $\varkappa = n + 2^{-N}$. This integral may be evaluated approximately by putting $\cos x = 1 - x^2/2$, $\sin x = x$, except for the final term of argument $2^{N-1}\pi\varkappa$.

Neglecting powers of $\varkappa$ higher than the second, the term in $T_2^{\frac{1}{2}}$, which will be found of special interest, becomes

$$-0 \cdot 61 \frac{a^2T_1^{N+\frac{1}{2}}T_2^{\frac{1}{2}}}{2^N}. \qquad \textbf{(2.6)}$$

The terms in T_2 integrate to a value of the order of $2^{-N}a^2T_1^NT_2$, while the term in T_1^{N+1} integrates to $0 \cdot 89 \times 2^{-N}a^2T_1^{N+1}$, provided $N > 3$. Hence

$$F_M \simeq \frac{T_1^{N+1}a^2}{2^N}\left[0 \cdot 89 - 0 \cdot 61\left(\frac{T_2}{T_1}\right)^{\frac{1}{2}} + \frac{T_2}{T_1}\right].$$

The fraction of the flux which lies outside the principal maximum band, namely,

$$f_s=\frac{F_T-F_M}{F_T}\simeq 0\cdot 11+0\cdot 61\left(\frac{T_2}{T_1}\right)^{\frac{1}{2}}+N\frac{T_2}{T_1}, \quad \ldots\ldots\ldots\ldots \quad (2.7)$$

we term stray light. With complete polarization, $f_s=0\cdot 11$, a result given previously by Evans (1949).

The lightest available " Polaroid " films have crossed densities of the order of 3, that is, $T_2/T_1\sim 10^{-3}$. With these, the stray light in a seven-" Polaroid " filter is $0\cdot 137$. With a crossed density of four, the stray light is reduced to $0\cdot 117$, but, as the transmitted flux varies as T_1^{N+1}, there is a considerable reduction in transmission if the stray light is to be held to this value.

(*f*) *The Reduction of Stray Light Using Incomplete Polarization*

With a suitable alignment of retardation plates, an important result can be obtained, namely, a reduction of stray light below that obtained with perfect polarizers.

From (**2.2**), it can be seen that a ray passing in the $r\rightarrow s\rightarrow r$ direction through two retardation plates with parallel optic axes finishes with a negative amplitude. But if the optic axes of the plates are at right angles (i.e. if the retardation of one is of opposite sign to that of the other), the amplitude is positive. Suppose all polarizers are parallel, but that the optic axis of each plate is perpendicular to that of its immediate predecessor, then A_1 is modified so that the sign of expression (**2.6**) is reversed, and the stray light is given by

$$f_s\simeq 0\cdot 11-0\cdot 61\left(\frac{T_2}{T_1}\right)^{\frac{1}{2}}+N\frac{T_2}{T_1}. \quad \ldots\ldots\ldots\ldots \quad (2.8)$$

If the element of highest retardation alone is at right angles to the rest, then it can be shown that the stray light is given by

$$f_s\simeq 0\cdot 11-0\cdot 39\left(\frac{T_2}{T_1}\right)^{\frac{1}{2}}+N\frac{T_2}{T_1}.$$

The optimum reduction in stray light is obtained when

$$\frac{T_2}{T_1}\simeq\frac{0\cdot 093}{N^2}, \quad \ldots\ldots\ldots\ldots \quad (2.9)$$

when the stray light is reduced to

$$f_s\simeq 0\cdot 11-0\cdot 09/N. \quad \ldots\ldots\ldots\ldots \quad (2.10)$$

Thus even with as many as 10 polarizers, the use of the lightest grade of " Polaroid " not only provides for high transmission but also yields a sensible improvement in stray light over that given by calcite polarizers.

(g) The Reduction of Stray Light with Lyot Type 1 Elements

By tracing the changes in the sign of the vibration on passing through Lyot type 1 elements, it is easy to verify that for a reduction in stray light

(1) consecutive Lyot type 1 elements should be arranged with the optic axes of their first components parallel, and

(2) the first component of a Lyot type 1 element should have its optic axis at right angles to that of a preceding or parallel to that of a following simple element.

III. Inexact Half-wave Plates

(a) General

For a Lyot type 1 element using parallel polarizers Evans (1949) has given the transmittance in a form reducing to

$$\tau=\cos^2 \pi\varkappa_j+\cos^2 \pi\varkappa_p\,(\sin^2 \pi\varkappa_j-\sin^2 \pi\delta_j),$$

where $\varkappa_p$ is the retardation of the nominal half-wave plate, and

$$\varkappa_j=\varkappa_m+\varkappa_q,$$
$$\delta_j=\varkappa_m-\varkappa_q,$$

$\varkappa_m$ and $\varkappa_q$ being the retardations of the two split plates, which we suppose to have equal thicknesses.

As shown by Evans, the effective retardation of a single plate is

$$\varkappa=\varkappa_0\left[1+\frac{\varphi^2}{2\omega}\left(\frac{\cos^2\theta}{\varepsilon}-\frac{\sin^2\theta}{\omega}\right)\right], \text{ for } \varepsilon>\omega, \quad \ldots\ldots \quad (\mathbf{3.1})$$

$\varkappa_0$ being the retardation for axial passage and φ and θ respectively the angles of incidence and the azimuth of the plane of incidence measured from the fast axis. Since the axes of the two plates are crossed, a ray at φ, θ to the first slab will enter the second at φ, $\theta+\pi/2$.

Then

$$\varkappa_j=\varkappa_{j0}\left[1+\frac{\varphi^2}{4\omega}\left(\frac{1}{\varepsilon}-\frac{1}{\omega}\right)\right], \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{3.2})$$

and

$$\delta_j=\varkappa_{j0}\left[\frac{\varphi^2}{4\omega}\left(\frac{1}{\varepsilon}+\frac{1}{\omega}\right)\cos 2\theta\right],$$

$$\varkappa_p=\varkappa_{p0}\left[1+\frac{\varphi^2}{4\omega}\left\{\left(\frac{1}{\varepsilon}-\frac{1}{\omega}\right)-\left(\frac{1}{\varepsilon}+\frac{1}{\omega}\right)\sin 2\theta\right\}\right].$$

The variation in $\varkappa_j$ with φ normally sets a limit to the field, involving a variation in λ_M, the wavelength of the principal maximum. If $\Delta\lambda_M$ is not to exceed $0\cdot1$ of the half-width of the principal band,

$$\Delta\varkappa_j\leqslant 2^{-N}/10,$$

which when combined with (**3.1**) yields the maximum value of φ.

An inexact half-wave plate introduces stray light, which we now discuss.

(b) *The Integrated Flux*

The flux transmitted by a birefringent filter of N elements, the first J of which are unsplit and the remainder Lyot type 1 elements, is

$$F=a^2\int \cos^2 \pi\varkappa \cos^2 2\pi\varkappa \ldots \cos^2 2^{J-1}\pi\varkappa . G(2^J\pi\varkappa)G(2^{J+1}\pi\varkappa) \ldots G(2^{N-1}\pi\varkappa)\mathrm{d}\varkappa,$$

where

$$G(2^x\,\pi\varkappa)=\cos^2 2^x\pi\varkappa+\gamma(\sin^2 2^x\pi\varkappa-\sin^2 2^x\beta\pi\varkappa),$$

$$\gamma=\cos^2 \pi\varkappa_p,$$

and, to a good approximation,

$$\beta=\frac{\varphi^2}{4\omega}\left(\frac{1}{\varepsilon}+\frac{1}{\omega}\right)\cos 2\theta.$$

Provided γ is small and varies negligibly over the range of integration, we find, as in Section II, that the flux between two successive principal maxima, F_T, may be written, for $2^{N-1}\beta< 1/4$,

$$F_T=\frac{a^2}{2^N}\left[1+\gamma\sum_{x=J+1}^{N}\frac{\cos 2^x\beta\pi\varkappa_0 \sin 2^{x-1}\beta\pi}{2^{x-1}\beta\pi}\right].$$

Similarly, with the same approximations, we find the flux in a principal maximum band to be given by

$$F_M=\frac{a^2}{2^N}\left[0\cdot 89+\gamma\left(\frac{2}{3}-\frac{4^{J-N+1}}{9}\right)-\gamma\sum_{x=J+1}^{N} y_x \sin^2 2^{x-1}\beta\pi\varkappa_0\right],$$

where

$$y_x=\frac{8}{9}+\frac{4^{x-N}}{3},\quad x\neq N,$$

$$y_N=1\cdot 45.$$

The stray light fraction on the axis is thus

$$f_s=\frac{0\cdot 11+\gamma\left(M-\frac{2}{3}+\frac{4^{1-M}}{9}\right)}{1+M\gamma},$$

where $M=N-J$ is the number of wide-field elements.

As an example consider a six-element filter designed to operate at the coronal red and green lines. If provided with mica half-wave plates adjusted for the sodium D line, then at the coronal line wavelengths $\gamma\sim 0\cdot 03$. The stray light fraction is given in Table 1.

To investigate off-axis effects, suppose the last two elements split. At angles such that $\sin 2^{x-1}\beta\pi\simeq 2^{x-1}\beta\pi$, the stray light is

$$f_s=\frac{0\cdot 11+\gamma(0\cdot 57+0\cdot 51\cos 2^5\beta\pi\varkappa_0+0\cdot 28\cos 2^6\beta\pi\varkappa_0)}{1+\gamma(\cos 2^5\beta\pi\varkappa_0+\cos 2^6\beta\pi\varkappa_0)}.$$

This varies around the field through its dependence on β and hence on θ, but *diminishes* away from the axis and so imposes no limit to the field.

IV. Incorrect Orientation of Components about a Normal to the Surface

In an otherwise perfect simple element, let the angle between the first polarizer axis r and the fast axis x of the crystal plate be ξ, the final polarizer being at an angle ζ to x. The first polarizer transmits a vibration $r=a \sin \psi$, the vibration transmitted through the whole element being plane polarized and represented by

$$a \cos \xi \cos \zeta \,.\, \sin \psi + a \sin \xi \sin \zeta \,.\, \sin (\psi - 2\pi\varkappa).$$

The transmittance of the element is thus

$$\tau = A \cos^2 \pi\varkappa + B \sin^2 \pi\varkappa,$$

where $A = \cos^2 (\xi - \zeta)$ and $B = \cos^2 (\xi + \zeta)$.

Table 1

Stray light at λ5303 with mica half-wave plates adjusted at λ5890

Number of Split Elements	Stray Light Fraction
0	0·110
1	0·119
2	0·142
3	0·165
4	0·188
5	0·208
6	0·229

The transmittance of a simple birefringent filter is the product of a set of such terms

$$\tau = (A_1 \cos^2 \pi\varkappa + B_1 \sin^2 \pi\varkappa)(A_2 \cos^2 2\pi\varkappa + B_2 \sin^2 2\pi\varkappa) \ldots$$
$$(A_N \cos^2 2^{N-1}\pi\varkappa + B_N \sin^2 2^{N-1}\pi\varkappa).$$

In practice, the A's will be almost unity and the B's small, so that terms involving higher powers than B may be neglected. As before, the flux in the whole transmission band is obtained by integrating from $\varkappa = n$ to $n+1$, yielding

$$F_T = \frac{\Pi_x}{2^N}\left\{1 + \sum_{x=1}^{N} \frac{B_x}{A_x}\right\},$$

where $\Pi_x = A_1 A_2 \ldots A_N$.

The flux in a principal maximum band is given by

$$F_M = \frac{\Pi_x}{2^N}\left[0 \cdot 89 + \frac{1}{3}\sum_{x=1}^{N-1} \frac{B_x}{A_x} 4^{x-N} + \frac{5}{9}\frac{B_N}{A_N}\right].$$

If the ratio B/A is constant and equal to ρ,

$$F_T = \frac{\Pi_x}{2^N}(1 + N\rho)\,; \quad F_M = \frac{\Pi_x}{2^N}\left(0 \cdot 89 + \frac{2}{3}\rho\right) \quad \text{for } N > 3.$$

These results show that if $\xi+\zeta=\pi/2$, that is, $B=0$, the effect is simply to reduce the flux uniformly over the whole spectrum ; there is no change in the proportion of scattered light.

With B/A constant, the stray light fraction is

$$f_s=\frac{0\cdot 11+\left(N-\frac{2}{3}\right)\rho}{1+N\rho},$$

which, for readily achievable orientations and normal numbers of elements, approximates to

$$f_s=0\cdot 11+N\rho\simeq 0\cdot 11+N\left(\frac{\pi}{2}-\xi-\zeta\right)^2.$$

If the errors in orientation ξ and ζ are all equal to x (radians) and are of the same sign,

$$f_s=0\cdot 11+4Nx^2. \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.1)$$

V. Misalignment of the Axes with Respect to the Surface of a Plate

(a) *General Equations*

Here the field size of a simple element is unchanged, as shall be shown, but may be displaced with respect to the normal to the surface of the element.

Following Evans's (1949) method, consider a plane parallel uniaxial crystal of thickness e, whose axes form a coordinate system, the x axis of which is parallel to the crystal optic axis. Choose units of time and distance which make the velocity of light *in vacuo* unity. Suppose the crystal surfaces do not lie in a plane containing two of the axes, the equations of the surfaces being

$$\alpha x+\beta y+\gamma x \quad =0, \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5.1)$$

$$\alpha x+\beta y+\gamma z-e=0, \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5.2)$$

where α, β, and γ are the direction cosines of the normals to the surfaces. An incident ray has a wave front

$$ax+by+cz-t=0, \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5.3)$$

where a, b, and c are the direction cosines of the ray.

At the origin, on the first surface, a secondary wavelet spreads into the crystal, its equation being that of the ellipsoid

$$\zeta^2x^2+\eta^2y^2+\nu^2z^2-t^2=0, \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5.4)$$

where ζ, η, and ν are the reciprocals of the velocities in the x, y, and z directions, that is, the refractive indices.

A tangent plane to the ellipsoid at x_1, y_1, z_1 is

$$x_1\zeta^2x+y_1\eta^2y+z_1\nu^2z-t=0. \qquad \ldots\ldots\ldots\ldots\ldots \quad (5.5)$$

The condition for this to be a wave front is that the three planes (**5.1**), (**5.3**), and (**5.5**) meet along a common line, that is,

$$\begin{vmatrix} \alpha & \gamma & 0 \\ a & c & -t \\ x_1\zeta^2 & z_1\nu^2 & -t^2 \end{vmatrix} = 0 = \begin{vmatrix} \beta & \gamma & 0 \\ b & c & -t \\ y_1\eta^2 & z_1\nu^2 & -t^2 \end{vmatrix},$$

from which

$$\left.\begin{aligned} x_1 &= (\gamma\zeta^2)^{-1}(\alpha\nu^2 z_1 - \alpha ct + \gamma at), \\ y_1 &= (\gamma\eta^2)^{-1}(\beta\nu^2 z_1 - \beta ct + \gamma bt). \end{aligned}\right\} \quad \ldots\ldots\ldots (\mathbf{5.6})$$

Substituting these in (**5.4**) we find

$$\nu z_1{}^2 = t^2 - (\gamma^2\zeta^2)^{-1}(\alpha\nu^2 z_1 - \alpha ct + \gamma at)^2 - (\gamma^2\eta^2)^{-1}(\beta\nu^2 z_1 - \beta ct + \gamma bt)^2, \quad (\mathbf{5.7})$$

whose solution is $z_1 = t.F$, where, restricting consideration to small errors in alignment, so that α and β are small and $\gamma \simeq 1$,

$$F = \frac{1}{\nu}\left(1 - \frac{a^2}{\zeta^2} - \frac{b^2}{\eta^2} + \frac{2a\alpha c}{\zeta^2} + \frac{2b\beta c}{\eta^2}\right)^{\frac{1}{2}} - \frac{\alpha a}{\zeta^2} - \frac{\beta b}{\eta^2}.$$

The time t_1 for the ray to reach the second surface is found by putting $x = x_1$, $y = y_1$, $z = z_1$ in (**5.2**), so

$$\alpha(\gamma\zeta^2)^{-1}(\alpha\nu^2 t_1 F - \alpha ct_1 + \gamma at_1) + \beta(\gamma\eta^2)^{-1}(\beta\nu^2 t_1 F - \beta ct_1 + \gamma bt_1) + \gamma t_1 F = e,$$

which yields to the first order,

$$\left.\begin{aligned} t_1 &= e\left(F + \frac{\alpha a}{\zeta^2} + \frac{\beta b}{\eta^2}\right)^{-1} \\ &= e\nu\left(1 - \frac{a^2}{\zeta^2} - \frac{b^2}{\eta^2} + \frac{2a\alpha c}{\zeta^2} + \frac{2b\beta c}{\eta^2}\right)^{\frac{1}{2}}. \end{aligned}\right\} \quad \ldots\ldots (\mathbf{5.8})$$

The emerging wave front, parallel to (**5.3**), is

$$ax + by + cz - (t - \Delta) = 0,$$

where

$$\Delta = t_1 - ax_1 - by_1 - cz_1.$$

The distance of the wave from the origin is

$$p = t - \Delta = t - t_1 + ax_1 + by_1 + cz_1. \quad \ldots\ldots\ldots\ldots (\mathbf{5.9})$$

For the extraordinary wave,

$$\zeta = \omega, \quad \eta = \nu = \varepsilon, \quad t_1 = t_\varepsilon.$$

For the ordinary wave,

$$\zeta = \eta = \nu = \omega, \qquad t_1 = t_\omega.$$

Hence we find for the retardation, $\varkappa=(p_\omega-p_\varepsilon)/\lambda$,

$$\varkappa=\frac{1}{\lambda}\left\{(F_\omega t_\omega-F_\varepsilon t_\varepsilon)\left(\frac{\beta b}{\gamma}+c\right)+\frac{F_\omega t_\omega \alpha a}{\gamma}-\frac{F_\varepsilon t_\varepsilon \alpha a \varepsilon^2}{\gamma\omega^2}+(t_\omega-t_\varepsilon)\left(\frac{a^2}{\omega^2}-\frac{a\alpha c}{\gamma\omega^2}-1\right)\right.$$

$$\left.+\left(\frac{t_\omega}{\omega^2}-\frac{t_\varepsilon}{\varepsilon^2}\right)\left(b^2-\frac{b\beta c}{\gamma}\right)\right\},$$

which after some reduction becomes, to the first order,

$$\varkappa=\varkappa_0\left\{1-\frac{a^2}{2\omega^2}+\frac{b^2}{2\omega\varepsilon}+\frac{\alpha a}{\omega^2}(c-\omega-\varepsilon)-\frac{b\beta c}{\omega\varepsilon}\right\}. \quad \ldots\ldots\ldots \quad (\mathbf{5.10})$$

(b) *The Simple Element*

The limit of the field is reached when $\varkappa-\varkappa_0$ reaches the tolerable limit. In the perfect element, where $\alpha=\beta=0$,

$$\frac{\varkappa-\varkappa_0}{\varkappa_0}=\frac{b^2}{2\omega\varepsilon}-\frac{a^2}{2\omega^2},$$

which has maxima at $b=0$ and $a=0$, if $b^2+a^2=\text{constant}$, and is zero when $b^2/a^2=\varepsilon/\omega$.

(i) *Case* 1.—If $a=0$, (**5.10**) becomes

$$\frac{\varkappa-\varkappa_0}{\varkappa_0}=\frac{b^2}{2\omega\varepsilon}-\frac{\beta b(1-b^2)^{\frac{1}{2}}}{\omega\varepsilon} \quad (\text{as } a^2+b^2+c^2=1).$$

For a given change in retardation, let $b=b_0$ for $\beta=0$, and $b=B$ for $\beta\neq0$. Then

$$b_0^2=B^2-2\beta B(1-B^2)^{\frac{1}{2}},$$

which, for $\beta\ll b_0$, and B small, reduces to

$$B-\beta=\pm b_0.$$

As the angle which the limiting ray makes with the plane containing the optic axis and the normal to the surface is given closely by $B-\beta$, the field in the plane $x=0$ is unchanged and symmetrical.

(ii) *Case* 2.—If $b=0$, the corresponding result is

$$A-\alpha=\pm a_0-\alpha(\omega+\varepsilon).$$

The angle which the limiting ray makes with the plane containing the y-axis and the normal to the surface is given closely by $A-\alpha$, so that the field in the plane $y=0$ is unchanged in size but asymmetrical by an amount $\alpha(\omega+\varepsilon)$.

(c) *Lyot Type* 1 *Element*

The large field of a perfect Lyot type 1 element results from a compensation of off-axis effects. We shall now show that this compensation is reduced and the field size diminished if the axes of the two plates are not parallel to the surfaces.

Suppose that axes of the first plate of a Lyot type 1 element are not parallel to the surface of the plate, the second plate being free from error.

With respect to the coordinate system established by the crystal axes of the first plate, those of the second plate have direction cosines $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, and $(\alpha_3, \beta_3, \gamma_3)$, where $(\alpha_3, \beta_3, \gamma_3)=(\alpha, \beta, \gamma)$ are the direction cosines of the normal to the surface. Hence the direction cosines of a ray which in the first system are (a, b, c) become in the coordinate system of the second plate (a', b', c') where

$$\begin{aligned} a' &= \alpha_1 a + \beta_1 b + \gamma_1 c, \\ b' &= \alpha_2 a + \beta_2 b + \gamma_2 c, \\ c' &= \alpha_3 a + \beta_3 b + \gamma_3 c. \end{aligned}$$

The retardation of the complete element is thus

$$\varkappa = \varkappa_1 \left\{ 1 - \frac{a^2}{2\omega^2} + \frac{b^2}{2\omega\varepsilon} + \frac{\alpha a}{\omega^2}(c - \omega - \varepsilon) - \frac{b\beta c}{\omega\varepsilon} \right\} + \varkappa_2 \left\{ 1 + \frac{a'^2}{2\omega\varepsilon} - \frac{b'^2}{2\omega^2} \right\}.$$

Since $\alpha_1 \simeq 1$, β_1 and γ_1 are very small, $\beta_2 \simeq 1$, α_2 and γ_2 are very small, and $c \simeq 1$, it readily follows that $\gamma_1 \simeq -\alpha$ and

$$\begin{aligned} \varkappa = \varkappa_1 &\left\{ 1 - \frac{a^2}{2\omega^2} + \frac{b^2}{2\omega\varepsilon} + \frac{\alpha a}{\omega^2}(1 - \omega - \varepsilon) - \frac{\beta b}{\omega\varepsilon} \right\} \\ &+ \varkappa_2 \left\{ 1 + \frac{a^2 + 2a(\beta_1 b - \alpha)}{2\omega\varepsilon} - \frac{b^2 - 2b(\beta_1 a + \beta)}{2\omega^2} \right\}. \end{aligned}$$

Suppose the plates are adjusted during construction so that in each the retardation along the normal is equal to $\frac{1}{2}K$. In the first plate, the direction of the normal is $a=\alpha$, $b=\beta$; in the second, $a'=0=b'$. Thus very closely

$$\begin{aligned} \varkappa_1 &= \tfrac{1}{2}K \left\{ 1 - \frac{\alpha^2}{\omega^2}(\tfrac{1}{2} - \omega - \varepsilon) + \frac{\beta^2}{2\omega\varepsilon} \right\}, \\ \varkappa_2 &= \tfrac{1}{2}K, \end{aligned}$$

whence very closely

$$\begin{aligned} \frac{\varkappa - K}{K} = \tfrac{1}{2} \Bigg\{ &- \frac{\alpha^2}{\omega^2}(\tfrac{1}{2} - \omega - \varepsilon) + \frac{\beta^2}{2\omega\varepsilon} - \frac{a^2}{2\omega^2} + \frac{b^2}{2\omega\varepsilon} + \frac{\alpha a}{\omega^2}(1 - \omega - \varepsilon) - \frac{\beta b}{\omega\varepsilon} + \frac{a^2 + 2a(\beta_1 b - \alpha)}{2\omega\varepsilon} \\ &- \frac{b^2 - 2b(\beta_1 a + \beta)}{2\omega^2} \Bigg\}. \qquad \text{(5.11)} \end{aligned}$$

(i) *Case* 3.—If $a=0$, (**5.11**) becomes

$$\frac{\varkappa - K}{K} = \tfrac{1}{2} \left\{ -\frac{\alpha^2}{\omega^2}(\tfrac{1}{2} - \omega - \varepsilon) + \frac{\beta^2}{2\omega\varepsilon} + \frac{1}{\omega^2\varepsilon}(\omega - \varepsilon)\left(\frac{b^2}{2} - \beta b\right) \right\}.$$

For a given change in retardation, let $b=b_0$ for $\alpha=\beta=0$, and $b=B$ for $\alpha \neq 0$, $\beta \neq 0$. Then it readily follows that

$$B - \beta = \pm \left\{ b_0^2 + \beta^2 - \frac{\beta^2\omega - 2\alpha^2\varepsilon(\frac{1}{2} - \omega - \varepsilon)}{\omega - \varepsilon} \right\}^{\frac{1}{2}}. \qquad \text{(5.12)}$$

Hence the field is symmetrical and reduced.

(ii) *Case* 4.—If $b=0$, the corresponding result is

$$A-\alpha=\frac{\alpha\varepsilon(\omega+\varepsilon)}{\omega-\varepsilon}\pm\left\{a_0^2+\alpha^2\left[\frac{\varepsilon(1-\omega-\varepsilon)-\omega}{\omega-\varepsilon}\right]^2-\frac{\beta^2\omega-2\alpha^2\varepsilon(\frac{1}{2}-\omega-\varepsilon)}{\omega-\varepsilon}\right\}^{\frac{1}{2}}$$

$$\simeq\frac{4\cdot5\alpha}{\omega-\varepsilon}\pm\left\{a_0^2+\left(\frac{4\cdot5\alpha}{\omega-\varepsilon}\right)^2-\frac{1\cdot5\beta^2+7\cdot5\alpha^2}{\omega-\varepsilon}\right\}^{\frac{1}{2}}. \qquad \ldots\ldots\ldots\ldots \quad (5.13)$$

Hence in this plane the field is unsymmetrical by about $4\cdot5\alpha/(\omega-\varepsilon)$, and since $(\omega-\varepsilon)^{-1}$ is -110 for quartz and $5\cdot8$ for calcite, the asymmetry will be large unless α is very small indeed, that is, unless the optic axis lies almost exactly in the surface.

Unless $\beta\gg\alpha$, the field is increased from $2a_0$ to approximately $2[a_0^2+\{4\cdot5\alpha/(\omega-\varepsilon)\}^2]^{\frac{1}{2}}$, the minimum angle between the edge of the field and the normal being $[a_0^2+\{4\cdot5\alpha/(\omega-\varepsilon)\}^2]^{\frac{1}{2}}-4\cdot5\alpha/(\omega-\varepsilon)$. This is $0\cdot4a_0$ when $\alpha=a_0(\omega-\varepsilon)/4\cdot5$, or $\alpha\simeq a_0/500$ for quartz and $a_0/25$ for calcite.

Nor may β be allowed to be very large. From both (**5.12**) and (**5.13**) it is evident that if $\alpha=0$, there is a 10 per cent. reduction in field if $\beta\simeq(\omega-\varepsilon/5\omega)^{\frac{1}{2}}b_0$, or about $b_0/30$ for quartz or $b_0/7$ for calcite.

It should be noted that the field is given by the above expressions only when $4\cdot5\alpha/(\omega-\varepsilon)$ is not too large; for large enough values, the retardation in the centre of the asymmetric field, which is opposite in sign to that at the edge, also exceeds the tolerable limits.

VI. Discussion

(*a*) *Stray Light*

(i) *Defective Polarizers.*—In the design of a birefringent filter, the question arises as to the type of polarizer, calcite or "Polaroid", and if the latter, what shall be its characteristics. While calcite prisms undoubtedly give good results, properly chosen "Polaroid" will often be preferable because of availability and cheapness. Usually, high transmission is an important factor; but this is obtainable in "Polaroid" only with incomplete polarization. However, by alternating the orientation of the retardation plates, the stray light can be reduced. A good compromise in a seven-"Polaroid" filter would be obtained with "Polaroid" for which $T_1/T_2=500$. Two pieces of this when parallel have a transmittance 250 times that when crossed. Such material would normally have a parallel transmittance (almost exactly $\frac{1}{2}k^2T_1^2$) of about $0\cdot40$—the term k allowing for surface reflections—and a seven-"Polaroid" filter would have a peak transmittance 82 per cent. of that of a similar filter equipped with calcite polarizers. The stray light would be reduced from $0\cdot11$ to $0\cdot10$. These results are, however, simplifications derived on the assumption of constant T_2/T_1, which in practice varies throughout the spectrum.

(ii) *Half-wave Plates.*—A second design consideration concerning the introduction of stray light is the use of non-achromatic half-wave plates in Lyot type 1 wide-field elements. If the filter is intended for a range of wavelengths, and mica half-wave plates are adjusted for the middle of the range, the stray light introduced depends on the range of wavelengths and number of wide-field elements, and can be serious. Even in a filter designed for a single wavelength and containing several wide-field elements the tolerance on the accuracy of the half-wave plate is quite exacting.

(iii) *Alignment between Retardation Plates and Polarizers.*—A further source of stray light is purely constructional, namely, the incorrect alignment between the retardation plates and polarizers. Its influence will be negligible if the errors are of the order of $0\cdot5°$ or less.

We have not discussed the effects of errors in thickness of retardation plates ; these have been treated by others, a tolerance of $\lambda/20$ in retardation being normally acceptable.

(*b*) *Field*

The major effect of faulty components on the field is an asymmetry due to lack of alignment of the optic axis in the surface of an element. The effect is particularly marked in the case of Lyot type 1 elements, where the asymmetry is some 500 times the error in alignment for quartz and 17 times for calcite.

Errors in the other crystal axes are not very important in simple elements ; in Lyot type 1 elements, the field is reduced by some 10 per cent. for an alignment error of about 3 per cent. of the semi-field for quartz or 15 per cent. of the semi-field for calcite.

In Lyot type 1 elements, one way of compensating for loss of field is to provide in the filter cell adjustment for aligning the individual components. Since the field of a simple element is asymmetrical by about 3α, the available adjustment should be at least three times the maximum error expected in aligning the crystal axes in the plane of the surface. However, the large asymmetry in field introduced by errors in alignment provides a means of checking this alignment during construction of the elements, and in this way avoiding the necessity for added complexity in the filter cell.

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