DIPOLE RESONANT MODES OF AN IONIZED GAS COLUMN

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Summary

It is shown by a quasi-static treatment there is not one but a multiplicity of plasma resonant modes which will emit dipole radiation in a cylindrical column of ionized gas with a radially symmetrical but non-uniform electron density. It is shown that the multiple resonances obtained by Romell in scattering of short waves from a gas discharge column may be qualitatively explained assuming a reasonable electron distribution. Approximate values for the resonant frequencies for a Gaussian distribution, relevant to meteor trails, are found and some of these may be observable.

The status of the finite energy loss and phase shift, shown by Herlofson and other authors to arise where the real part of the dielectric constant vanishes, is examined and mechanisms pointed out which modify the expressions given.

I. Introduction

It was shown by Tonks (1931) and emphasized by Herlofson (1951) that the frequencies of plasma oscillations depend on the shape of the boundaries and that putting a dielectric constant

$$K \equiv 1 - ne^2/(\pi m \nu^2) = 0,$$

where n is an electron density, does not always serve to locate these frequencies. Not much study has yet been given to the oscillations occurring with non-uniform densities ; it does not, for example, appear to be realized that in this case there will in general be a number of dipole modes of different frequencies apart from a multiplicity of quadrupole and higher modes. This is shown in the following paragraphs and related to the experiments of Romell (1951) where such modes were found in reflection from a gas discharge column.

Closer examination is also needed of the paradoxical conclusion that with vanishing collision frequency there is still a finite energy loss and phase shift (Makinson *et al.* 1951 ; Kaiser and Closs 1952) arising near the region where the real part of K goes through zero.

II. The Field Equations

If the quantities varying with time have frequency ν, the electron density is $n_0(\mathbf{r}) + n(\mathbf{r}, t)$ and the force on an electron is taken as $-\mathbf{E}e/m$ to the first order,

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we obtain from Maxwell's equations when streaming and thermal velocities are supposed zero :

$$\operatorname{div} \mathbf{E} = -4\pi n e, \qquad (1)$$

$$\operatorname{div} (K\mathbf{E}) = 0, \quad K = 1 - \frac{n_0 e^2}{\pi m \nu^2}, \qquad (2)$$

$$K \operatorname{div} \mathbf{E} = \frac{e^2}{\pi m \nu^2} \mathbf{E} \cdot \operatorname{grad} n_0, \qquad (3)$$

$$\frac{\partial^2 n}{\partial t^2} = \frac{-4\pi n_0 e^2}{m} n + \frac{e}{m} \mathbf{E} \cdot \operatorname{grad} n_0, \qquad (4)$$

so that K is the dielectric constant of the plasma. We will neglect at first energy losses due to collisions ; this is known to be a useful approximation in many problems. The positive ions in the plasma are effectively stationary at the frequencies of interest and the mean net charge density is taken as zero.

(*a*) *Uniform Density*

It follows from (2) and (3) that if n_0 is uniform over a region, *either* (i) div $\mathbf{E}=0$ over that region so that, in any forced oscillations produced by external fields of arbitrary frequency, the electron cloud moves without density changes, *or* (ii) $K=0$ and oscillations may take place only at the corresponding (Langmuir) frequency $\nu_L = (n_0 e^2/\pi m)^{\frac{1}{2}}$, there being oscillation of the electron density. The name " plasma electron resonance " was given by Tonks (1931) to such motion. Both types may coexist at the Langmuir frequency.

In case (i) there are restoring forces produced at the boundaries of the plasma, and there are resonant frequencies in general quite different from ν_L which depend on the shape of the boundaries and the orientation and order of the multipole electric moment produced. Tonks called these plasma resonances. They are discussed also by Herlofson (1951).

If the free-space wavelength is much greater than the extent of the plasma, as will now be assumed, we may find such plasma-resonant frequencies simply from the well-known expressions in electrostatics by noting the values of K for which the field inside the region is infinite, that is, finite for vanishing external field (Kaiser and Closs 1952). If the external field is uniform, frequencies of the dipole modes result ; from the expressions for a non-uniform field, frequencies of the multipole modes as well.

(*b*) *Non-uniform Density*

Except in the special case where $\mathbf{E}$ is everywhere perpendicular to grad n_0, div $\mathbf{E} \neq 0$, the density no longer remains constant and we lose any such distinction as between types of oscillation (i) and (ii) above. The Langmuir frequency varies from point to point and has no special meaning.

The quasi-static treatment is again applicable to find resonant frequencies but there is no body of standard electrostatic results for non-uniform media to draw upon.

Since div $\mathbf{D}=0$, $\mathbf{D}=K\mathbf{E}$, continuous lines of electric displacement may be drawn through such a plasma, their spacing giving the magnitude of $\mathbf{D}$. It is now necessary to suppose the fields sinoidal and to give K the complex form $K_1-\mathrm{j}K_2$, the imaginary term arising from energy losses due to processes such as collisions, cf. Section V. We suppose K_2 very small, consequently unimportant except near where $K_1=0$. Near a surface where K_1 goes through zero, E changes suddenly from large positive to large negative values and a phase shift is introduced, while n becomes very large according to equations (1)–(4). There is an oscillatory piling up of charge towards and away from this surface, with smaller piling up throughout the plasma. The restoring forces governing the frequency of a particular resonant mode arise from the free charges appearing at the boundaries as well as from such piling up.

III. Dipole Modes of Non-uniform Circular Cylinder

If the cylinder has axial symmetry and the vanishingly small applied uniform electric field is transverse, putting $\mathbf{E}=-\text{grad } V$, $V=R(r)\cos\theta$ in cylindrical coordinates, R must satisfy (Kaiser and Closs 1952)

$$R''+\left(\frac{K'}{K}+\frac{1}{r}\right)R'-\frac{R}{r^2}=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

with the boundary condition for resonance (in the case where $K\rightarrow 1$ as $r\rightarrow\infty$) $R(0)=0$, real part of $R\sim\text{const.}/r$ for large R. If we put $K=1-\lambda f(r)-\mathrm{j}K_2$ so that $f(\infty)=0$, only for certain eigenvalues of λ can the boundary conditions be satisfied. We now show that there can be no eigenvalues (and hence no plasma resonance) unless K is negative for some range of R.

From (5) and the condition $R(0)=0$, if R is, say, positive at $r=0$, R can never turn downward if K is everywhere positive, since at such turning point we would require R'' negative which is contrary to (5). Thus R increases monotonically and is proportional to r, not r^{-1}, for large r. If, however, K_1 passes through zero, R turns down again at that point in such a way that it *may* for some values of λ behave at infinity like r^{-1}.

It is apparent from (5) that if $K(r)$ is replaced by $K(mr)$, $R(r)$ by $R(mr)$, where m is any multiplier, the eigenvalues will be unchanged. Thus the plasma resonances depend on the shape of the distribution, not at all on its scale (provided its extent is much less than a wavelength).

The question now is how many eigenvalues there are and what is their distribution for forms of $n_0(r)$ of interest. We approach an answer to these questions by considering stepped distributions approximating to (i) that inside a long discharge tube (Killiam 1930 ; Howe 1953), (ii) that in a meteor trail.

It is necessary first to show that resonant frequencies obtained from a stepped distribution approximating to a given smooth distribution will not differ significantly from those appropriate to the latter if the number of steps is large enough.

If K were everywhere >1, as in ordinary dielectric materials, the internal field would certainly not be grossly affected by a transition from finely-stepped to smooth distribution. Thus one would only expect such effects (if any) in the present case where the mathematical features differ, namely, for eigenvalues crowded near $\lambda=1$ (see Section IV and Fig. 2) or arising from the behaviour of the solution near $K_1=0$, which latter we now consider.

If R in (5) is supposed integrated out from the origin from a zero value and arbitrary real slope at the origin, at $r=a$ where $K_1=0$ an imaginary part R^I is introduced into R proportional to $1/K'(a)$ (Kaiser and Closs 1952, equation (46)) but the real part R^R is the same at points A, B on either side of a such that $K_1(A)=-K_1(B)$ and both are near zero.

Thus the only effect of a finite $K'(a)$, as compared with the infinite $K'(a)$ with a stepped distribution, is on R^I. However, the condition of resonance is that $R^R \sim \text{const}/r$ and R^R in $r>a$ is quite independent of R^I if K_2 is negligible, since (5) is linear.

We may therefore calculate values of λ corresponding to resonance without considering R^I at all and for sufficiently many steps the stepped distribution should give the same answers as the smooth distribution to which it approximates.

IV. Stepped Distributions

In the regions 0, 1, 2, . . . n (Fig. 1), $V_p=(B_p r+A_p r^{-1})\cos\theta$, say, where $B_0=0$, $A_n=0$, and from the joining conditions at r_p,

$$B_{p-1}r_p+A_{p-1}r_p^{-1}=B_p r_p+A_p r_p^{-1},$$

$$K_{p-1}(B_{p-1}-A_{p-1}r_p^{-2})=K_p(B_p-A_p r_p^{-2}).$$

The condition for a non-zero solution is then

$$D_n\equiv\begin{vmatrix} r_1^{-1} & -r_1 & -r_1^{-1} & 0 & 0 & . & . & & & & \\ -r_1^{-2} & -K_1 & K_1 r_1^{-2} & 0 & 0 & . & . & & & & \\ 0 & r_2 & r_2^{-1} & -r_2 & -r_2^{-1} & 0 & . & & & & \\ 0 & K_1 & -K_1 r_2^{-2} & -K_2 & K_2 r_2^{-2} & 0 & . & & & & \\ . & . & . & . & . & . & . & & & & \\ 0 & 0 & 0 & . & . & . & . & 0 & r_n & r_n^{-1} & -r_n \\ 0 & 0 & . & . & . & . & . & 0 & K_{n-1} & -K_{n-1}r_n^{-2} & -K_n \end{vmatrix}=0. \qquad (6)$$

Putting $K=1-\lambda f_r$, this is an equation of degree n to determine λ. This will have n roots, but only real roots and their distribution are here of interest, these giving the dipole mode frequencies. The case of two steps was considered by Kaiser and Closs (1952).

(a) *Discharge Tube Distribution*

The measured distribution of electron density across a discharge tube of cylindrical section given by Killian (1930) for a temperature of 18·6 °C can be fitted approximately by the parabolic curve

$$n_0 = N_0\left\{1 - 0\cdot 6\left(\frac{r}{r_1}\right)^2\right\}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

where N_0 is the density at the centre and r_1 the radius of the tube.

We have taken stepped distributions approximating to this curve with successive values 2, 3, . . . 7 of n, the number of steps. The steps are at equal radial intervals and, except in the case $n=6$, the height of each is proportional to the value of (7) at its mid point, with $f=1$ at $r=0$. The values of λ for resonance and the corresponding values of K_p are shown in Table 1.*

It will be noticed that at least one of the K_p is negative (as required from Section III and as can be generally shown from the form of the determinant).

TABLE 1

VALUES OF λ FOR RESONANCE AND THE CORRESPONDING DIELECTRIC CONSTANTS K_p IN THE ANNULI BETWEEN STEPS, FOR VARIOUS NUMBERS n OF STEPS IN THE APPROXIMATING DISTRIBUTION AT RADII r_p.

n	λ	Dielectric Constant, K_p							
		r_p	1	1/2					
		f_p	0·6883	1·0					
2	1·226		0·1560	—0·2262					
	2·684		—0·8477	—1·6844					
		r_p	1	2/3	1/3				
		f_p	0·5932	0·8644	1·0				
	1·0888		0·3541	0·0588	—0·0888				
3	1·4442		0·1433	—0·2484	—0·4442				
	2·8846		—0·7111	—1·4934	—1·8846				
		r_p	1	3/4	1/2	1/4			
		f_p	0·5457	0·7729	0·9243	1·0			
	1·048		0·4280	0·1900	0·0313	—0·0481			
4	1·2001		0·3451	0·0725	—0·1092	—0·2001			
	1·5973		0·1283	—0·2345	—0·4764	—0·5973			
	2·9745		—0·6233	—1·2989	—1·749	—1·974			

* General expressions for the coefficients of the powers of λ in (6) were found, but for six or more steps it was found necessary for accurate determination of the smaller eigenvalues to evaluate the determinant (using Crout's method) for neighbouring values of λ and interpolate to the eigenvalues.

TABLE 1 (*Continued*)

n	λ	Dielectric Constant, K_p							
		r_p	1	4/5	3/5	2/5	1/5		
		f_p	0·5171	0·7103	0·8551	0·9517	1·0		
5	1·0321		0·4663	0·2670	0·1174	0·01777	—0·03211		
	1·1202		0·3472	0·1034	—0·0795	—0·2014	—0·2624		
	1·3205		0·3172	0·06208	—0·1292	—0·2568	—0·3205		
	1·7102		0·1157	—0·2147	—0·4624	—0·6276	—0·7102		
	3·0217		—0·5625	—1·1462	—1·584	—1·876	—2·022		
		r_p	1	5/6	2/3	1/2	1/3	1/6	
		f_p	0·5	0·66	0·8	0·9	0·96	1·0	
6	1·215		0·3925	0·1981	0·0280	—0·0935	—0·1664	—0·2150	
	1·375		0·3125	0·0925	—0·100	—0·2375	—0·3200	—0·375	
	1·75		0·125	—0·155	—0·400	—0·575	—0·680	—0·75	
	3·061		—0·5305	—1·0203	—1·449	—1·755	—1·9386	—2·061	
		r_p	1	6/7	5/7	4/7	3/7	2/7	1/7
		f_p	0·4841	0·6315	0·7543	0·8526	0·9263	0·9754	1·0
7	1·027		0·5029	0·3515	0·2254	0·1245	0·0488	—0·0016	—0·0268
	1·146		0·4449	0·2760	0·1352	0·0225	—0·0620	—0·1183	—0·1465
	1·321		0·3605	0·1658	0·0035	—0·1262	—0·2236	—0·2885	—0·3209
	1·924		0·0685	—0·2151	—0·4514	—0·6405	—0·7823	—0·8768	—0·9241
	3·068		—0·4854	—0·9376	—1·3144	—1·6159	—1·842	—1·993	—2·068

The number of real eigenvalues is, for many steps, less than the number of steps and with increasing n the further modes introduced have values of λ crowding in near $\lambda=1$. For $n=7$ it was not feasible to determine accurately the eigenvalues closest to 1. Figure 2 shows the eigenvalues for increasing n; while the dotted lines have no significance in detail (since the approximating stepped distributions are to a certain extent arbitrary) they do show a general trend.

In Figure 1 the lines of electric displacement **D** corresponding to two values of λ are sketched and it is seen that in the mode with highest λ (i.e. highest frequency when the distribution is fixed, or highest density when the frequency is fixed and the density generally increased by raising the discharge current) the lines reach out furthest. This feature has been found generally in the modes studied and it shows that the highest λ mode will be that most strongly coupled to an external exciting field, hence that most strongly excited if losses are not greatly different for the neighbouring modes. However, the damping arising from energy loss near $K_1=0$ is least for the modes with largest K_1' there.

The experiments of Romell (1951) may be explained qualitatively by the foregoing. He found a number of resonant peaks in the reflection of transversely polarized waves from a cylindrical discharge tube as the discharge current was varied and showed that their angular dependence was that of dipole, not multipole, modes. The highest peak was that with the greatest discharge current, corresponding to our greatest eigenvalue of λ. Quantitative comparison is not possible because the electron distribution across the tube was not measured by Romell.

Stepped electron distributions approximating to other shapes, e.g. spherical, may be similarly discussed with the same conclusion that there is in general a multiplicity of dipole modes, as there is also of multipole modes.

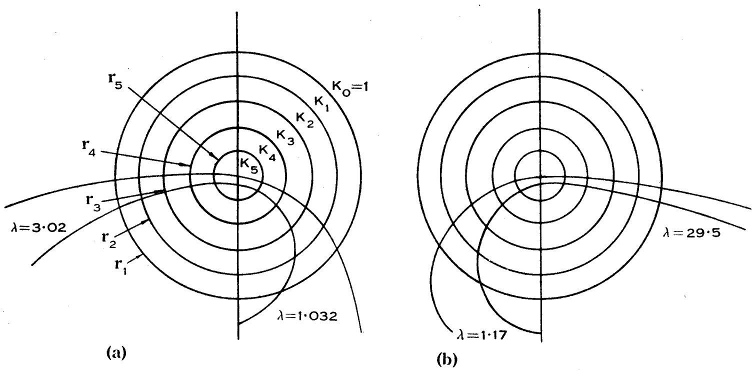

Fig. 1.—Lines of electric displacement in resonant modes for stepped distributions approximating to (*a*) that found in a discharge tube by Killian, (*b*) a Gaussian distribution. The left and right sides of each diagram correspond to the particular eigenvalues indicated.

(*b*) *Gaussian Distribution*

Taking $f(r)=\exp(-r^2)$ and approximating to this by five steps ($f_0, \ldots f_5=0$, $0\cdot0425$, $0\cdot1489$, $0\cdot4043$, $0\cdot7447$, $1\cdot0$; $r_1, \ldots r_5=1$, $0\cdot8$, $0\cdot6$, $0\cdot4$, $0\cdot2$) the equation for λ is

$$\lambda^5-44\cdot619\lambda^4+515\cdot084\lambda^3-2135\cdot7\lambda^2+3475\cdot9\lambda-1888\cdot7=0,$$

and the five real roots are: $1\cdot17$, $1\cdot81$, $3\cdot44$, $8\cdot73$, $29\cdot5$. The corresponding lines of induction are sketched in Figure 1 (*b*). Again the greater reach of the lines for the higher eigenvalues indicates stronger coupling to the exciting field. However, in considering an actual smooth Gaussian distribution, if the above values of λ are taken to give the resonant conditions, we note that at the radius where $K=0$ the slope of the curve for $K(r)$ is greatest for the eigenvalues $1\cdot81$, $3\cdot44$, consequently these modes will suffer least damping (see Section VI).

Kaiser and Closs discovered only one dipole mode by numerical integration with (in our notation) $\lambda=2\cdot4$. This corresponds presumably to our value $1\cdot81$. Their Figure 7 based on calculations with a smooth Gaussian distribution and numerical integration of (5) shows no evidence of more than one resonance, yet there is no reason to believe that the stepped distribution manufactures resonances with no counterparts for the smooth distribution. We can only suggest that the intervals of λ (their f) chosen by those authors were not closely spaced enough to make the finer structure of the resonant behaviour apparent when the diffuseness damping (Section V) is present to the extent they assume.*

We would thus expect at least two maxima to be apparent in transversely polarized echoes from meteor trails (of diameter small enough for the quasi-static approximation) unless other conditions of the problem obscure their separateness.

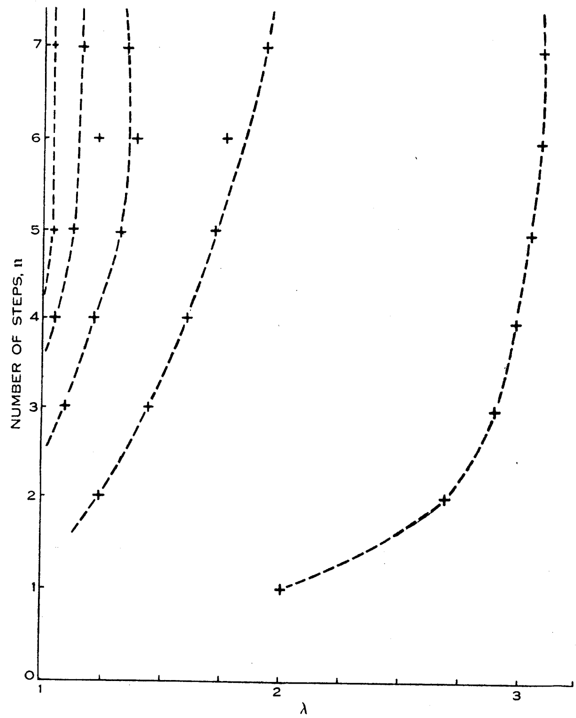

Fig. 2.—The values of λ for resonance as the number of steps is increased in a stepped distribution approximating to (7) A trend is suggested by the dotted lines.

(c) *Nearly Uniform Cylinder*

Some light on the distribution of the eigenvalues in general may be obtained as follows. We suppose the electron density distribution perturbed from a simple step, putting $f_r = 1 + \varepsilon g_r$, ε small, $r = 1, \ldots n$, so that in (6) $K_r = 1 - \lambda - \lambda\varepsilon g_r$. When $\varepsilon = 0$, $D_n(\lambda) = \frac{1}{2}(-2)^n(1-\lambda)^{n-1}(2-\lambda)(r_1 \ldots r_n)^{-1}$, so that $D_n(\lambda)$ has an $(n-1)$-fold root at $\lambda = 1$, as well as the single root at $\lambda = 2$ corresponding to the

* *Note added in Proof.*—Dr. Kaiser has informed us that the intervals were quite close (namely, $\lambda = 0$, $0 \cdot 5$, $1 \cdot 0$, $1 \cdot 5$, $2 \cdot 0$, $2 \cdot 4$, $2 \cdot 5$, $3 \cdot 0$, $4 \cdot 0$, $6 \cdot 0$, $8 \cdot 0$, $18 \cdot 0$, $20 \cdot 0$, 1000). This suggests that the diffuseness damping was sufficient to prevent resolution of separate resonances. However, in the actual physical case the diffuseness damping may well be less than that calculated, see Section V.

eigenvalue $K=-1$ for a uniform cylinder. If we now consider terms in ε, ε^2, . . . in succession it is seen by inspection of D_n that it is of the form

$$\tfrac{1}{2}(-2)^n(r_1 \ldots r_n)^{-1}(1-\lambda)^{n-1}(2-\lambda)+b_1\varepsilon\lambda(1-\lambda)^{n-2}+b_2\varepsilon^2\lambda^2(1-\lambda)^{n-3} + \ldots + b_{n-1}\varepsilon^{n-1}\lambda^{n-1}+b_n\varepsilon^n\lambda^n,$$

where the b_r do not contain the factor $(1-\lambda)$. Thus in the first order perturbation there is an $(n-2)$-fold root at $\lambda=1$, one of the coincident roots having been split off and lying near 1 while the root near 2 is displaced slightly. Similarly each successive order of perturbation splits off a root from the remaining coincident roots at 1 and displaces those already split off. Not all these roots are necessarily real.

We thus expect, for a smooth distribution not too far removed from a simple step, a distribution of eigenvalues of λ in which there is one somewhere near 2, the remainder crowding up towards unity and perhaps becoming infinitely dense near unity (of course, in experimental observation, the latter would be obscured). It is apparent from Figure 2 that this general behaviour is there exemplified.

(*d*) *Cavity Resonator Behaviour*

If the distribution contains an annulus of finite width between r_p and r_{p+1} in which $K_p=0$, no lines of induction can penetrate this region and the oscillations (if any) inside the annulus are quite uncoupled with any taking place outside. The corresponding property of D_n is that it can then be expressed as the product of two determinants, the first containing only terms in $r_1, \ldots r_p$, $K_1, \ldots K_{p-1}$, the second only terms in $r_{p+1}, \ldots, K_{p+1}, \ldots$ For example, if there are two steps inside the annulus with $K_p=0$, the resonant frequency of oscillations inside the annulus is given by

$$K_p=0,, \qquad \ldots\ldots\ldots\ldots \qquad (8)$$

$$\frac{K_{p+1}}{K_{p+2}}=-\left(1+\frac{r^2_{p+2}}{r^2_{p+1}}\right). \qquad \ldots\ldots\ldots\ldots \qquad (9)$$

Regarded as equations to determine λ, (8) and (9) cannot in general both be satisfied, but given, say, f_p and f_{p+2}, there will be a certain value of f_{p+1} for which resonance is possible, namely,

$$f_p+(1+r^2_{p+2}/r^2_{p+1})(f_p-f_{p+2})/(1-r^2_{p+2}/r^2_{p+1}).$$

Experimentally such a resonance might be sought with two concentric hollow cylinders of plasma, the central core having no plasma so that $K=1$ there. The average density of one plasma would need to be adjusted until resonance at some frequency near that expected was found. It is interesting to note that the dimensions of such a cavity resonator may be as small as we please in relation to a free-space wavelength.

V. Energy Loss near the Region where $K=0$

It is well known that if one supposes energy losses, due, for example, to collisions, by the electrons to be adequately described by assigning a complex value K_1-jK_2 to K there is a finite energy loss in a vanishingly thin region near where $K_1=0$ (and E therefore is very large) even though K_2 is vanishingly small (e.g. Makinson *et al.* 1951). If $K_1'(a)$ is the value of the gradient of K_1 where it vanishes and $\mathbf{D}$ is the amplitude of the (sinoidal) electric displacement, the rate of energy loss is readily seen to be proportional to

$$\frac{\mathbf{D}^2}{K_1'(a)}. \qquad (10)$$

This loss is responsible for the damping of the resonances as discussed by Herlofson (1951) and Kaiser and Closs (1952).

Herlofson (1952) has pointed out that a more careful discussion of such results is needed and that a finite effect from vanishing collision frequency is not credible. We indicate below some limitations of the simple formal treatment. Any mechanism other than collisions which limits the amplitude of E at $K_1=0$ will remove or modify the energy loss derived as above.

Now firstly, one can describe the properties of a region containing free electrons by assigning to it a dielectric constant (real or complex) only as an approximation. As shown by Salpeter and Makinson (1949), if the velocities of the electrons (due to streaming or thermal motion) are large enough for their " transit time " (in travelling a distance over which E varies considerably) to be comparable with a period, such an assignment is not valid except as an approximation. In that approximation the values of K_1 and K_2 properly to be assigned to a point depend not only on the density and collision frequency at that point, but on the variation of E in the neighbourhood, which itself depends on the variation of K_1 and K_2. Electrons may then acquire or lose energy where the field is strong and lose it elsewhere, (*a*) at the walls of a discharge tube, or (*b*) by collisions with gas molecules, or (*c*) by interaction with the electric field. Thus, if collision losses are very small, in the region near where K_1 vanishes E becomes very large and rapidly varying, and the proper value of K_2 is non-zero even if collision losses are supposed zero.

Mechanisms (*a*) and (*b*) lead to a net loss of energy from the field, while (*c*) does not, energy being merely transported from one part of the field to another by the electrons, giving a positive contribution to K_2 in some places, negative in others.

The energy loss near where $K_1=0$ will thus depend on the whole configuration in a complicated way but calculation might be attempted using the relations derived by Salpeter and Makinson (1949) in the special case where walls are remote, collisions are negligible, streaming is zero, and the electron temperature is supposed known and not too high.

One can however say that because at least of mechanism (*c*) the energy loss given by (**10**) is an overestimate and the " diffuseness " damping of resonances calculated on that basis is in some degree excessive.

A second reason for failure of (**10**) has been indicated by Herlofson (1952), namely, the neglect of second order effects arising from the very large values of E near $K_1=0$. For example, the amplitude of an electron motion will take it through regions of varying E. However, for sufficiently small applied field it would seem that the processes described in the preceding paragraphs must be more important.

VI. Acknowledgments

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VII. References

Herlofson, N. (1951).—*Ark. Fys.* **3**: 247.
Herlofson, N. (1952).—U.R.S.I. Conference, Sydney.
Howe, R. M. (1953).—*J. Appl. Phys.* **24**: 881.
Kaiser, T. R., and Closs, R. L. (1952).—*Phil. Mag.* **43**: 1.
Killian, T. J. (1930).—*Phys. Rev.* **35**: 1238.
Makinson, R. E. B., Thonemann, P. C., King, R. B., and Ramsay J. V. (1951).—*Proc. Phys. Soc. Lond.* B **64**: 665.
Romell, D. (1951).—*Nature* **167**: 243.
Salpeter, E. E., and Makinson, R. E. B. (1949).—*Proc. Phys. Soc. Lond.* B **62**: 180.
Tonks, L. (1931).—*Phys. Rev.* **38**: 1219.