AN INVESTIGATION OF FINITE STRAIN IN AN ISOTROPIC MATERIAL SUBJECTED TO HYDROSTATIC PRESSURE AND ITS SEISMOLOGICAL APPLICATIONS

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Summary

An attempt is made to reduce the generality of the usual theories of finite strain and to concentrate on the case of hydrostatic pressure. The method used has been based on the works of Murnaghan (1951) and Birch (1952), and yields a general relation giving the elastic constants in terms of the strain and the derivatives of pressure with respect to strain.

It becomes evident that a law in lieu of Hooke's law is needed to proceed with a mathematical theory; and so various elasticity equations in the literature are investigated to determine the special assumptions used to derive them from the present general equations. A linear relation between the second and third order elastic constants is then proposed and the resulting relation between pressure and density compared with Bridgman's experimental results for the alkali metals. It is found that the proposed law of finite hydrostatic strain agrees favourably with experiment and also with deductions from the atomic theory of solids.

The paper concludes with a few relevant remarks on the implications of the present theory to certain seismological problems.

I. Introduction

The aim of this paper is to investigate the elasticity equations governing large hydrostatic strains, so that the effect of the pressure existing in the deep interior of the Earth can be assessed. Considerable simplification results on treating this special problem instead of developing a general theory.

Following the terminology suggested by Murnaghan (1937), we say that if the displacements are referred to the initial state as standard we are using the Lagrangian approach, while if the final state is taken as standard we have the Eulerian approach. Throughout this paper the Lagrangian viewpoint will be adopted.

In the first part of the paper the assumption is made that the pressure can be expressed as a power series in the strain. This is seen to be equivalent to the usual assumption that the elastic strain energy can be so expanded.

All the equations herein can easily be derived on the Eulerian approach, which apparently has neither advantage nor disadvantage except that, in approximations like that made by Birch (1952), the two methods give widely different results.

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II. THE COMPRESSIBILITY RATIO

Since we are concerned with hydrostatic strain we may refer a particle to the same set of Cartesian axes before and after the application of strain. In the following we take the coordinates of a typical particle in the initial state as (a, b, c) and in the final state as (x, y, z) where $(x, y, z) = (a - ka, b - kb, c - kc)$. We denote the displacement of the particle by (u, v, w). Regarding the medium as isotropic the initial state may be previously subjected to hydrostatic strain, according to the work of Murnaghan (1951, p. 64).

Using the Lagrangian approach the components of the strain tensor for hydrostatic strain are

$$\varepsilon_{aa} = \varepsilon_{bb} = \varepsilon_{cc} = \varepsilon_L, \quad \varepsilon_{bc} = \varepsilon_{ca} = \varepsilon_{ab} = 0,$$

where

$$\varepsilon_{aa} = \frac{\partial u}{\partial a} + \tfrac{1}{2}\left(\frac{\partial u}{\partial a}\right)^2,$$

so that

$$\varepsilon_L = -k + \tfrac{1}{2}k^2 \quad \text{or} \quad 1 - k = (1 + 2\varepsilon_L)^{\frac{1}{2}}.$$

If we take V_0 as the volume contained in a rectangular parallelepiped of sides a, b, c, in the initial state and V as the volume in a rectangular parallelepiped of sides x, y, z in the final state, then

$$V = xyz = abc(1 - k)^3 = V_0(1 - k)^3,$$

or

$$\frac{V}{V_0} = (1 + 2\varepsilon_L)^{3/2},$$

where the strain ε_L is intrinsically negative. It is usual to write $\varepsilon_L = -e_L$ so that the equation reduces to

$$\frac{\rho}{\rho_0} = (1 - 2e_L)^{-3/2}, \quad \dots\dots\dots\dots\dots\dots (1)$$

where ρ, ρ_0 are the densities in the final and initial states respectively.

III. THE GENERAL STRESS-STRAIN RELATION

We assume that, for an isotropic medium subjected to an initial hydrostatic pressure p_0, the total pressure p is given by

$$p = p_0 + A_1\varepsilon + A_2\varepsilon^2 + \dots + A_n\varepsilon^n + \dots, \quad \dots\dots\dots\dots (2)*$$

where the parameters $A_n = A_n(p_0)$ are functions of the initial pressure and the strain ε is measured from the configuration in which $p = p_0$. Hence if the initial pressure is zero

$$p = A_{01}e + A_{02}e^2 + \dots + A_{0n}e^n + \dots, \quad \dots\dots\dots\dots (3)$$

where the coefficients A_{0n} are constants and the strain e is measured from the configuration in which $p = 0$. Note that in this section the strains ε and e are taken as intrinsically positive, corresponding to equation (1), and the symbols are not to be confused with those of Section II.

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* It has been noted by the referee that (2) with coefficients given by (5) follows from (3) and $e = e_0 + (1 - 2e_0)\varepsilon$ by Taylor's theorem.

Let the density when $p=0$, p_0, p be $\rho=\rho_0$, ρ_{p_0}, ρ respectively. If the strain at pressure $p=p_0$ be measured from the configuration in which the pressure is zero, we have

$$\frac{\rho_{p_0}}{\rho_0}=(1-2e_0)^{-3/2}, \quad \frac{\rho}{\rho_0}=(1-2e)^{-3/2}, \quad \frac{\rho}{\rho_{p_0}}=(1-2\varepsilon)^{-3/2},$$

so that

$$1-2e=(1-2e_0)(1-2\varepsilon)$$

or

$$de=(1-2e_0)d\varepsilon. \quad \dots\dots\dots\dots\dots\dots \quad (4)$$

On differentiating equation (2) n times we obtain

$$\frac{d^np}{d\varepsilon^n}=n!A_n+(n+1)!A_{n+1}\varepsilon+\dots,$$

which, on using (4), reduces to

$$(1-2e_0)^n\frac{d^np}{de^n}=n!A_n+(n+1)!A_{n+1}\varepsilon+\dots.$$

On taking $\varepsilon=0$ so that $e=e_0$ we have

$$A_n=\frac{(1-2e_0)^n}{n!}\left(\frac{d^np}{de^n}\right)_{e=e_0},$$

or, regarding A_n as a function of p instead of p_0 (since the initial pressure may be considered as variable), we obtain the relation

$$A_n=\frac{(1-2e)^n}{n!}\frac{d^np}{de^n}. \quad \dots\dots\dots\dots\dots \quad (5)$$

On differentiating (5) with respect to p, it is easily shown that

$$\frac{dA_n}{dp}=\frac{(n+1)A_{n+1}-2nA_n}{A_1}, \quad \dots\dots\dots\dots \quad (6)*$$

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* Murnaghan (1951, p. 69) has derived the equation

$$p=p_0+(3\lambda+2\mu+p_0)e+\left(3\lambda+2\mu-9l-n+\frac{3}{2}p_0\right)e^2+\dots$$

from which, on using equation (6) we derive that

$$\frac{d}{dp}\left(\lambda+\frac{2}{3}\mu\right)=\frac{-\frac{l}{3}\left(\lambda+\frac{2}{3}\mu\right)-2\left(l+\frac{1}{9}n\right)}{\lambda+\frac{2}{3}\mu+\frac{1}{3}p},$$

which is in conflict with the result obtained by Murnaghan. The slip in Murnaghan's argument is in the change of variable given by the equation

$$\xi=\ln\frac{1}{V},$$

which implies the infinitesimal relation

$$\frac{\delta V}{V}=-\delta\xi+\frac{(\delta\xi)^2}{2!}-\frac{(\delta\xi)^3}{3!}+\dots$$

Defining the incompressibility $\varkappa$ by the relation

$$\varkappa = \rho \frac{\mathrm{d}p}{\mathrm{d}\rho}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

it follows from equations (1) and (5) that $A_1 = 3\varkappa$.

It becomes evident at this stage that for further advance in the mathematical theory we require $A_1(p)$ as a function of p or $A_2(p)$ as a function of $A_1(p)$ since we can then determine all the other parameters by the use of equation (6). The law of finite hydrostatic strain proposed in the present paper is that $A_2(p)$ should be taken as a linear function of $A_1(p)$. However, we will first investigate the various pressure-density equations found in the literature, in order to determine the relations imposed on $A_1(p)$ and $A_2(p)$ by them.

IV. INCOMPRESSIBILITY A CONSTANT

On taking A_1 constant we find from equation (6) that $A_n = \left(\frac{2^{n-1}}{n}\right) A_1$, so that, on substituting in equation (2) and summing the series, we obtain

$$p - p_0 = \varkappa \ln \frac{\rho}{\rho_0}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

The same relation can be obtained from equation (7), the definition of incompressibility. This equation is equivalent to the finite form of the infinitesimal strain relation and is known to be valid only in the case of small pressure increments.

V. INCOMPRESSIBILITY A LINEAR FUNCTION OF PRESSURE

If we assume $\varkappa = \varkappa_0 + sp$ we have on integrating the equation defining incompressibility

$$p = \frac{\varkappa_0}{s}\left\{ \left(\frac{\rho}{\rho_0}\right)^s - 1 \right\}, \quad \ldots\ldots\ldots\ldots\ldots (9)$$

where we have taken the initial state as being subjected to zero pressure. We can also obtain this relation by using equation (6) to find A_{0n} in terms of A_{01} and substituting in (3). From (9) we have immediately

$$\varkappa = \varkappa_0 \left(\frac{\rho}{\rho_0}\right)^s. \quad \ldots\ldots\ldots\ldots\ldots (10)$$

These equations are those obtained from Murnaghan's (1951) integrated linear theory of elasticity, a method which gives fairly good results for small ranges of pressure, or, as will be shown later, for extremely high pressure ranges. The special case $s = 2/3$ gives the same result as Seth's (1950) method.

VI. A GENERALIZATION OF BIRCH'S EQUATION

From equation (2) we easily derive that

$$p = \frac{1}{(1-2\varepsilon)^\beta}\left\{ p_0 + (A_1 - 2\beta p_0)\varepsilon + \left(A_2 - 2\beta A_1 + \frac{\beta(\beta-1)}{1.2}.2^2 p_0\right)\varepsilon^2 + \ldots \right\}.$$

Assuming now that the coefficient of ε^2 vanishes, so that

$$A_2 = 2\beta A_1 - \frac{\beta(\beta-1)}{1.2} \cdot 2^2 p_0, \quad \ldots\ldots\ldots\ldots (11)$$

it can be demonstrated by the use of equation (**6**) that all the coefficients of higher powers of ε vanish. Hence

$$p = \frac{1}{(1-2\varepsilon)^\beta}\{p_0 + (A_1 - 2\beta p_0)\varepsilon\},$$

or, if the initial pressure is zero,

$$p = \frac{A_{01}e}{(1-2e)^\beta},$$

where β is given by $A_{02}/2A_{01}$.

In this case the pressure-density relation is

$$p = \frac{3}{2}\varkappa_0 \left(\frac{\rho}{\rho_0}\right)^{\frac{1}{3}(\beta-1)}\left\{\left(\frac{\rho}{\rho_0}\right)^{2/3} - 1\right\}, \quad \ldots\ldots\ldots\ldots (12)$$

and it can be shown that

$$\frac{d\varkappa}{dp} = \frac{2\beta^2\left(\frac{\rho}{\rho_0}\right)^{2/3} - 2(\beta-1)^2}{3\beta\left(\frac{\rho}{\rho_0}\right)^{2/3} - 3(\beta-1)},$$

so that

$$\left.\begin{array}{l} \left(\dfrac{d\varkappa}{dp}\right)_{p=0} = \dfrac{4\beta-2}{3}, \\[3mm] \left(\dfrac{d\varkappa}{dp}\right)_{p\to\infty} = \dfrac{2\beta}{3}. \end{array}\right\} \quad \ldots\ldots\ldots\ldots (13)$$

The equation (**12**) reduces to that of Seth when $\beta=1$. Birch (1952), working from a different approach, obtained equation (**12**) with $\beta=7/2$. This value of β gives reasonable agreement with the results of Bridgman's experiments on the alkali metals but Birch's formula needs modification as will become apparent in Sections VIII and IX. It should be remarked that the equation

$$p = \frac{3}{2}\varkappa_0\left\{\left(\frac{\rho}{\rho_0}\right)^{7/3} - \left(\frac{\rho}{\rho_0}\right)^{5/3}\right\}, \quad \ldots\ldots\ldots\ldots (14)$$

which has been referred to as Birch's equation, is not considered by Birch as an exact relation but merely as the best single-constant formula describing the behaviour of all substances subjected to large hydrostatic strain.

In support of Birch's contention it should be noted that, even if the special relation (**11**) is not valid, equation (**12**) is still correct to the second order in e if β is given by $A_{02}/2A_{01}$.

VII. A_2 A LINEAR FUNCTION OF A_1

The present method of approach to the problem of finite hydrostatic strain immediately suggests the assumption $A_2 = \lambda A_1 + \mu$. From equation (**6**) we obtain

$$\frac{d\varkappa}{dp} = \frac{(\alpha-N)\varkappa_0 + N\varkappa}{\varkappa}, \quad \ldots\ldots\ldots\ldots (15)$$

where the constants α, N have been introduced as being more convenient than λ, μ. It follows from (15) that

$$\left.\begin{array}{l} \left(\dfrac{d\varkappa}{dp}\right)_{p=0}=\alpha, \\[2mm] \left(\dfrac{d\varkappa}{dp}\right)_{p\to\infty}=N. \end{array}\right\} \quad \dotfill \quad (16)$$

Integrating equation (15) we obtain

$$p=\frac{\varkappa-\varkappa_0}{N}-\frac{(\alpha-N)\varkappa_0}{N^2}\ \ln\ \left\{1+\frac{N(\varkappa-\varkappa_0)}{\alpha\varkappa_0}\right\}. \quad \dotfill \quad (17)$$

The definition of incompressibility combined with (15) gives

$$\frac{\rho}{\rho_0}=\left\{1+\frac{N(\varkappa-\varkappa_0)}{\alpha\varkappa_0}\right\}^{\frac{1}{N}}, \quad \dotfill \quad (18)$$

so that (17) reduces to

$$p=\frac{\alpha\varkappa_0}{N^2}\left\{\left(\frac{\rho}{\rho_0}\right)^N-1\right\}-\frac{(\alpha-N)\varkappa_0}{N}\ \ln\ \frac{\rho}{\rho_0}. \quad \dotfill \quad (19)$$

This equation is a generalization of Murnaghan's equation (9) and reduces to it when $\alpha=N$. The assumption of a linear law is very appealing since it is in the spirit of Hooke's law, and further, the success of Murnaghan's integrated linear theory suggests possibilities in its generalization.*

VIII. COMPARISON OF THEORY WITH EXPERIMENT

Birch (1952) has compared equation (14) with Bridgman's experimentally determined values of p v. v/v_0 for the alkali metals by determining the value of $\varkappa_0$ for each value of pressure. The results of the calculation are shown in Table 1.

The most noticeable feature of these results is that the calculated value of $\varkappa_0$ increases with the pressure and over the range under consideration varies by more than 20 per cent. This suggests that Birch's formula will not be useful for extrapolation since $\varkappa_0$ in this formula varies with pressure. However, as the pressure increases the calculated value of $\varkappa_0$ becomes nearly constant so that the suggestion is that the formula may be valid at high pressures but is invalid at low pressures. This invalidates its use for extrapolating from densities obtained over a range of high pressures to the density at zero pressure. It further suggests that the value $(d\varkappa/dp)_{p=0}=4$ obtained from the formula needs modification.

The fitting of the experimental data to equation (19) is laborious and difficult since the curve of best fit can only be obtained by a method of trial and error. In the accompanying Tables 2, 3, and 4 are shown examples of

* It is to be noted that the present solution is also a generalization of the results obtained by the writer (1953) as an approximate solution of the equation derived by Murnaghan (1951, p. 70) for $d\varkappa/dp$. In view of the footnote to Section III, it follows that the previous results are in error ; but it is evident that the modification to Murnaghan's equation in no way alters the method of approximation used nor does it greatly alter the deductions made therefrom.

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curves of best fit to Bridgman's data for lithium, sodium, and potassium.   It is significant that the variation of the calculated values of pressure using equation (19) have only half the variation found in the calculated values of Birch.   Since Bridgman's two values for the density at 40×10^9 dyn/cm² for sodium corresponds to an error in the calculated pressure of 6 per cent., the greatest error of 7 per cent. in the calculated pressures is almost within the accuracy of the experimental data.

TABLE 1

THE RATIO $2/3\varkappa_0$ FOR THE ALKALI METALS CALCULATED FROM EQUATION (14) AND USING BRIDGMAN'S EXPERIMENTAL VALUES (BIRCH 1952)

p (10^3 kg/cm²)	Lithium		Sodium		Potassium	
	v/v_0	$2/3\varkappa_0$	v/v_0	$2/3\varkappa_0$	v/v_0	$2/3\varkappa_0$
5	0·957	6·41	0·929	11·3	0·884	21·0
10	0·926	5·99	0·883	10·7	0·817	20·2
15	0·899	5·84	0·852	9·8	0·770	19·6
20	0·875	5·81	0·818	10·0	0·732	19·4
25	0·855	5·73	0·791	10·0	0·699	19·6
30	0·835	5·77	0·767	10·0	0·671	19·7
35	0·816	5·81	0·746	10·0	0·647	19·9
40	0·798	5·91	0·727	10·1	0·627	19·9
45	0·782	5·97	0·710	10·1	0·604	20·6
30	0·833	5·84	0·770	9·81	0·668	20·2
40	0·801	5·76	0·737	9·38	0·628	19·7
50	0·773	5·76	0·708	9·21	0·595	19·7
60	0·748	5·78	0·683	9·11	0·568	19·6
70	0·727	5·76	0·661	9·05	0·546	19·5
80	0·707	5·79	0·641	9·05	0·528	19·2
90	0·689	5·83	0·623	9·07	0·513	19·0
100	0·672	5·88	0·606	9·13	0·500	18·7

The values obtained for the density at 45×10^9 dyn/cm² in 1938 for lithium, sodium, and potassium are seen to be unreasonably high, and it is perhaps safe to say that the high values in the 1948 table are not very accurate.   This is pertinent since the error in the present calculated values is 4 per cent. for 100×10^9 dyn/cm².

Apart from the variation at 100×10^9 dyn/cm², the greatest error in the tables is for sodium at 15×10^9 dyn/cm².   It can be seen from differencing the values of v/v_0 that the experimental value at 15×10^9 dyn/cm² is too low ; this conclusion also follows from Birch's calculations of $\varkappa_0$.

In view of the above discussion it would appear that formula (19) fits the experimental data to within 4 per cent., which corresponds to a range of less than 8 per cent., while Birch's formula gives a range of from 10 to 20 per cent. in the calculated values of $\varkappa_0$.

Murnaghan (1951, p. 78) has compared his second-order theory with experiment and shown that the calculated values of v/v_0 agree with Bridgman's experimentally determined values within 1 per cent. It can be seen from Table 3 for the two values of v/v_0 at 40×10^9 dyn/cm² that an error of 1 per cent. in v/v_0 is equivalent to an error of $5 \cdot 5$ per cent. in p. Similarly from Table 1 it can

TABLE 2

COMPARISON OF EQUATION (19) WITH BRIDGMAN'S EXPERIMENTAL
RESULTS FOR LITHIUM

v/v_0	p (Bridgman) (10^9 dyn/cm²)	p (calc.)* (10^9 dyn/cm²)
Bridgman's 1938 values		
0·957	5	5·2
0·926	10	9·9
0·899	15	14·6
0·875	20	19·6
0·855	25	24·2
0·835	30	29·4
0·816	35	34·9
0·798	40	40·7
0·782	45	46·5
Bridgman's 1948 values		
0·833	30	30·0
0·801	40	39·7
0·773	50	49·9
0·748	60	60·4
0·727	70	70·6
0·707	80	81·5
0·689	90	92·5
0·672	100	104·0

* Pressure calculated from the formula

$$p = 79 \cdot 8 \left\{ \left(\frac{\rho}{\rho_0} \right)^{2 \cdot 6} - 1 \right\} - 101 \cdot 6 \ \ln \frac{\rho}{\rho_0}$$

so that $\varkappa_0 = 10 \cdot 6 \times 10^{10}$ dyn/cm², and $\alpha = 5 \cdot 1$.

be seen that an error of 1 per cent. in v/v_0 corresponds to an error of $5 \cdot 5$ per cent. in $\varkappa_0$. It thus appears that Murnaghan's variation of 2 per cent. in the calculated values of v/v_0 corresponds to a variation of 11 per cent. in either the pressure or the initial incompressibility.

IX. DEDUCTIONS FROM ATOMIC THEORY

The atomic theory of solids (Seitz 1940 ; Fürth 1944) gives some important evidence on the compression of materials at high pressures. In its simplest form the theory will give only indicative results and cannot be considered as rigorous.

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Assuming inverse laws of attraction and repulsion we have

$$w(x) = -\frac{a}{x^m} + \frac{b}{x^n}$$

TABLE 3

COMPARISON OF EQUATION (19) WITH BRIDGMAN'S EXPERIMENTAL
RESULTS FOR SODIUM

v/v_0	p (Bridgman) (10^9 dyn/cm^2)	p (calc.)* (10^9 dyn/cm^2)
Bridgman's 1938 values		
0·929	5	5·0
0·883	10	9·8
0·852	15	14·0
0·818	20	19·6
0·791	25	25·0
0·767	30	30·6
0·746	35	36·4
0·727	40	42·3
0·710	45	48·1
Bridgman's 1948 values		
0·770	30	29·9
0·737	40	39·1
0·708	50	48·9
0·683	60	59·1
0·661	70	69·5
0·641	80	80·5
0·623	90	91·8
0·606	100	103·9

* Pressure calculated from the formula

$$p = 38 \cdot 5 \left\{ \left(\frac{\rho}{\rho_0} \right)^3 - 1 \right\} - 61 \ln \frac{\rho}{\rho_0}$$

so that $\varkappa_0 = 5 \cdot 45 \times 10^{10}$ dyn/cm^2 and $\alpha = 6 \cdot 3$.

as the potential between two atoms from which it can be derived that at the
absolute zero of temperature

$$\frac{d\varkappa}{dp} = \frac{\theta^2 \left(\frac{\rho}{\rho_0} \right)^\theta - \varphi^2 \left(\frac{\rho}{\rho_0} \right)^\varphi}{\theta \left(\frac{\rho}{\rho_0} \right)^\theta - \varphi \left(\frac{\rho}{\rho_0} \right)^\varphi},$$

where $\theta = n/3 + 1$, $\varphi = m/3 + 1$.
Thus as

$$\rho \to \rho_0, \quad \frac{d\varkappa}{dp} \to \theta + \varphi,$$

$$\rho \to \infty, \quad \frac{d\varkappa}{dp} \to \theta.$$

The significant deductions to be made from these results are that

$$\left(\frac{d\varkappa}{dp}\right)_{p=0} < 2\left(\frac{d\varkappa}{dp}\right)_{p\to\infty},$$

and

$$\left(\frac{d\varkappa}{dp}\right)_{p=0} > \left(\frac{d\varkappa}{dp}\right)_{p\to\infty} + 1.$$

TABLE 4

COMPARISON OF EQUATION (19) WITH BRIDGMAN'S EXPERIMENTAL
RESULTS FOR POTASSIUM

v/v_0	p (Bridgman) (10^9 dyn/cm^2)	p (calc.)* (10^9 dyn/cm^2)
Bridgman's 1938 values		
0·884	5	5·0
0·817	10	10·0
0·770	15	14·6
0·732	20	19·5
0·699	25	24·7
0·671	30	30·2
0·647	35	35·6
0·627	40	41·0
0·604	45	48·0
Bridgman's 1948 values		
0·668	30	30·8
0·628	40	40·7
0·595	50	51·1
0·568	60	61·7
0·546	70	72·0
0·528	80	81·9
0·513	90	91·3
0·500	100	100·2

* Pressure calculated from the formula

$$p = 16\left\{\left(\frac{\rho}{\rho_0}\right)^3 - 1\right\} - 17\ \ln\frac{\rho}{\rho_0}$$

so that $\varkappa_0 = 3\cdot1\times10^{10}$ dyn/cm^2 and $\alpha = 4\cdot65$.

The first result follows from the fact that $n>m$ since otherwise the crystal would collapse, and the second follows since $\varphi = m/3 + 1$.

It has been shown (Seitz 1940) that for lithium fluoride $n=6$ so that $\theta=3$, and further it has been determined that θ increases with increasing atomic weight. It appears that the values obtained for the alkali metals in Tables 2, 3, and 4 are not inconsistent with these results.

Equation (14) is seen to correspond to a law of force between the atoms in the alkali crystal lattice given by

$$w(x) = -\frac{a}{x^2} + \frac{b}{x^4}.$$

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Apart from the fact that such a law is only a first approximation, the value $n=4$ is somewhat too low and its assumed constancy for the alkali metals is inconsistent with other evidence. It follows that Birch's formula for high pressures should best fit lithium and become increasingly less accurate for sodium and potassium. This is supported by the results of Table 1. Birch (1952, p. 252) has considered other aspects of the problem.

The constancy of $\varkappa_0$ as calculated from Birch's formula for rubidium at high pressures is in conflict with the present arguments. However, due to an internal variation of more than 12 per cent. in the experimental data, it appears that no definite deductions can be made. Another indication of incompatibility is the large value of $\varkappa_0$, which is nearly equal to the value found for potassium.

Within the accuracy of the experimental data the results for the alkali metals obtained on using equation (17) satisfy the following three conditions imposed by the atomic theory of solids.

(i) $\left(\dfrac{d\varkappa}{dp}\right)_{p=0} < 2\left(\dfrac{d\varkappa}{dp}\right)_{p\to\infty}$,

(ii) $\left(\dfrac{d\varkappa}{dp}\right)_{p=0} > \left(\dfrac{d\varkappa}{dp}\right)_{p\to\infty} + 1$,

(iii) $\left(\dfrac{d\varkappa}{dp}\right)_{p\to\infty}$ increases with atomic weight.

X. Seismological Deductions

(1) For pressures above $0\cdot5\varkappa_0$ the value of $d\varkappa/dp$ is somewhat constant. This is in agreement with deductions made by Ramsey (1950) from atomic theory. Graphs of incompressibility against depth and of pressure against depth obtained by Bullen (1947, p. 220) for the interior of the Earth lead to the inference that $\varkappa$ is not a linear function of p in the Earth's inner core and further that $d\varkappa/dp$ increases with pressure. Both these facts suggest that Bullen's values need modification in this region. Perhaps the assumed constancy of $d\varkappa/dp$ would give the calculations for the inner core a greater degree of determinateness.

(2) On extrapolating the ratio $\varphi=\varkappa/\rho$ for layer D of the Earth to zero pressure, Birch (1952) has found that the value of φ_0 is too high to agree with the experimentally determined values for any of the most likely components of this layer. On the basis of the discussion given earlier it would appear that on extrapolating with Birch's formula the value of $\varkappa_0$ would be too high while at the same time ρ_0 would be too low. Thus on using Birch's formula the extrapolated value of φ_0 would be increased markedly from its true value.

(3) Figure 1 shows the variation of $\varkappa/\rho$ with pressure for potassium. The large initial increase may help to explain part of the large increase in seismic velocities in layer C of the Earth.

(4) Both equations (12) and (19) when fitted to Bridgman's results give values of $(d\varkappa/dp)_{p\to\infty}$ greater than $5/3$. This is in conflict with Elsasser's (1951)

smooth interpolation of Bridgman's experimental results to those of quantum mechanics and Bullen's (1952) caution, that considerable care should be shown in any such interpolation, must be supported.
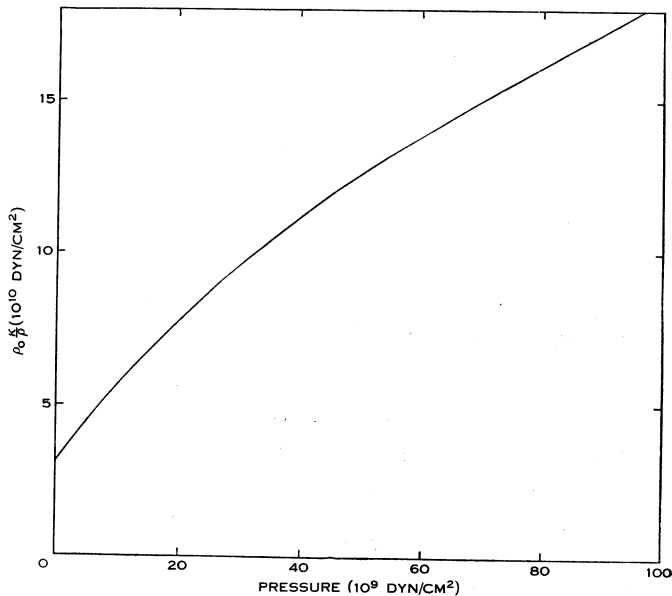

Fig. 1.—Variation of $\varkappa/\rho$ with pressure for potassium.

XI. ACKNOWLEDGMENTS

Professor G. Bosson of the New South Wales University of Technology has read through the manuscript at various stages of its development and made many helpful suggestions. For this assistance the author wishes to express his appreciation. He is also indebted to Professor F. D. Murnaghan, previously of the Instituto Tecnologico de Aeronautica, Brazil, and Professor Francis Birch of Harvard University, U.S.A., for helpful replies to some of his queries.

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