SHORT COMMUNICATIONS

THE CAPACITANCE OF AN ANCHOR RING*

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The surface generated by the circumference of a circle which is rotated about a coplanar axis not intersecting the circle is usually called an anchor ring. Its capacitance in free space will depend on the radius $CP=r$ of the generating circle (Fig. 1) and the radius $OC=R$ of the centre of the generating circle. The

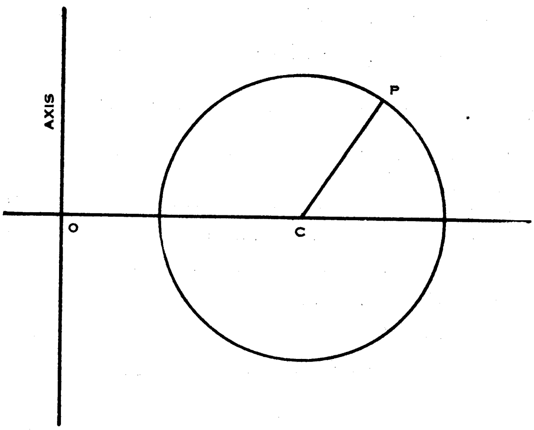

Fig. 1

solution of Laplace's equation in toroidal coordinates is discussed by Hobson (1931). Hicks (1881) has shown that the capacitance C is given by a formula involving a convergent infinite series the terms of which contain toroidal functions. The formula is

$$C=2R\left\{1-\left(\frac{r}{R}\right)^2\right\}^{\frac{1}{2}}\left[\frac{Q_0}{P_0}+2\sum_1^{\infty}\frac{Q_n}{P_n}\right], \qquad \ldots\ldots\ldots\ldots\ (1)$$

where

$$\left.\begin{aligned} P_n&=\int_0^{\pi}\frac{\mathrm{d}v}{(\cosh u-\sinh u\cos v)^{n+\frac{1}{2}}},\\ Q_n&=\int_0^{\infty}\frac{\mathrm{d}v}{(\cosh u+\sinh u\cosh v)^{n+\frac{1}{2}}},\\ \cosh u&=R/r. \end{aligned}\right\} \qquad \ldots\ldots\ldots\ (2)$$

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These integrals are the same as the Laplace and Heine integrals for Legendre functions of the first and second kind and order $n-\frac{1}{2}$. When $n=0$ and $n=1$ the integrals can be transformed into formulae involving complete elliptic integrals of the first and second kind, K and E, to modulus k where

$$k^2=\frac{2\sqrt{R^2-r^2}}{R+\sqrt{R^2-r^2}}. \quad \ldots\ldots\ldots\ldots \quad (3)$$

K' and E' signify the same integrals to modulus k' where $k'^2=1-k^2$. The formulae are:

$$\left.\begin{aligned} P_0&=2\sqrt{k'}K, & Q_0&=2\sqrt{k'}K', \\ P_1&=\frac{2E}{\sqrt{k'}}, & Q_1&=\frac{2(K'-E')}{\sqrt{k'}}. \end{aligned}\right\} \quad \ldots\ldots \quad (4)$$

When the first two terms in the expansion have been found, the remaining terms can be found by successive applications of the recurrence formulae:

$$(2n+1)P_{n+1}-4n\cosh u\, P_n+(2n-1)P_{n-1}=0, \quad \ldots\ldots \quad (5)$$

$$(2n+1)Q_{n+1}-4n\cosh u\, Q_n+(2n-1)Q_{n-1}=0. \quad \ldots\ldots \quad (6)$$

As C is proportional to R when r/R is constant, it will be sufficient to compute C for $R=1$ and different values of r/R. The results are given in Table 1. Terms in which $Q_n/P_n<0\cdot0001$ were neglected. In the table C is given in c.g.s. electrostatic units, $n+1$ is the total number of terms used in the expansion, and X the sum of the terms for which $n>1$.

When $k^2 \simeq 1$ the elliptic integrals can be obtained from the formulae

$$K=\ln(4/k')K_1-K_2, \quad \ldots\ldots\ldots\ldots \quad (7)$$

$$E=\ln(4/k')E_1+E_2, \quad \ldots\ldots\ldots\ldots \quad (8)$$

as tables giving the values of the constants K_1, K_2, E_1, E_2 to a high degree of accuracy have been prepared by Airey (1935).

When k' is small enough $K'-E'$ is best obtained from the expansion

$$K'-E'=\frac{\pi}{2}\left[\frac{1}{2}k'^2+\frac{3}{16}k'^4+\frac{15}{128}k'^6+\ldots\right]. \quad \ldots\ldots \quad (9)$$

Integrals which could not be found from Airey's tables were obtained from Dwight's tables (1941).

If only the first two terms are of importance the formula

$$C=2\sqrt{R^2-r^2}\left\{\frac{K'}{K}+2\frac{K'-E'}{E}\right\}=2\sqrt{R^2-r^2}\left\{\frac{3K'}{K}-\frac{\pi}{EK}\right\} \quad \ldots \quad (10)$$

can be used. The error in using this is given by X in Table 1 and it will be seen that the relative error is less than 1 per cent. if $r/R<0\cdot46$. When only the first term is significant, elliptic integrals can be eliminated by the approximations $K=\ln 8R/r$, $K'=\pi/2$, and $(r/R)^2 \ll 1$. The approximate formula thus obtained is

$$C=\frac{\pi R}{\ln (8R/r)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

The relative error in using this is less than 1 per cent. if $r/R<0\cdot12$ and less than $4\cdot5$ per cent. if $r/R<0\cdot30$.

When $r/R>0\cdot30$ the relative error is less than 1 per cent. when the linear equation

$$C=0\cdot68R+1\cdot07r \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

is used.

When C is divided by D, where $D=2(R+r)$ is the outside diameter, it is found that C/D increases very gradually from $0\cdot381$ when $r/R=0\cdot30$ to $0\cdot426$ when $r/R=0\cdot80$, so the simple rule $C=0\cdot4D$ will give the capacitance with an error of less than 6 per cent. if $r/R>0\cdot30$. It is of interest to note that in this case the capacitance is about four-fifths that of a sphere of the same diameter.

TABLE 1

r/R	C (c.g.s., e.s.u.)	$n+1$	X
0·05	0·616	1	—
0·10	0·722	2	—
0·15	0·800	2	—
0·20	0·868	2	—
0·25	0·932	3	0·0009
0·30	0·992	3	0·0019
0·35	1·050	3	0·0037
0·40	1·106	3	0·0060
0·45	1·161	4	0·0105
0·50	1·216	4	0·0166
0·60	1·323	5	0·0409
0·70	1·429	7	0·080
0·80	1·534	7	0·163
0·90	1·638	10	0·350
1·00	1·74	—	—

In the case of a full ring with $r=R$ the Hicks formula cannot be used. However, as a full ring is the inverse surface of an infinitely long cylinder, a formula can be found by Kelvin's method of inversion (Smythe 1950).

The surface density on an earthed cylinder due to a point charge on the axis is known (Smythe 1950), and from this the surface density at corresponding points on a freely charged anchor ring can be found, together with the potential of the ring. The total charge is then found by integrating over the surface of the ring. In this way the capacitance of a full ring is found to be

$$C=4R\sum_{s=1}^{\infty}\frac{1}{J_1(2\mu_s R)}\int_0^{\infty}\frac{\exp(-\mu_s z)\mathrm{d}z}{\sqrt{4R^2+z^2}},$$

where $2\mu_s R$ is given by the roots of $J_0(2\mu_s R)=0$. On substituting $z=2R\sinh\varphi$ this becomes

$$C=4R\sum_{s=1}^{\infty}\frac{1}{J_1(2\mu_s R)}\int_0^{\infty}\exp\left(-2\mu_s R\sinh\varphi\right)d\varphi. \quad \ldots\ldots (13)$$

This result is given as a problem by Smythe. As $J_1(2\mu_s R)$ is alternatively positive and negative the terms in the series will also alternate. It is obvious that $\int_0^{\infty}\exp\left(-2\mu_s R\sinh\varphi\right)d\varphi$ is always less than $\int_0^{\infty}\exp\left(-2\mu_s R\varphi\right)d\varphi=1/2\mu_s R$ and that the difference diminishes as s increases. The first four terms have been evaluated by numerical integration and it is found that the relative error is less than 1 per cent. when $s>3$. The series can be summed by Euler's transformation for a slowly converging series and, using the approximate formula for the integrals when $s>4$, the capacitance of a full ring is found to be $C=1\cdot 74R$. This agrees with the value obtained by extrapolating the tabulated values of the Hicks formula.

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