ELECTRON SCREENING AND THERMONUCLEAR REACTIONS

By E. E. SALPETER*

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Summary

In the interior of stars most atoms are ionized, but the electrostatic potential of a bare nucleus induces a spherically symmetric polarization of the surrounding electrons and nuclei. The effect of this screening charge cloud on the rate of thermonuclear reactions is investigated for the case of complete ionization of all atoms.

The charge distribution and potential of the screening cloud is calculated for two limiting cases where the electrostatic interaction energy between neighbouring nuclei is small or large compared with the thermal energy (weak or strong screening). The charge cloud is also investigated for intermediate strength of screening, for nuclear species which are rare and have a large charge.

Under most stellar conditions the impact parameter for a thermonuclear collision is much smaller than the radius of a screening cloud. For such cases, a simple formula is given relating the increase in the reaction rate to the potential of the screening cloud. Numerical values are presented for a few typical reactions. For conditions typical for the interior of ordinary main sequence stars the increase in the reaction rate is fairly small, usually less than a factor of two.

I. INTRODUCTION

The rates of many thermonuclear reactions, which can take place under various conditions in the interior of the Sun and stars, have been and are being calculated.† These reactions can be pictured as follows. At the high temperatures in stellar interiors all (or practically all) the atoms are ionized. Two bare nuclei (of charge Z_1 and Z_2 respectively) collide with each other with relative kinetic energy E, arising from the thermal motion of the gas. For the two nuclei to undergo a nuclear transformation they must approach to distances of the order of 10^{-13} cm (nuclear radius). As the particles approach each other they experience a Coulomb repulsion and the Coulomb barrier (electrostatic potential for a separation of the order of a nuclear radius) is very large compared with the mean thermal energy kT. An important factor in the reaction rate then is the barrier penetration factor, the probability of the nuclei approaching sufficiently closely for the nuclear forces to come into play.

The reaction rate is proportional to the following integral

$$\int_0^\infty \mathrm{d}E[E^{\frac{1}{2}}\mathrm{e}^{-E/kT}]P(E)\sigma_{\text{nuc.}}(E). \qquad \ldots\ldots\ldots\ldots \quad (1)$$

* Research School of Physical Sciences, Australian National University, Canberra; on leave of absence from Cornell University, Ithaca, N.Y., U.S.A.

† See, for example, Bethe (1939), Gamow and Critchfield (1949), and Salpeter (1953).

The first term in this integrand is the Maxwell-Boltzmann distribution factor (probability of kinetic energy being E). The second factor P is the barrier penetration factor which depends very strongly on E and on Z_1Z_2. The last factor $\sigma_{\text{nuc.}}$ is a purely nuclear factor which depends on the details of the interaction after barrier penetration and usually (but not always) varies fairly slowly with E.

In the usual calculations of reaction rates the barrier penetration factor $P(E)$ in (1) is evaluated by assuming the electrostatic interaction energy between the two nuclei to be purely $Z_1Z_2e^2/r_{12}$, the Coulomb potential between two positive unscreened charges. But in stellar interiors the gas density ρ is high and the average distance a between a nucleus and neighbouring electrons and nuclei is small. Each nucleus, even though completely ionized, attracts neighbouring electrons and repels neighbouring nuclei and thus polarizes the surrounding gas somewhat. The nucleus is then completely screened by a spherically symmetric negative charge cloud. The radius R of this charge cloud is of the same order as the interparticle distance a or larger, depending on the ratio of Coulomb repulsion between neighbouring charges to the mean thermal energy. Hence, when two nuclei approach each other in a collision, each of them carries its screening charge cloud with it and this screening affects the interaction energy between the nuclei. We write the total interaction energy as

$$U_{\text{tot.}}(r_{12})=Z_1Z_2e^2/r_{12}+U(r_{12}). \qquad \dots\dots\dots\dots \quad (2)$$

The main aim of this paper is to discuss the screening term $U(r_{12})$ and its effect on the barrier penetration factor P.

II. Definition of Parameters

We shall now define a number of dimensionless parameters representing the ratios of various physical quantities.

(a) *Impact Parameter and Radius of Charge Cloud*

The rate of a thermonuclear reaction depends on (1), which involves an integral over E, the relative kinetic energy of the two colliding nuclei. For all reactions likely to occur in stellar interiors the integrand of (1) has a fairly sharp maximum at a particular energy $E_{\text{max.}}$. The exact value of $E_{\text{max.}}$ depends on a number of factors, including the temperature, Z_1 and Z_2, where the relevant energy levels of the resulting compound nucleus lie, etc. But in all cases of practical interest $E_{\text{max.}}$ is very large compared with the mean thermal energy kT, in most cases larger than $10kT$.

Let r_c be the classical turning point for energy $E_{\text{max.}}$ in the collision between two nuclei of charge Z_1 and Z_2, defined by

$$E_{\text{max.}}=\frac{Z_1Z_2e^2}{r_c}. \qquad \dots\dots\dots\dots\dots\dots \quad (3)$$

Let r_n be the nuclear radius, that is, the distance where the nuclear attractive forces overcome the Coulomb repulsion.

Let a be a distance defined by the relation

$$4\pi a^3 \rho N_0 = 1\ ;\quad a = \rho^{-1/3}(0\cdot 51 \times 10^{-8}\ \text{cm}), \qquad (4)$$

where ρ is the gas density in g/c.c. and N_0 is Avogadro's number. The distance a is a measure of the interparticle distance, an average mass of $\frac{1}{3}$ a.m.u. being contained inside a sphere of radius a. Let R be the radius of the charge cloud surrounding a nucleus, that is, a distance beyond which an appreciable fraction of the nuclear charge is screened by the polarization charge cloud. We shall show in later sections that R is larger than a.

Let us consider a case in which the classical impact parameter r_c is very small compared with the charge cloud radius R. The nuclear radius r_n is always very much smaller than r_c. The barrier penetration factor $P(E)$ essentially depends only on the expression

$$\left[E - U(r_{12}) - \frac{Z_1 Z_2 e^2}{r_{12}}\right] \qquad (5)$$

for values of r_{12} between r_n and r_c, hardly at all on the potential for distances larger than r_c. Now $U(r_{12})$ must be a function which is small for $r_{12} \gg R$ and which approaches a constant value U_0 as r_{12} becomes small compared with R. Further the value of U_0 will be of the order of magnitude of $Z_1 Z_2 e^2/R$. From this expression and (3) it follows that, for the case considered,

$$\frac{r_c}{R} \sim \frac{U_0}{E_{\text{max.}}} \ll 1. \qquad (6)$$

Both in (5) and in the treatment of the nuclear factor, $U(r_{12})$ is needed only for $r_{12} \leqslant r_c$. If the inequality (6) is satisfied, $U(r_{12})$ can then be replaced by the potential at the origin, U_0, which is independent of both E and r_{12}. Therefore the penetration factor P and the nuclear factor $\sigma_{\text{nuc.}}$ for energy E without screening are equal to the correct factors with screening for an energy $(E+U_0)$. The integral (1) is then replaced by

$$\int_0^\infty dE[(E+U_0)^{1/2} e^{-E/kT} e^{-U_0/kT}] P(E) \sigma_{\text{nuc.}}(E). \qquad (1a)$$

Since U_0 is much smaller than $E_{\text{max.}}$ (but not necessarily smaller than kT) the term $(E+U_0)^{\frac{1}{2}}$ can be approximated by $E^{\frac{1}{2}}$. The whole effect of screening in this case is then that the reaction rate with screening neglected has to be multiplied by the factor

$$e^{-U_0/kT}. \qquad (7)$$

Throughout this paper we shall consider only cases in which the inequality (6) holds. The problem is then reduced merely to evaluating U_0 for substitution into (7), without having to consider any details of the actual nuclear reactions involved. For most practical cases (6) is in fact satisfied. For the proton-proton reaction in the solar interior (the Sun's main source of energy), for instance, $r_{\text{nuc.}} \sim 2 \times 10^{-13}$ cm, $r_c \sim 2 \times 10^{-11}$ cm, $a \sim 10^{-9}$ cm and $R \sim 3 \times 10^{-9}$ cm. The effect of screening at extremely high densities, where (6) no longer holds, is discussed by Schatzman (1948).

(b) *Degree of Ionization*

Let the parameter I_z be the ratio of the ionization potential of a K-shell electron in a hydrogen-like atom of charge Z to the mean thermal energy. We have

$$I_z=\left(\frac{Ze^2}{2a_{0z}}\right)(kT)^{-1}=\frac{(h/a_{0z})^2}{8\pi^2m}(kT)^{-1}, \qquad \ldots\ldots\ldots\ldots \quad (8)$$

where a_{0z} is the Bohr radius for such an atom and m is the electron mass. Throughout this paper we shall consider only cases for which the parameter I_z for all relevant values of the atomic charge is very small compared with unity. Expressing the temperature T in units of 10^6 °K, we have

$$I_z=0\cdot 16\frac{Z^2}{T}\ll 1. \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

The problem is greatly simplified if the inequality (9) holds. All atoms are completely ionized and further we shall be able to treat the electrons by means of semi-classical approximations. In fact we shall find that our final results essentially do not depend on Planck's constant (except indirectly, the results depending slightly on the degree of degeneracy of the electrons).

For most cases of stellar interest the ionization parameter I_z is indeed much smaller than unity, with one important exception, the reactions of the carbon-nitrogen cycle in the Sun and other main sequence stars. For these reactions I_z is only slightly smaller than unity and the approximations of this paper are not very accurate. However, the effect of electron screening on these particular reactions has been discussed by Keller (1953) without assuming the inequality (9).

(c) *Strength of Screening Effect*

Let Z_1 be the larger of the two charges Z_1 and Z_2 of the two interacting nuclei and let z be the atomic charge of the main constituent of the gas (in most cases $z=Z_2$). The nature of the polarization charge cloud surrounding a nucleus Z_1 depends on whether the screening is " weak " or " strong ". By " weak " screening we mean that the Coulomb interaction energy between this nucleus and the nearest few electrons and nuclei of the gas is small compared with the thermal energy kT. In this case the average positions of surrounding electrons and nuclei are displaced only very slightly from each other. The polarization charge cloud will then have a large radius R, containing many electrons and nuclei, the small difference between total negative and positive charge being Z_1. This case will be discussed in Sections III and IV.

By " strong " screening we mean that the Coulomb interaction between the nucleus Z_1 and nearby nuclei z is large compared with kT. In this case the nucleus is surrounded in its immediate vicinity by electrons only, the nuclei z staying outside a sphere containing nearly Z_1 electrons which effectively screen the nucleus. This case is dealt with in Section V. The more difficult case of intermediate strength of screening is not treated exactly in this paper, but is discussed in Section VI.

(*d*) *Degree of Electron Degeneracy*

Let ξ be the average number of electrons per atomic mass unit in the stellar gas,

$$\xi=\sum_i \frac{x_i z_i}{A_i}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{10})$$

where x_i is the fractional abundance (by mass) of nuclei of charge z_i and mass number A_i. The Fermi energy of the electron gas is defined as the thermodynamic potential at zero temperature. It is

$$E_F=\frac{h^2}{8\pi^2 m}(3\pi^2 N_0\xi\rho)^{2/3}=\left(\frac{3\pi\xi}{4}\right)^{2/3}\frac{(h/a)^2}{8\pi^2 m}, \qquad \ldots\ldots \quad (\mathbf{11})$$

where a is the measure of interparticle distance defined in (**4**). Let D be the ratio of Fermi energy to the mean thermal energy kT. For all numerical work throughout this paper we shall express density in g/c.c. and temperature T in units of 10^6 °K. In these units

$$D=[26\cdot 0(\xi\rho)^{2/3}eV](kT)^{-1}=0\cdot 30(\xi\rho)^{2/3}T^{-1}. \qquad \ldots\ldots \quad (\mathbf{12})$$

The Fermi energy E_F is given by (**11**) only when the electrons are non-relativistic, that is, when E_F is much less than the electron restmass energy, which is the case for $\xi\rho \ll 10^6$. For $\xi\rho \gg 10^6$, equation (**12**) is replaced by

$$D=\frac{60(\xi\rho)^{1/3}}{T}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{12a})$$

In Section III we consider the special case of the electron gas being non-degenerate, $D \ll 1$. In Section IV the formulae are generalized for arbitrary values of D. All nuclei will be assumed to be completely non-degenerate, which is true for all but extremely high density ρ. The case of a degenerate nuclear gas for extremely large ρ is discussed by Schatzman (1948).

(*e*) *Ratio of Abundances and Charges*

The formulae derived in this paper for the two limits of very weak and very strong screening, hold quite generally for all values of Z_1 and Z_2 and any composition of the gas. But for many (although by no means all) cases of stellar interest, one of the two reacting nuclei Z_1 has a very low abundance x_1 and a large charge compared with that of the other reacting nucleus and with the average charge z of the gas nuclei.

For this special case,

$$Z_2,\ z \ll Z_1\,;\quad x_1 \ll 1, \qquad \ldots\ldots\ldots\ldots\ldots \quad (\mathbf{13})$$

the problem of electron screening is simplified and numerical solutions are also presented for intermediate strength of screening in Section VI. The two earlier papers by Schatzman (1948) and Keller (1953) are mainly concerned with cases for which (**13**) holds.

III. Weak Screening: Electrons Non-degenerate

We shall now consider the case of the screening being " weak ", in the sense discussed in Section II (*c*). For simplicity we also assume in this section that the electrons are non-degenerate and consider the main constituent of the gas to be atoms of atomic charge z and atomic weight A. Our aim is to calculate the interaction energy $U(r)$, defined in (**2**), between nuclei of charges Z_1 and Z_2 at a distance r.

We now assume that $U_{\text{tot.}}(r)$ is of form

$$U_{\text{tot.}}(r)=Z_1Z_2e^2\psi_{\text{tot.}}(r)\,;\quad \psi_{\text{tot.}}(r)=r^{-1}+\psi(r), \qquad \ldots\ldots \quad (\mathbf{14})$$

where $\psi(r)$, the part due to the screening cloud, is independent of both Z_1 and Z_2 (but does depend on z). We shall show later that this assumption is justified if the screening is weak. It should be remembered that $U(r)$ represents a statistical average of the interaction energy. Only the two nuclei Z_1 and Z_2 are at a fixed separation r, the position of all other nuclei being averaged over.

Consider the nucleus Z_1 fixed at the origin and let $V(r)$, $\bar{\rho}(r)$ be the electrostatic potential and electric charge density at a point r, averaged over all particles except the nucleus at the origin. If $U(r)$ is of the form of (**14**), depending on the charges only through the factor Z_1Z_2, then the interaction energy of an infinitesimal " test charge " δZ at the point r is $\delta Z Z_1 e^2\psi(r)$ and hence $V(r)$ is $Z_1e\psi(r)$. Since V and $\bar{\rho}$ are related by Poisson's equation, we have

$$\nabla^2[Z_1e\psi_{\text{tot.}}(r)]=-4\pi\bar{\rho}(r)-4\pi Z_1e\delta^{(3)}(r). \qquad \ldots\ldots\ldots \quad (\mathbf{15})$$

Another relation between ψ and $\bar{\rho}$ is furnished by statistical mechanics, namely, that the density of particles (nuclei z or electrons) in a region in which each particle has potential energy $U_e(r)$ is the field-free density times the Boltzmann factor $\exp[-U_e(r)/kT]$. Since the potential energy of electrons has opposite sign to that of nuclei, their electric charge densities no longer cancel each other exactly and we have

$$\bar{\rho}(r)=\left(\frac{\rho N_0 ze}{A}\right)\left\{\exp\left[\frac{-Z_1ze^2\psi_{\text{tot.}}(r)}{kT}\right]-\exp\left[\frac{+Z_1e^2\psi_{\text{tot.}}(r)}{kT}\right]\right\}. \quad \ldots (\mathbf{16})$$

Substituting the expression (**16**) on the right-hand side of equation (**15**) gives an inhomogeneous second order differential equation in $\psi(r)$, the Poisson-Boltzmann equation.

According to our assumption of weak screening, the exponents of the two exponential terms in (**16**) are small compared with unity for r of the order of magnitude of a or larger. We can then expand these two exponential factors. The first terms in the two factors cancel each other and we keep only the second term of each, linear in Z_1. Substituting this approximation into equation (**15**) we get the linear approximation to the Poisson-Boltzmann equation,

$$\nabla^2\psi(r)=4\pi\rho N_0\left(\frac{z^2+z}{A}\right)\frac{e^2}{kT}\left[\frac{1}{r}+\psi(r)\right]. \qquad \ldots\ldots\ldots \quad (\mathbf{17})$$

The boundary conditions for (**17**) are imposed by the physical condition that $\psi(r)$ represents the potential due to the screening charge distribution, which has a finite radius and a total charge equal and opposite to that of the nucleus at the origin. Hence $\psi(r)$ approaches $-r^{-1}$ as r approaches infinity and approaches a finite limit $\psi(0)$ as r approaches zero.

We define the "radius" of the charge cloud, R, by

$$R^2=\left(\frac{kT}{4\pi\rho N_0 e^2}\right)\left(\frac{A}{z^2+z}\right)=\left(\frac{A}{z^2+z}\right)\left(\frac{kT}{e^2/a}\right)a^2. \quad \ldots\ldots \quad \mathbf{(18)}$$

The solution of (**17**) is then

$$\psi(r)=r^{-1}(\mathrm{e}^{-r/R}-1)\;;\;\;\psi(0)=-R^{-1}. \quad \ldots\ldots\ldots\ldots \quad \mathbf{(19)}$$

Our approximate equation and its solution, (**17**) and (**19**) respectively, are mathematically equivalent to those derived by Debye and Hückel (1923) in their theory for dilute solutions of electrolytes. A number of assumptions and approximations are involved in our derivation of (**17**) above. We shall now discuss the validity of these approximations.*

(1) We have used a continuous (average) charge density $\bar{\rho}(r)$ and in its evaluation have used the statistical Boltzmann factor $\exp[-U(r)/kT]$ for particles at the point r. For this procedure to be strictly valid many nuclei and electrons should be contained in a volume small enough so that $\bar{\rho}(r)$ and $U(r)$ do not vary appreciably over this volume. Now the linear dimension of the charge distribution is of order R, the average distance between particles of order a. Hence our procedure is valid as long as $a \ll R$. For the case of weak screening which we are considering $z^2e^2/a \ll kT$. It then follows from (**18**) that R is indeed large compared with a.

(2) In our whole derivation we have treated the electrons classically and considered their charge density at a definite distance r from the origin. This assumes that we can specify the position of an electron to within a distance r without considering the corresponding Heisenberg uncertainty energy. The neglect of this uncertainty energy is justified as long as it is small compared with the mean thermal energy of the electrons. Hence our classical treatment is justified only for values of r larger than a critical distance $r_{\mathrm{min.}}$, which is roughly given by

$$h^2/8\pi^2 m r^2_{\mathrm{min.}} \sim kT. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \mathbf{(20)}$$

We are at present only considering cases where the ionization potential for an atom Z_1 and the Fermi energy of the electron gas are both small compared with kT. A comparison of (**8**) and (**11**) with (**20**) then shows that the limiting distance $r_{\mathrm{min.}}$ is much smaller than the Bohr orbit a_{0z} and also much smaller than a, which in turn is smaller than R.

(3) As was discussed above, we have expanded the two exponential factors in (**16**), keeping only the first two terms in the expansion. Since $\psi_{\mathrm{tot.}}$ is of order r^{-1} for small distances, this linear approximation is valid for r larger than r_e, where $(Z_1 z e^2/r_e) \sim kT$ for nuclei z. For the case of weak screening r_e is small

* See also the discussions by Fowler and Guggenheim (1949) and by Keller and Meyerott (1952).

compared with R, the distance from which most of the screening effects stem. For the electrons one finds, using (**8**) and (**20**), that r_e is not only smaller than R, but also smaller than $r_{\text{min.}}$.

(4) In deriving the Poisson-Boltzmann equation we have assumed that the interaction energy between two particles has the form of (**14**), with $\psi(r)$ independent of Z_1 and Z_2. The solution (**19**) for $\psi(r)$, obtained by means of the linear approximation, satisfies this requirement, R depending only on z, but not on Z_1 and Z_2.

Our final answer for the screening potential between nuclei Z_1 and Z_2 is then

$$-\frac{U_0}{kT}=\frac{(Z_1Z_2e^2)}{R}(kT)^{-1}=0\cdot188Z_1Z_2\zeta\rho^{\frac{1}{2}}T^{-3/2}, \quad \ldots\ldots \quad (\mathbf{21})$$

where

$$\zeta=(z^2+z)^{\frac{1}{2}}A^{-\frac{1}{2}}.$$

The conditions of validity for (**21**) are that (**9**) and (**12**) be satisfied and that the screening is weak. More exactly, the weak screening condition requires the interaction energy between nuclei Z_1 and z at separation R to be smaller than kT; that is, the expression (**21**), with Z_2 replaced by z, must be smaller than unity.

IV. Weak Screening: General Case

We first introduce a trivial generalization of the work of the preceding section. We consider the gas as made up of different nuclear species of charge z_i and mass number A_i, their fractional abundance (by mass) being x_i. The average number of electrons per a.m.u. is then the quantity ξ, defined in (**10**) (instead of the z/A of Section III). The first exponential factor in (**16**) is then replaced by a sum of exponentials, one for each nuclear species. Expanding these exponentials as before, the first terms again cancel the first term from the electron factor. The second terms are quadratic in z_i and finally in (**17**) the expression z^2/A is simply replaced by $\sum_i x_i z_i^2/A_i$.

The effect of electron degeneracy is slightly more complicated. We consider now the case of arbitrary degree of degeneracy, but weak screening. It should be noted that it is still possible to have a degenerate electron gas but weak screening if, and only if, the inequality (**9**) is satisfied. We again want to consider the electron charge density $\bar{\rho}_e(r)$ at a point r and to neglect the corresponding uncertainty energy which is roughly $(h/2\pi r)^2/2m$. This neglect is justified if the uncertainty energy is much smaller than the average electron kinetic energy, which is of order $E_F\sim(h/2\pi a)^2/2m$, if the electrons are degenerate. This is the case if r is larger than a. We again will be most interested in distances of the order of R, again large compared with a, which justifies our procedure.

Let $f(\eta)$ be the Fermi-Dirac function,

$$f(\eta)=\int_0^\infty \mathrm{d}x\; x^{\frac{1}{2}}[\mathrm{e}^{(x-\eta)}+1]^{-1}, \quad \ldots\ldots\ldots\ldots\ldots \quad (\mathbf{22})$$

and D be the ratio of the Fermi energy E_F to kT. For the field-free case the thermodynamic potential $\mu=\eta kT$ of the electron gas* is determined by the equation

$$f(\eta)\doteq\tfrac{2}{3}D^{3/2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (23)$$

If electrons near the point r experience an interaction energy $U_e(r)$, the electron density will no longer be uniform throughout space but will adjust itself such that the total thermodynamic potential is uniform. The ratio of the actual density to the field-free density is then $f(\eta-U_e(r)/kT)/f(\eta)$, where η is still given by (23). If the screening is weak and if $a<r\sim R$, then $U_e(r)\ll kT$. We can then use a Taylor expansion for the expression $f(\eta-U_e/kT)$. Retaining only the first two terms in this expansion, the ratio of actual density to field-free density becomes

$$1-\left[\frac{U_e(r)}{kT}\right]\left[\frac{f'(\eta)}{f(\eta)}\right], \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (24)$$

where f' is the first derivative of $f(\eta)$. This expression is our generalization of the linear approximation $(1-U_e/kT)$ to the Boltzmann factor, which we used in Section III.

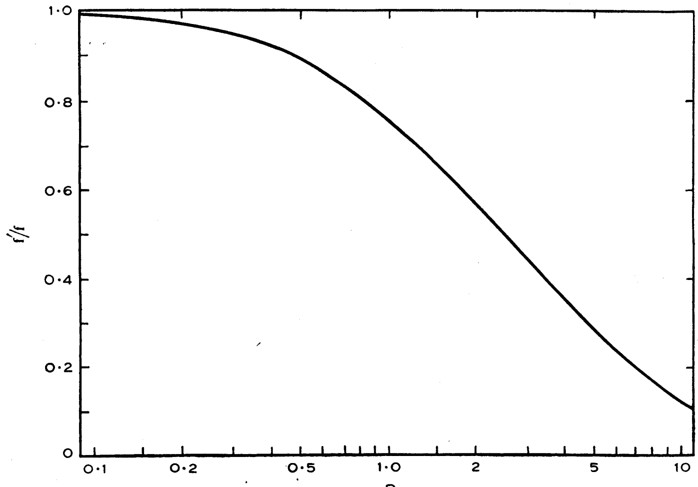

Fig. 1.—The quantity $f'(\eta)/f(\eta)$, to be substituted into equation (24), plotted against the degeneracy parameter D.

The factor f'/f takes on simple forms in two limiting cases. For extreme non-degeneracy, $D\ll 1$ and η is negative and very large. In this case f'/f becomes equal to unity and the expression (24) reduces to the expression of Section III. For extreme degeneracy, $D\gg 1$, η approaches D and f'/f approaches $(3/2D)$. In this case the polarization of the electron gas is negligible compared with that of the nuclear gas. For intermediate values of D, η and then f'/f were obtained from (22) and (23), using the numerical tabulation of the Fermi-Dirac function given by McDougall and Stoner (1938). The results of these numerical calculations are given in Figure 1, the quantity $f'(\eta)/f(\eta)$ being plotted against D.

* See, for instance, Mayer and Mayer (1940).

Our final formula for U_0/kT is then still (**21**), but with the quantity ζ now having the general form

$$\zeta = \left\{ \sum_i x_i \frac{z_i^2}{A_i} + \left(\frac{f'}{f}\right) \sum_i x_i \frac{z_i}{A_i} \right\}^{\frac{1}{2}}. \qquad \ldots\ldots\ldots\ldots \quad (\mathbf{25})$$

V. Strong Screening

We shall now investigate the screening charge cloud surrounding a nucleus Z_1 imbedded in an ionized gas in the limit of strong screening. We shall see later that the radius R of the charge cloud in this case is approximately $(3Z_1/\xi)^{1/3}a$. The condition for strong screening is then that the Coulomb repulsion between charges Z_1 and z (average charge of gas nuclei) at distance R be large compared with kT. For the sake of simplicity we restrict ourselves slightly more to the case where this Coulomb energy is also considerably larger than the ionization potential for charge Z_1. We shall show that the electron density is then essentially uniform, which simplifies the problem greatly.

We thus assume the following conditions to apply :

$$2zI_{z1} \ll \frac{Z_1 z e^2}{a}(kT)^{-1} \gg \left(\frac{3Z_1}{\xi}\right)^{1/3}, \qquad \ldots\ldots\ldots\ldots \quad (\mathbf{26})$$

where I_z and ξ are defined in (**8**) and (**10**) respectively. From the first part of this inequality and (**8**) it follows that $a \ll a_{0z}$ and that

$$\frac{Z_1 e^2}{a} \ll \left(\frac{h}{2\pi a}\right)^2 m^{-1} \sim E_F.$$

Hence the Coulomb energy of an electron at distance a or greater is small compared with the Fermi energy and at distances less than a small compared with the corresponding uncertainty energy. It then follows from the discussion of Section IV that the electron density at all relevant distances from the origin differs only slightly from the field-free density. We shall therefore take the electron density to be uniform.

The second inequality in (**26**) expresses the condition that the Coulomb repulsion experienced by nuclei z at distances between a and R is much larger than kT, unless a very large fraction of the charge Z_1 is screened by the electron cloud. For these distances then the density of gas nuclei is very small compared with the field-free density. We can therefore use the following approximate picture for the screening charge cloud. The nucleus Z_1 is surrounded by electrons of uniform density up to a sphere of radius R_1, where

$$R_1 = \left(\frac{3Z_1}{\xi}\right)^{1/3} a. \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{27})$$

No nuclei z are present within this sphere and the total charge of the electrons inside the sphere is $-Z_1$. Outside this sphere the nucleus Z_1 is effectively screened, the density of nuclei z also has its field-free value and there is no net charge density outside the sphere.

On this simple picture the total electrostatic energy of the system of nucleus Z_1 and its surrounding electron sphere is

$$W_1=-\left(\frac{3}{2}-\frac{3}{5}\right)\frac{Z_1^2e^2}{R_1}=-\frac{9\xi^{1/3}Z_1^{5/3}e^2}{3^{1/3}\times10a}, \quad \ldots\ldots\ldots\ldots \quad (28)$$

where the first term is the interaction energy between the nucleus and the electron cloud and the second term is the electrostatic self-energy of the electron cloud. When two nuclei, Z_1 and Z_2 respectively, approach each other in a collision, each of them carries its own electron sphere with it until the two spheres interpenetrate. When interpenetration occurs the gas nuclei rearrange themselves, the density of electrons still remaining uniform. For intermediate distances of approach the shape of the electron cloud is rather complicated. But for distances between the nuclei small compared with R_1 and R_2, the electron cloud is again spherical and of the same radius as the electron cloud for a nucleus of charge (Z_1+Z_2).

As was discussed in Section II, we are interested only in the interaction energy between the two nuclei and their charge clouds for separations less than a and therefore less than R_1 and R_2. On our present approximation the screening contribution to this energy is simply the interaction energy (**28**), for a nucleus of charge (Z_1+Z_2), minus the sum of the equivalent energies for the two nuclei Z_1 and Z_2. This screening potential is

$$\begin{aligned}-\frac{U_0}{kT}&=\frac{9e^2}{10akT}\left(\frac{\xi}{3}\right)^{1/3}[(Z_1+Z_2)^{5/3}-Z_1^{5/3}-Z_2^{5/3}]\\&=0\cdot205[(Z_1+Z_2)^{5/3}-Z_1^{5/3}-Z_2^{5/3}](\xi\rho)^{1/3}T^{-1}. \quad \ldots (29)\end{aligned}$$

Numerically, the condition (**26**) can be written as

$$Z_1\ll\rho^{1/3}; \quad 0\cdot23Z_1^{2/3}z(\xi\rho)^{1/3}T^{-1}\gg1. \quad \ldots\ldots \quad (26a)$$

Besides assuming this inequality we have made some further approximations:

(1) As in previous sections we have used continuous charge densities. Since the number of particles making up the screening charge cloud is now only of order Z_1, the fluctuations omitted in this treatment may not be negligible.

(2) In reality the electron density is not completely uniform, but slightly larger near the central nucleus. Similarly the density of gas nuclei does not jump discontinuously from zero inside the radius R to the field-free value just outside, but changes gradually over a certain " skin-depth ". But the more strongly the two inequalities in (**26**) are satisfied, the more nearly uniform is the electron density and the smaller is the ratio of skin-depth to radius R.

(3) In the strong screening case the screening cloud surrounding a charge (Z_1+Z_2) is no longer a linear superposition of the clouds for charges Z_1 and Z_2, the radii of the clouds being different. We have neglected all dynamic effects ensuing from the rearrangement of the gas nuclei, which may in fact not be negligible. (**29**) may therefore not be a very accurate approximation.

VI. Nuclei of Low Abundance and Large Charge

We have so far considered the general case of arbitrary abundance and charge of the nuclei Z_1. For this general case we have seen that the use of the Poisson-Boltzmann equation is not self-consistent for intermediate and strong screening, largely due to the non-linearity of its solution. For the case of strong screening we derived an approximation independent of the Poisson-Boltzmann equation, but its accuracy is limited by the non-linearity of the screening potential and by the small number of particles involved in the charge clouds.

As was pointed out by Keller and Meyerott (1952), these difficulties are removed if the abundance of the nuclear species of charge Z_1 is low and if Z_1 is much larger than Z_2 and z (equation (**13**)). Since $Z_1 \gg Z_2$ we need only consider the interaction of the nucleus Z_2 with the charge cloud surrounding the nucleus Z_1, without explicitly considering the charge cloud around Z_2. In this approximation the radius R_1 of the charge cloud does not alter during a nuclear collision, even if the screening is strong. We are then justified in neglecting the dynamic effects of the rearrangement of gas nuclei, mentioned in Section V. Since the abundance of the nuclei Z_1 is low, their average distance from each other is large compared with the radius of their charge cloud. We can then consider the charge cloud around a single such nucleus. Since $Z_1 \gg z$, we can also neglect the charge cloud around all nuclei z. In this case the condition of linearity (**14**) is no longer required for the validity of the Poisson-Boltzmann equation. Finally, the charge cloud contains at least Z_1 electrons, even for strong screening. Since Z_1 is large, our neglect of fluctuations from average distributions is justified.

Keller (1953) has described numerical methods for solving the Poisson-Boltzmann equation for arbitrary screening strength, if the inequality (**13**) holds. For the sake of simplicity we shall consider here only cases satisfying two further conditions: (1) the bulk of the gas consists of only one type of nuclear species of charge z and mass A, (2) the electrons are degenerate or $z \gg 1$. In either case the departure from uniform density is much smaller for the electrons than for the nuclei z. We thus take the electron density to have its field-free value.

Writing the electrostatic potential at distance r from the nucleus Z_1 as $Z_1 e/r + V(r)$, the Poisson-Boltzmann equation becomes

$$\nabla^2 V(r) = 4\pi z e \frac{\rho N_0}{A}\left\{1 - \exp\left[-\frac{Z_1 z e^2}{rkT} - \frac{ze}{kT}V(r)\right]\right\}. \quad \ldots . (30)$$

The exponential term in parentheses arises from the Boltzmann factor for the nuclei, the constant term from the assumed uniform electron density. Making the substitutions

$$Y(r) = \frac{ze}{kT}V(r),\quad R^2 = \frac{AkT}{4\pi\rho N_0 z^2 e^2},\quad x = \frac{r}{R},\quad F = \frac{Z_1 z e^2}{RkT},$$

equation (**30**) takes the form

$$\nabla_x^2 Y(x) = 1 - \exp\left[-\frac{F}{x} - Y(x)\right]. \quad \ldots\ldots\ldots\ldots\ldots (31)$$

The boundary conditions for (**31**) are that $Y(x)$ approaches a finite value at the origin and approaches $-F/x$ as x tends to infinity.

Following the discussion at the beginning of this section, we assume that the interaction energy U_0 for a collision between nuclei Z_1 and Z_2 (to be substituted into (**7**)) is simply $Z_2eV(0)$. Hence

$$\frac{U_0}{kT}=\frac{Z_2}{z}Y(0). \qquad (32)$$

$Y(0)$ is the value at the origin of $Y(x)$, obtained by solving (**31**) for the appropriate value of the parameter F. Numerically,

$$F=0\cdot 188\frac{Z_1z^2\rho^{1/2}}{A^{1/2}T^{3/2}}. \qquad (33)$$

Equation (**31**) has simple analytic solutions in two limiting cases. If $F\ll 1$, it is

$$Y(x)=\frac{F}{x}(\mathrm{e}^{-x}-1)\,;\quad Y(0)=-F. \qquad (34)$$

This solution is identical with our approximation for weak screening, derived in Sections III and IV. The factor ζ of (**21**) and (**25**) is here $z/A^{\frac{1}{2}}$, appropriate if the electrons are degenerate or if $z\gg 1$. If $F\gg 1$, the solution becomes

$$\left.\begin{aligned} Y(x)&=-\frac{3^{2/3}}{2}F^{2/3}+\frac{x^2}{6}, &&\text{for } x<(3F)^{1/3},\\ Y(x)&=-\frac{F}{x}, &&\text{for } x>(3F)^{1/3}.\end{aligned}\right\} \qquad (35)$$

This solution is identical with our approximation for strong screening, derived in Section V. The quantity U_0 is then given by

$$-\frac{U_0}{kT}=\frac{3^{2/3}}{2}\frac{Z_2}{z}F^{2/3}=0\cdot 34\frac{Z_1^{2/3}Z_2z\rho^{1/3}}{A^{1/3}T}. \qquad (36)$$

If we expand the expression in parentheses in (**29**) in powers of (Z_2/Z_1) and retain only the lowest order term, (**29**) reduces to (**36**).

Equation (**31**) was solved numerically for a few intermediate values of the parameter F. It was found convenient to transform this equation into one involving $y(x)=xY(x)$. It is

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=x\left\{1-\exp\left[-\frac{(F+y)}{x}\right]\right\}, \qquad (37)$$

with the boundary conditions $y(0)=0$ and y approaching $-F$ as x tends to infinity. The expression wanted is $Y(0)$, the first derivative of $y(x)$ at the origin. For each value of F separately, $Y(0)$ was found by trial and error, the equation being integrated from the origin outwards with different guesses for $Y(0)$. This procedure was continued until y for large x approached $-F$ for a particular value of $Y(0)$.

A graph of $Y(0)$ v. F is given in Figure 2. A few numerical results follow: for $F=1\cdot5$, the numerical result for $Y(0)$ is about $0\cdot98$, as compared with $1\cdot5$ on the weak screening approximation and $1\cdot36$ on the strong screening approximation; for $F=0\cdot25$, $Y(0)$ is $0\cdot21$, as compared with $0\cdot25$ on the weak screening approximation; for $F=15$, $Y(0)$ is $5\cdot7$ as compared with $6\cdot3$ on the strong screening approximation.

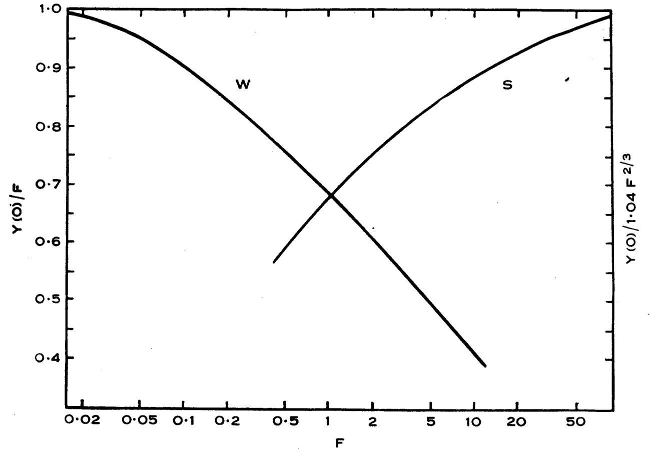

Fig. 2.—The ratio of the exact value of $Y(0)$ to its weak screening and strong screening approximations, as a function of F. The curve W gives $Y(0)/F$, the curve S gives $Y(0)/1\cdot04F^{2/3}$.

VII. Some Numerical Examples

We conclude this paper with a few specific examples of nuclear reactions, which are of importance in different types of stars. According to (**7**), the effect of electron screening consists solely in multiplying the reaction rate by a factor $\exp(-U_0/kT)$. We give below numerical values for the exponent $(-U_0/kT)$.

(*a*) *Proton-Proton Chain*

This reaction chain provides the main energy source for our Sun and for all cooler main sequence stars. The reaction determining the rate of energy production involves the collision between two protons. The rate of this reaction, without any screening correction, is the most accurately known one of all thermonuclear reactions of stellar interest. It is therefore important to calculate the effect of screening fairly accurately. Fortunately the screening for this reaction in the interior of main sequence stars is quite weak and the ionization potential of hydrogen very much less than kT. Equation (**21**) should therefore be a good approximation.

(i) Approximate central conditions in the Sun are $\rho=100$, $T=13$. From (**8**) the ionization parameter I_z is about $0\cdot01$, which certainly satisfies the inequality (**9**). The solar interior consists mainly of hydrogen, with the abundance of helium, x_{He}, of order $0\cdot2$ or less. From (**12**) the degeneracy parameter D is about $0\cdot5$. From Figure 1 the factor f'/f is about $0\cdot90$. Hence

ζ, defined by (**25**), is about $1 \cdot 38(1-0 \cdot 12 x_{\mathrm{He}})$. Finally, using this value of ζ and putting $Z_1=Z_2=1$ for the proton-proton reaction, (**21**) gives

$$-\frac{U_0}{kT}=0 \cdot 055(1-0 \cdot 12\, x_{\mathrm{He}}). \qquad (38)$$

(ii) For a typical red dwarf star (a main sequence star cooler than the Sun) approximate central conditions are $\rho=100$, $T=8$. In this case $D=0 \cdot 8$, $f'/f=0 \cdot 8$, and $\zeta=1 \cdot 34$. Equation (**21**) then gives a value of about $0 \cdot 11$ for $-U_0/kT$.

(*b*) *Carbon-Nitrogen Cycle*

This chain of reactions provides the energy source for the hotter main sequence stars. The reaction determining the rate of energy production is one involving the collision between a nitrogen nucleus ($Z_1=7$) and a proton.

(i) We first consider a case for which an accurate numerical computation has been carried out by Keller* (1953), namely for $\rho=122$ and $T=11 \cdot 6$. In this case the ionization parameter $I_z=0 \cdot 68$. Since this value is not very small compared with unity we should not expect the formulae of the present paper to be very accurate. With the gas mainly consisting of hydrogen, $D=0 \cdot 64$ and $f'/f=0 \cdot 85$. Using the weak screening formula (**21**) we get $-U_0/kT=0 \cdot 50$. Keller's accurate value for this quantity is $0 \cdot 43$.

(ii) The actual conditions in the deep interior of a star which derives its energy from the carbon cycle involve lower densities and higher temperature than in the above example. Hence I_z is somewhat smaller and the screening slightly weaker and (**21**) should be a reasonably good approximation. Conditions typical of the interior of Sirius, say, are $\rho=80$ and $T=20$. In this case I_z is about $0 \cdot 34$, D is about $0 \cdot 3$, and f'/f almost unity. (**21**) then gives $-U_0/kT=0 \cdot 19$.

We can conclude from these few examples that the effect of electron screening on the rate of energy production in all main sequence stars is fairly small, less than a factor of two in most cases.

(*c*) *Formation of* ^{8}*Be*

The more luminous main sequence stars convert hydrogen into helium at such a rapid rate that their central regions may be completely without hydrogen after a lifespan much less than the age of our Galaxy. The subsequent life history of such a star is not yet completely understood but it is likely that the core of such a star, consisting almost entirely of helium, will contract and both density and temperature will increase. When a temperature of 1 to 2×10^8 °K is reached, helium begins to be transformed into carbon and heavier nuclei. The first step in these processes is the formation of the short-lived ^{8}Be nucleus from the collision of two α-particles. The heavier nuclei are then built up by successive radiative captures of α-particles, starting from $^8\mathrm{Be}\,(\alpha,\gamma)^{12}\mathrm{C}$.

We give now values of the quantity U_0, the screening energy, for the collision of two helium nuclei in a gas consisting largely of ^{4}He. The order of magnitude

* See also Schatzman (1954).

of densities in stellar interiors at very high temperature is not yet known, so we consider a few different values.

(i) $\rho=10^4$, $T=150$: The ionization parameter I_z is negligibly small, $\xi=z/A$ is $\frac{1}{2}$, the degeneracy parameter $D=0\cdot58$, and f'/f is about $0\cdot87$. The parameter ζ is then about $1\cdot20$ and (**21**) gives a value of $0\cdot048$ for $-U_0/kT$.

(ii) $\rho=10^6$, $T=150$: The electrons are highly degenerate ($D>10$) and f'/f is practically zero, ζ practically unity. Equation (**21**) gives a value of $0\cdot40$ for $-U_0/kT$, but, since this number is not very small, our weak screening approximation is probably not very accurate.

(iii) $\rho=10^8$, $T=150$: The electrons are highly degenerate (relativistically) and their density is practically uniform. The screening is fairly (but not very) strong. Hence (**29**) should give an approximation which is fairly poor, but at least better than (**21**). The result is $-U_0/kT=1\cdot9$.

(*d*) *Formation of* ^{20}Ne

We take as our final example the collision between an oxygen nucleus and an α-particle. The reaction $^{16}O\ (\alpha,\gamma)^{20}Ne$ is one in the reaction chain beginning with the formation of ^8Be. We again consider a gas consisting mainly of helium at a temperature of $1\cdot5\times10^8$ °K.

(i) $\rho=10^4$, $T=150$: In this case the screening is still reasonably weak. The calculation proceeds as for the collision between two α-particles (Example (*a*) (i)) with Z_1 being 8 instead of 2. This gives a value of $0\cdot19$ for $-U_0/kT$.

(ii) $\rho=10^6$, $T=150$: The screening is neither weak nor strong. But the electrons are highly degenerate, the ^{16}O nuclei are rare and their charge at least fairly large compared with that of an α-particle. The conditions of Section VI are then satisfied. Equation (**33**) gives a value of $1\cdot64$ for the parameter F. Figure 2 then gives a value of $0\cdot65$ for $Y(0)/F$. Since $Z_2=z$ we find $-U_0/kT=Y(0)\approx1\cdot1$.

(iii) $\rho=10^8$, $T=150$: Again using (**33**) and Figure 2, we find $F=16\cdot4$ and $-U_0/kT=6\cdot1$.

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