

# THE ANGULAR DISTRIBUTION OF HIGH ENERGY ELECTRONS IN AIR SHOWERS

## I. LANDAU APPROXIMATION

By M. H. KALOS\* and J. M. BLATT†

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### Summary

The angular distribution of high energy ( $>5 \times 10^8$  eV) electrons in air showers is calculated on a track length basis, using approximation A of Rossi and Greisen (1941) (no ionization loss) and the Landau (1940) multiple scattering approximation. We start with a discussion of the approximations used and an estimate of their validity. The basic equations are written down; qualitative results and very rough solutions are developed and applied to the Furry model of a cascade. For the Furry cascade the qualitative arguments lead directly to an Ansatz which yields an exact solution. In the actual cascade the corresponding Ansatz does not yield an exact solution. We then perform an iteration, employing a general method of Friedman. The final (iterated) solution is compared with the exact (in the Landau approximation) solution by means of their moments, and appears to be within 10 per cent. of the correct solution for  $E\theta/E_s < 1$ . Our solution compares well with earlier work on this problem. Appendices contain a short derivation of the angular moments, a general inversion formula for going from the distribution-in-projected-angle to the distribution-in-angle-with-the-shower-axis, and a derivation of the Friedman variation principle in vector space terminology.

## I. INTRODUCTION

The interactions of extremely high energy cosmic rays can usually be observed only from the cascades they produce in the atmosphere (air showers). A theory of the cascade process including the angular and lateral distribution of the particles is necessary to interpret the observations and infer the nature of the primary events. For example, the knowledge of the expected lateral structure of the shower of electrons and photons produced by a single source will help determine whether an observed shower comes from one particle or from a number of lower energy sources. In this paper we discuss a rather limited aspect of the more general problem, namely, the angular distribution of monoenergetic electrons in a single-source shower.

Except for some work by Molière (1946) and by Belenky (1944), calculations on the angular distribution (Roberg and Nordheim 1949; Eyges and Fernbach 1951; Green and Messel 1952) have centred on the moments of this distribution function. This approach is simpler than a direct attack on the distribution function. Many functions can be determined over much of their range from

\* University of Illinois, Urbana, Ill., U.S.A.; present address: Laboratory for Nuclear Studies, Cornell University, Ithaca, N.Y., U.S.A.

† School of Physics, University of Sydney.

the sequence of moments. Unfortunately, it is quite difficult to find the behaviour near zero in this way. Although there exist some mathematical theorems about the uniqueness of the moment inversion, these theorems are of little practical help.

The work described here was undertaken in an attempt to calculate the so-called "track length" values of the angular distribution with particular emphasis on the behaviour of the function for small argument. The track length distribution is obtained by integrating over the whole depth of the shower; it naturally contains less information than the original function with depth dependence included. It is considerably easier to work with, however, and can be used to analyse experiments in which all electrons above a given energy are detected. In such a situation, the distribution-in-angle at any one depth  $t$  is determined primarily from that part of the energy range which gives the largest number of particles at that depth. The distribution of these electrons is very close to the track length distribution for that same energy.\*

We shall use the set of approximations which Rossi and Greisen (1941) call "Approximation A"; that is, ionization loss, Compton collisions, and knock-on collisions are ignored, and the cross sections for pair production and bremsstrahlung are replaced by their extreme relativistic values. This set of approximations is reasonable for electrons of energy much higher than the critical energy (much higher than 100 MeV in air). Furthermore, to simplify the mathematics, we shall use, instead of the actual extreme relativistic approximations for the cross sections, some mathematically more tractable expressions, namely,

$$\sigma_{\text{pair}}(E, E') = 7/(9E), \quad \sigma_{\text{brems}}(E, E') = (E - E')^{-1}. \quad \dots \quad (1.1)$$

Here  $E$  is the energy of the initial particle or photon and  $E'$  is the energy of a secondary electron. In the case of bremsstrahlung, the secondary photon has energy  $(E - E')$ , in the case of pair creation the second member of the pair has this energy. The cross sections (1.1) are called "super-simplified cross sections" by Friedman (1949); the cross sections differ from the correct ones by about 30 per cent. at most (see Rossi and Greisen 1941, Figs. 8 and 10), but of course the error in the resulting angular distribution function is smaller than this. The choice of supersimplified rather than conventional cross sections leads to errors of less than 10 per cent. in the region  $E\theta/E_0 < 1$ .

As far as the angular spread of the shower is concerned, we shall ignore all spreading processes other than Coulomb scattering of electrons by the nuclei of the air. The most important processes neglected are the angular deviations in bremsstrahlung and pair creation, and the angular deviation due to the action

\* The use of track length quantities has been questioned by Green and Messel (1952). In our opinion the approximation suggested above, when applied to air showers, is at least as good as some of the basic approximations (neglect of the angular spread in bremsstrahlung and pair production, neglect of the effect of the magnetic field of the Earth) made in all calculations so far, including the calculations of Green and Messel. Furthermore, it is rather easy to find approximation methods for taking into account the deviations from the track length distribution once the track length distribution itself is known.

of the magnetic field of the Earth.\* All these processes have the same dependence on energy as the Coulomb scattering ( $E^{-1}$ ). Their importance can be estimated by considering the mean square angles of deviation per radiation length :

$$\text{Multiple Coulomb scattering : } \langle \theta^2 \rangle_{\text{ave}} = (E_s/E)^2 \quad E_s = 21 \text{ MeV}, \quad \dots (1.2)$$

$$\text{Bremsstrahlung : } \langle \theta^2 \rangle_{\text{ave}} \sim [(mc^2/E) \ln (E/mc^2)]^2, \quad \dots (1.3)$$

$$\text{Pair production : } \langle \theta^2 \rangle_{\text{ave}} \sim [(mc^2/h\nu) \ln (h\nu/mc^2)]^2. \quad \dots (1.4)$$

The effect of the magnetic field of the Earth depends upon the direction of the shower axis and upon the air pressure, being nearly zero if the shower axis is parallel to the magnetic field lines. However, for showers with directions appreciably different from that of the magnetic field of the Earth,  $\langle \theta^2 \rangle_{\text{ave}}$  is of the same general magnitude as the  $\langle \theta^2 \rangle_{\text{ave}}$  for multiple Coulomb scattering, especially so at mountain altitudes where the lower air pressure implies a longer actual path for a given path length in radiation lengths. The factor  $\ln(E/mc^2)$  in the bremsstrahlung and pair production  $\langle \theta^2 \rangle_{\text{ave}}$  becomes larger as the energies increase. However, in practice we are not interested in energies larger than  $10^{10}$  eV, and for such energies  $\langle \theta^2 \rangle_{\text{ave}}$  for the radiative processes is smaller than  $\langle \theta^2 \rangle_{\text{ave}}$  for multiple Coulomb scattering by a factor larger than 16. Thus the neglect of the angular deviations in radiative processes should not lead to errors larger than 10 per cent. in the final angular distribution function. The neglect of the angular deviations due to the magnetic field of the Earth leads to errors larger than this unless the axis of the shower is substantially parallel to the direction of the magnetic field of the Earth. None of the work published so far has taken any of these processes into account.

Besides ignoring these other spreading processes, we shall use the Landau (1940) multiple scattering approximation for the Coulomb scattering of electrons. This approximation has been criticized lately by Green and Messel (1952), who point out that it leads to serious errors in the higher moments of the angular distribution function. Their calculations show that the Landau approximation leads to serious errors in the distribution function for values of  $u = E\theta/E_s > 1$ . Whether or not the Landau approximation also leads to serious errors in the distribution function for *small* values of  $u$  cannot be decided by a comparison of moments.

However, the qualitative arguments given by Rossi (1952), as well as the exact calculations of Snyder and Scott (1949) on the simpler diffusion problem without cascade multiplication, indicate strongly that the Landau approximation is in fact insufficient even for small values of  $E\theta/E_s$ . The angular distribution found by Snyder and Scott behaves roughly like a Gaussian curve for small angles, but the width of the Gaussian is not given correctly by the Landau approximation unless this width is larger than the maximum angle for single Coulomb scattering (Rossi's criterion of validity, eqn. (14), p. 72). For a

\* The importance of this latter effect has been pointed out recently by Cocconi (1954).

typical path length of about  $\frac{1}{2}$  radiation unit per electron in the shower\* this criterion is badly violated. We therefore feel that the Landau approximation cannot be trusted even for small distances and angles.

There is some point in making calculations with the Landau approximation, nevertheless. First, all of the previous work not concerned only with moments has used this approximation, and hence a comparison is in order. Second, once it is realized that the main error of the Landau approximation for small angles arises from the error in the width of the Gaussian (for no cascade), a simple method of correction presents itself fairly obviously. It consists in altering the value of the energy  $E_s$  in such a way that the width of the Landau Gaussian curve agrees approximately with the width of the Scott and Snyder Gaussian for depths of the order of 0.1–0.5 radiation lengths. The efficacy of this method is now being tested, and will be the subject of a later publication. It should be emphasized that this method of correction works only for values of  $E\theta/E_s \ll 1$ . For larger angles, the moments computed by Green and Messel (1952) are sufficient to define the behaviour of the distribution function to sufficient accuracy.

The general approach of this paper is adapted to the study of air showers, and several of the approximations made here would not be applicable to other types of showers. We have tried to obtain a distribution function which is within 10 per cent. of the correct value for all angles which are experimentally measurable, and for all energies larger than  $5 \times 10^8$  eV (at lower energies approximation A can not be trusted to this accuracy).

Section II of this paper gives the mathematical equations to be solved. In Section III we give some qualitative arguments; these arguments lead to a very rough guess about the behaviour of the distribution function. In Section IV these arguments are applied to the Furry model of a shower, and for this model it is shown that they lead naturally to an exact solution. Since the Furry cascade does not differ from the true cascade in features which are likely to have a strong influence on the angular distribution function, we can have some confidence that the qualitative arguments of Section III apply also to the actual shower. One result of Section IV is that the Furry cascade does not lead to any singularity in the distribution function  $f(u)$  at  $u=0$ . It is extremely likely that there is also no singularity of  $f(u)$  for the actual cascade. In Section V we apply the same qualitative arguments to obtain an approximate distribution function for the actual shower. Unlike the Furry cascade, this is not an exact solution. We obtain our final result in Section VI by using the approximate solution of Section V as a trial function in a variationally correct iteration procedure. The accuracy of the iterated function is tested in two ways: by comparison with the exact (in Landau approximation) moments, and by comparison with the results of Belenky (1944) and of Molière (1946). Both comparisons show that the iterated function is within 10 per cent. of the exact

\* This is the mean free path against bremsstrahlung collisions in which the electron loses more than 10 per cent. of its energy. The mean free path for photons is not relevant here because photons are not scattered at all under our approximations.

function (in the Landau approximation) for values of  $u < 1$ . Appendix I contains calculations of the moments of the distribution function; Appendix II gives an explicit relationship between the distribution-in-projected-angle  $\theta$  (as found in this paper) and the distribution-in-angle  $\Theta$  between the direction of motion of the shower particle and the shower axis; Appendix III contains a general statement of the variational method used.

We realize that experimentalists are not primarily interested in the track length angular distribution of monoenergetic electrons, but would like to know the theoretically predicted lateral distribution at a given depth  $t$  from the origin of the shower, integrated over all electron energies. Work on that problem is in progress.

II. DIFFUSION EQUATIONS FOR THE ANGULAR DISTRIBUTION

Let  $\varphi_\pi(E_0, E, t, \theta) dE d\theta$  be the average number of electrons in the energy interval  $E, E + dE$  at depth  $t$  making a projected angle with the shower axis (direction of initial electron) in the range  $\theta, \theta + d\theta$ ; the initial electron had energy  $E = E_0$  at the point  $t = 0$ . Let  $\varphi_\gamma(E_0, E, t, \theta)$  be the corresponding average distribution function for the photons. We introduce integral operators  $A, B, C$  as follows: let  $f(E)$  be an arbitrary function of  $E$ , other variables being suppressed for the moment; then

$$Af = \lim_{\epsilon \rightarrow 0} \left[ - \int_{E+\epsilon}^{\infty} f(E') \sigma_{\text{brems}}(E', E) dE' + f(E) \int_0^{E-\epsilon} \sigma_{\text{brems}}(E, E') dE' \right], \quad \dots (2.1)$$

$$Bf = 2 \int_E^{\infty} f(E') \sigma_{\text{pair}}(E', E) dE', \quad \dots (2.2)$$

$$Cf = \int_E^{\infty} f(E') \sigma_{\text{brems}}(E', E - E) dE'. \quad \dots (2.3)$$

The limiting process is necessary because the bremsstrahlung cross section contains the infra-red catastrophe. The signs are chosen in such a way that the quantities  $A(s)$ ,  $B(s)$ , and  $C(s)$  in Rossi and Greisen (1941) are the Mellin transform images of the operators  $A$ ,  $B$ , and  $C$  respectively. We also use the notation  $\sigma_0$  for the total pair production cross section; in approximation (1.1)  $\sigma_0 = 7/9$ . Finally  $E_s = 21$  MeV is the characteristic energy for multiple Coulomb scattering. In what follows all angles are projected angles, that is, the whole shower is projected onto a plane containing the shower axis. This corresponds experimentally to taking measurements of angles in a cloud chamber without stereoscopic photographs. Furthermore, all angles are assumed to be much less than 1 radian.\*

\* This approximation is excellent for high energy electrons ( $E \gg E_s$ ) and it has nothing to do with the question whether the moments of the angular distribution function determine the function for "large" or "small" angles. The characteristic variable in the angular distribution function is not  $\theta$  itself, but the combination  $E\theta/E_s$ . For energies  $E \gg E_s$  it is quite possible that  $E\theta/E_s > 1$  and yet  $\theta \ll 1$ . Approximation A, which neglects ionization loss, is valid only when  $E$  exceeds the critical energy, which is about 4 times  $E_s$  in air. Hence the condition  $E \gg E_s$  is automatically satisfied under the assumptions of our calculation.

In this paper we shall work with the Landau approximation throughout ; the limitations of this approximation will be discussed in a later publication. The Landau (1940) diffusion equations are :

$$\partial\varphi_{\pi}/\partial t = -A\varphi_{\pi} + B\varphi_{\gamma} + (E_s/2E)^2\partial^2\varphi_{\pi}/\partial\theta^2 + \delta(E_0 - E)\delta(t)\delta(\theta), \quad \dots (2.4)$$

$$\partial\varphi_{\gamma}/\partial t = C\varphi_{\pi} - \sigma_0\varphi_{\gamma}. \quad \dots (2.5)$$

The delta function in (2.4) represents a single incident electron at  $t=0$  with energy  $E=E_0$  and direction  $\theta=0$ . Note that these equations are invariant under translation in  $t$  even in the case of an inhomogeneous medium such as the air. The changing density of the air has no effect on the angular distribution, although it does affect the lateral distribution.\*

We now eliminate the photon distribution  $\varphi_{\gamma}$  from (2.4) by using (2.5). We get from (2.5) :

$$\varphi_{\gamma} = (\partial/\partial t + \sigma_0)^{-1}C\varphi_{\pi}, \quad \dots (2.6)$$

$$(\text{d}/\text{d}t + \sigma_0)^{-1}f(t) = \int_0^t f(t') \exp[\sigma_0(t' - t)] \text{d}t'. \quad \dots (2.7)$$

We then substitute (2.6) into (2.4) and define the operator  $L$  by

$$L = \partial/\partial t + A - B(\partial/\partial t + \sigma_0)^{-1}C \quad \dots (2.8)$$

to get an equation containing  $\varphi_{\pi}$  only

$$L\varphi_{\pi} - (E_s/2E)^2\partial^2\varphi_{\pi}/\partial\theta^2 = \delta(E_0 - E)\delta(t)\delta(\theta). \quad \dots (2.9)$$

This is the fundamental diffusion equation for the Landau theory. From this equation others can be derived. A particularly useful integral relation is suggested by a moment recursion relation found by Nordheim (1952). Let us use the Rossi and Greisen notation  $\pi(E_0, E, t)$  for the average number of electrons irrespective of angle, that is,

$$\pi(E_0, E, t) = \int_{-\infty}^{\infty} \varphi_{\pi}(E_0, E, t, \theta) \text{d}\theta. \quad \dots (2.10)$$

The integration extends over an infinite range as a result of the assumption of small angles throughout the calculation. Let us denote by  $\pi_n(E_0, E, t)$  the  $n$ th moment of the function  $\varphi_{\pi}$ , that is,

$$\pi_n(E_0, E, t) = \int_{-\infty}^{\infty} \theta^n \varphi_{\pi}(E_0, E, t, \theta) \text{d}\theta. \quad \dots (2.11)$$

Clearly  $\pi_0 = \pi$ . The Nordheim moment recursion relation then reads

$$\pi_n(E_0, E, t) = n(n-1) \int_0^t \text{d}t' \int_E^{E_0} \text{d}E' (E_s/2E')^2 \pi(E_0, E', t') \times \pi_{n-2}(E', E, t-t'). \quad \dots (2.12)$$

\* The magnitude of the effect on the lateral distribution depends upon how one interprets the calculations for a homogeneous atmosphere when applying them to the actual inhomogeneous atmosphere. The usual method consists in measuring all distances, radial as well as vertical, in radiation units. The effect is then of the order of 10-20 per cent. at mountain altitudes. By measuring vertical distances in radiation units but radial distances in centimetres, the effect of the changing density of the atmosphere is greatly magnified. As Messel and Green (1952) point out, this accounts for the 5000 per cent. correction found by them.

Consider the integral equation

$$\begin{aligned} \varphi_{\pi}(E_0, E, t, \theta) = & \frac{\partial^2}{\partial \theta^2} \int_0^t dt' \int_E^{E_0} dE' (E_s/2E')^2 \pi(E_0, E', t') \varphi_{\pi}(E', E, t-t', \theta) \\ & + \delta(\theta) \pi(E_0, E, t). \end{aligned} \quad (2.13)$$

By multiplying both sides of (2.13) by  $\theta^n$  and integrating over  $\theta$ , it is seen that (2.13) is consistent with (2.12). Equation (2.13) can also be proved directly from the Landau equation (2.9). The proof is rather lengthy and the reader is referred to a thesis by one of us (Kalos 1952).

We shall be working mostly with track length quantities. We define the track length  $v(E_0, E, \theta)$  by

$$v(E_0, E, \theta) = \int_0^{\infty} \varphi_{\pi}(E_0, E, t, \theta) dt. \quad (2.14)$$

The longitudinal track length is

$$z_{\pi}(E_0, E) = \int_{-\infty}^{\infty} v(E_0, E, \theta) d\theta = \int_0^{\infty} \pi(E_0, E, t) dt. \quad (2.15)$$

To the extent that the main contribution to the integral over the depth  $t$  comes from a narrow region of  $t$ , namely, that region over which electrons of energy  $E$  have their maximum number, the track length  $v$  is close to the value of  $\varphi_{\pi}(E_0, E, t, \theta)$  at  $t=t_{\max}$ . The advantage of the track length is of course that it is much easier to work with.

By integrating over  $t$  from 0 to infinity, we get from (2.8)

$$Fv - (E_s/2E)^2 \partial^2 v / \partial \theta^2 = \delta(\theta) \delta(E_0 - E), \quad (2.16)$$

where the operator  $F$  is the track length analogue of the operator  $L$ , (2.8), and is defined by

$$F = A - BC/\sigma_0. \quad (2.17)$$

A similar procedure applied to the integral equation (2.13) gives

$$v(E_0, E, \theta) = \frac{\partial^2}{\partial \theta^2} \int_E^{E_0} dE' \left( \frac{E_s}{2E'} \right)^2 z_{\pi}(E_0, E') v(E', E, \theta) + \delta(\theta) z_{\pi}(E_0, E). \quad (2.18)$$

### III. QUALITATIVE ARGUMENTS

Rather than plunge directly into the mathematics of finding solutions to these equations, let us first give some qualitative arguments to establish the general, rough features of the distribution functions we are looking for. We shall be particularly interested in the behaviour of  $v(E_0, E, \theta)$  for two extreme cases:  $E \ll E_0$ , and  $E$  very close to  $E_0$ .

It is clear physically that the nature and energy of the initial particle which started the cascade cannot have a great influence on the angular structure of the shower at energies  $E \ll E_0$ , even though the number of particles of this energy depends very much on  $E_0$ . The high energy particles all stay close to the core of the shower, and the angular (as well as lateral) deviations observed for particles of energy  $E$  arise during the last few radiation lengths, that is, during the period

when the particle or its ancestors had energies not tremendously much larger than  $E$ . If  $E \ll E_0$ , this means that the observed deflexions arise from particles all of which had energies  $E' \ll E_0$ , and hence the particular value of  $E_0$  cannot matter for the angular structure of the shower. Once this point is established, dimensional arguments applied to equation (2.16) show immediately that the angle  $\theta$  can enter only in the combination

$$u = E\theta/E_s. \dots\dots\dots (3.1)$$

The angular structure of the shower is defined essentially by the ratio  $v/z_\pi$ . Guided by the arguments above, we define the angular structure function  $f(E_0, E, \theta)$  of the shower by

$$f(E_0, E, \theta) = (E_s/E)(v/z_\pi). \dots\dots\dots (3.2)$$

We then expect,

$$f = f(u) \text{ only, } \int_{-\infty}^{\infty} f(u) du = 1, \text{ for } E \ll E_0. \dots (3.3)$$

This structure function  $f(u)$  is the information of primary interest. Its moments, in the Landau approximation, are given in Appendix I as well as in the paper by Eyges and Fernbach (1951). Unfortunately, it turns out that an understanding of  $v(E_0, E, \theta)$  for values of  $E$  close to  $E_0$  is necessary in order to get useful results for  $f(u)$  when  $E$  is much less than  $E_0$ .

In order to gain such understanding, let us for the moment ignore the infra-red divergence in the bremsstrahlung cross section; we shall assume that the total bremsstrahlung cross section is finite. It is then possible to expand according to successive collisions,\* that is, we shall group the particles according to the number of radiative collisions which have occurred in their ancestry. The "zero group" or "end group" consists of the initial particle. This initial particle is multiply scattered, and its probability of surviving against radiative (bremsstrahlung) collisions is  $\exp(-\sigma_1 t)$  where  $t$  is the thickness of matter traversed, and  $\sigma_1$  is the (assumed finite) total bremsstrahlung cross section. The angular distribution for multiple scattering without cascade multiplication, in the Landau approximation, was given by Fermi (as quoted in Rossi and Greisen (1941)). It is

$$G(E, t, \theta) = (E/E_s)(\pi t)^{-\frac{1}{2}} \exp[-t^{-1}(E\theta/E_s)^2]. \dots\dots (3.4)$$

Thus the distribution function for the end group of particles, that is, for the initial particle, is

$$\varphi_\pi^{(0)}(E_0, E, t, \theta) = G(E_0, t, \theta) e^{-\sigma_1 t} \delta(E_0 - E), \dots\dots\dots (3.5)$$

and the corresponding track length distribution function is

$$v^{(0)}(E_0, E, \theta) = \delta(E_0 - E) \int_0^\infty G(E_0, t, \theta) \exp(-\sigma_1 t) dt. \dots (3.6)$$

\* A much more complicated expansion in successive collisions is possible also with an infinite bremsstrahlung total cross section; this was given by H. J. Bhabha and W. Heitler in their fundamental paper on shower theory. In the Bhabha-Heitler expansion, the particles are grouped into generations according to the number of photons in their ancestry. The number of bremsstrahlung events is not counted, indeed that number is infinite.

The integration in (3.6) can be performed analytically by using an integral representation for Bessel functions of imaginary arguments given by Watson (1948, p. 183). The Bessel function involved is  $K_{-\frac{1}{2}}$  which is expressible in terms of elementary functions. The result is

$$v^{(0)}(E_0, E, \theta) = (v/\sigma_1) \exp(-2v|\theta|) \delta(E_0 - E);$$

$$v = (E_0/E_s)(\sigma_1)^{\frac{1}{2}}. \dots\dots\dots (3.7)$$

It is worth remarking that the discontinuity in the derivative of this function at  $\theta=0$  corresponds to a (weak) infinity of the corresponding distribution in the angle  $\Theta$  between the actual (not projected) motion of the particles and the shower axis. Indeed, the corresponding integration leads to the Bessel function  $K_0$  which has a logarithmic infinity at the origin. The same result follows also from the inversion formula discussed in Appendix II, which allows one to go directly from the distribution in the projected angle to the distribution in the actual angle  $\Theta = (\theta_x^2 + \theta_y^2)^{\frac{1}{2}}$ .

The next generation has an angular distribution which is harder to compute analytically, and we shall not do so here. If for some reason the behaviour of  $v$  for  $E=E_0$  (or for  $E$  very close to  $E_0$  in the true cascade) is sufficient, expression (3.7) can be used.

Unfortunately these considerations cannot be applied directly to the actual shower, since the bremsstrahlung cross section  $\sigma_1$  diverges. This has a considerable influence on the detailed form of the distribution function for  $E$  near  $E_0$ . Let us consider as the end group of the actual shower the initial electron which may have undergone any number of bremsstrahlung collisions, but has no photon in its ancestry. Then a good first approximation for the longitudinal distribution function of this end group is\*

$$\pi^{(0)}(E_0, E, t) = \frac{[\ln(E_0/E)]^{(t/\ln 2) - 1}}{E_0[(t/\ln 2) - 1]!} \dots\dots\dots (3.8)$$

This should be compared with the expression

$$\pi^{(0)}(E_0, E, t) = \delta(E_0 - E) \exp(-\sigma_1 t) \dots\dots\dots (3.9)$$

for the end group of a hypothetical shower with finite total bremsstrahlung cross section  $\sigma_1$ . Of course, the comparison is not really warranted because the expression "end group" denotes two different things in these two cases. But for our purposes, (3.8) is adequate for energies  $E$  close to  $E_0$  in the actual shower, (3.9) is adequate in the hypothetical shower, and in this sense the two are comparable. We see that, at least for reasonably small  $t$ , (3.8) is very sharply peaked near  $E=E_0$ , but that peak is not nearly as strong as the delta function peak in (3.9).

In the angular problem, a reasonably good approximation for the end group can be obtained by multiplying (3.8) by the Fermi function  $G(E_0, t, \theta)$ . Unfortunately, the track length  $v^{(0)}$  which then replaces (3.6) cannot be found in simple closed form. Since we need a simple closed form for our later work, we are forced to use a very much rougher approximation.

\* See Rossi (1952, p. 244). This formula was first found by Bethe and Heitler and is sometimes called the range straggling formula for bremsstrahlung.

We shall employ the expression (3.9) for the end group of the actual shower, with the constant  $\sigma_1$  adjusted in some "best" way. The adjustment follows the suggestion of Friedman (1949). It is well known that in the actual shower the longitudinal track length  $z_\pi(E_0, E)$  is given by

$$z_\pi(E_0, E) = \alpha E_0 / E^2, \quad \text{for } E \ll E_0, \dots \dots \dots (3.10)$$

where  $\alpha = 0.4368$  for "conventional" cross section and  $\alpha = 0.4662$  for "super-simplified" cross section. The longitudinal track length associated with (3.9) is

$$z_\pi^{(0)}(E_0, E) = (\sigma_1)^{-1} \delta(E_0 - E).$$

Combining these two expressions, we get the following track length formula for the shower as a whole

$$z_\pi(E_0, E) = \alpha E_0 / E^2 + (\sigma_1)^{-1} \delta(E_0 - E). \dots \dots \dots (3.11)$$

Friedman (1949) shows that the coefficient of  $E_0 / E^2$  in (3.11) must equal the coefficient of the delta function, that is, if we insist on using the very rough approximate form (3.9) at all, then we must make the choice

$$\sigma_1 = \alpha^{-1}. \dots \dots \dots (3.12)$$

It should be emphasized that (3.11) with the choice (3.12) for  $\sigma_1$  is the best that can be done for the actual shower if we insist on using (3.9) rather than (3.8) for the end group, but it is by no means a close approximation. The best that can be said for it is that the area under the delta function in (3.11) corresponds approximately to the true area underneath that part of the actual  $z_\pi$  which cannot be represented correctly by (3.10).

The reason for stressing the behaviour of  $v$  for  $E$  close to  $E_0$  is related to the integral equation (2.18). Let us see what region of  $E'$  contributes most to the integral in (2.18). There is a factor  $(E_s / 2E')^2$  as well as  $z(E_0, E')$  which itself is proportional to  $(E')^{-2}$ . These two factors favour low values of  $E'$ . The factor  $v(E', E, \theta)$  favours high values of  $E'$ , but presumably it does so no more strongly than  $z_\pi(E_0, E')$  favours low values of  $E'$ . The net result is that the maximum in the integrand of (2.18) occurs for low values of  $E'$ , that is, for  $E'$  very close to  $E$ . Thus approximation (3.11) can be used for  $z_\pi(E_0, E')$  and approximation (3.7) (with (3.12) for  $\sigma_1$ ) or a slightly improved version of it for  $v(E', E, \theta)$ .

#### IV. THE FURRY SHOWER

We can find an exact solution to an approximate model of a cascade first introduced by Furry (1937). We can test the qualitative arguments by applying them to the Furry cascade and comparing with the exact solution. It should be noted that there is no qualitative difference between the Furry and actual cascades as far as the angular development of the shower is concerned.

The Furry model contains only one kind of particle which can split into two particles of the same kind. The probability  $\sigma_f(E, E')$  for a splitting event in which one of the emerging particles has energy  $E'$ , the other energy  $E - E'$ , is

$$\sigma_f(E, E') = 1/E, \dots \dots \dots (4.1)$$

and the diffusion equation for the average number of particles is

$$\frac{\partial \varphi_{\pi}}{\partial t} = 2 \int_E^{E_0} \varphi_{\pi}(E_0, E', t, \theta) \left( \frac{1}{E'} \right) dE' - \varphi_{\pi} + \left( \frac{E_s}{2E} \right)^2 \frac{\partial^2 \varphi_{\pi}}{\partial \theta^2} + \delta(E_0 - E) \delta(t) \delta(\theta). \quad (4.2)$$

This equation replaces (2.4) and (2.5) for the actual shower. By integrating over  $t$  from 0 to infinity we get the track length equation which replaces (2.16). We are most interested in the analogue of (2.18). Since (2.18) does not contain any reference to the photons in the true cascade, it actually holds also for the Furry shower, without modification. The only change is the expression for  $z_{\pi}$ . The longitudinal track length in the Furry shower was found by Nordsieck, Lamb, and Uhlenbeck (1940). They give

$$z_{\pi}(E_0, E) = 2E_0/E^2 + \delta(E_0 - E). \quad (4.3)$$

According to the qualitative arguments of Section III, the end group (corresponding to the  $\delta(E_0 - E)$  in (4.3)) should have a track length given by  $v^{(0)}$ , (3.7), with  $\sigma_1 = 1$ ; low energy particles should have a track length given by  $v = (E/E_s) z_{\pi} f(u)$ , according to (3.2) and (3.3). We therefore make the initial assumption

$$v(E_0, E, \theta) = v^{(0)}(E_0, E, \theta) + (E/E_s)(2E_0/E^2) f(E\theta/E_s), \quad (4.4)$$

where  $f$  is so far an unknown function. We substitute (4.3) and (4.4) into the integro-differential equation (2.18) to get an equation for  $f$ . We make use of the fact that

$$\frac{d^2 \exp(-a|\theta|)}{d\theta^2} = a^2 \exp(-a|\theta|) - 2a\delta(\theta) \quad (4.5)$$

in order to get the following differential equation for  $f(u) = f(E\theta/E_s)$ :

$$\frac{d^2 f}{du^2} - 4f(u) = -4 \exp(-2|u|). \quad (4.6)$$

The fact that this is an equation not involving the energy variables  $E, E_0$  explicitly shows that the initial assumption (4.4) is consistent with the equations defining the Furry cascade. The differential equation (4.6) is solved easily, subject to the boundary conditions that  $f(u)$  vanish for  $u = +\infty$  and  $u = -\infty$ . The solution is

$$f(u) = \frac{1}{2} e^{-2|u|} + |u| e^{-2|u|}. \quad (4.7)$$

Expression (4.4) with (4.7) for  $f(u)$  is an exact solution for the Furry cascade in the Landau approximation. Since the Ansatz (4.4) was constructed on the basis of the qualitative arguments in Section III, these qualitative arguments are thereby strengthened considerably, and we can have some confidence in their application to the actual shower.

It is worth remarking that, unlike the expression (3.7) for the angular distribution of the end group of particles, the angular distribution for energies  $E < E_0$  (4.7) does not have a cusp at  $u = 0$ . Correspondingly, the distribution in  $U = E\Theta/E_s$ , ( $\Theta = (\theta_x^2 + \theta_y^2)^{1/2}$ ) implied by (4.7) is finite for all  $U$ , indeed it is

$$F(U) = (2U/\pi) K_1(2U). \quad (4.8)$$

Thus, in the Landau approximation at least, the analytic behaviour of the end group near  $\theta=0$  does not give a reliable guide to the analytic behaviour of the desired solution.

V. THE ACTUAL SHOWER : ROUGH SOLUTION

In order to get a first orientation about the angular distribution in the actual shower, we use the integro-differential equation (2.18) with the approximation (3.11) for  $z_\pi(E_0, E)$  and with the following Ansatz for  $v(E_0, E, \theta)$  :

$$v(E_0, E, \theta) = v^{(0)}(E_0, E, \theta) + (E/E_s)(\alpha E_0/E^2)f(E\theta/E_s). \dots\dots (5.1)$$

In this expression  $v^{(0)}$  is taken from (3.7) with the choice  $\sigma_1=1/\alpha$  for the "effective radiative cross section" (see the discussion in connexion with formula (3.12)).  $f(u)$  is an unknown function.

Substitution into (2.18) and performance of the integration over  $E'$  leads to the following result :

$$f(u) = (\alpha/8)[1 + (E^2/E_0^2)]f''(u) + \exp(-2\alpha^{-\frac{1}{2}}|u|). \dots (5.2)$$

The presence of the ratio  $(E/E_0)^2$  in this equation means that the Ansatz (5.1) is not consistent with the shower equations. When  $E \ll E_0$ , however, the coefficient of  $f''$  becomes approximately independent of energy. We propose to take this limit in solving the equation. In that case the solution of (5.2) is, with

$$\mu = \alpha^{-\frac{1}{2}} = 1.46, \dots\dots\dots (5.3)$$

given by

$$f(u) = 2^{\frac{1}{2}}\mu[2^{\frac{1}{2}} \exp(-2\mu|u|) - \exp(-8^{\frac{1}{2}}\mu|u|)]. \dots (5.4)$$

This expression has the property that  $f(0)$  is finite and  $f'(0)=0$ , just as the Furry shower  $f(u)$ , (4.7). The integral of  $f(u)$  over all values of  $u$  is 1, as it should be. This relation depends upon the choice (3.12) for the effective cross section  $\sigma_1$ . Any other choice would have led to an incorrect value for  $\int_{-\infty}^{\infty} f(u)du$ .

Since  $\sigma_1$  was chosen on the basis of considerations about the longitudinal development of the shower, the fact that this same  $\sigma_1$  makes sense for the angular development is encouraging.

(5.4) can not be an exact solution. We have used an approximate formula for  $z_\pi$ , expression (3.10), and we have ignored a term in the differential equation (5.2). Direct comparison of the moments of (5.4) with the precise moments (in the Landau approximation) shows the same thing ; the moments disagree, increasingly so as we go to higher and higher moments. (These moments are given in Section VI, Table 3.)

VI. THE ACTUAL SHOWER : ITERATED SOLUTION

The best way to test the accuracy of an approximate solution of some complicated equation is to compare it with the exact solution. If no exact solution is available, the next best thing is to iterate on the approximate solution in some systematic manner. If the iterated function does not differ greatly from the first approximate function, then the difference between the two is

probably of the order of the absolute error of the original. In addition, the iterated function is presumably a better approximation to the truth than the original.

The best method to judge the accuracy of the iterated function is to perform a second iteration, with the first iterate as trial function. As will be seen later, this is practically impossible here. However, this does not mean that we have no way of testing the accuracy of the iterated function. We shall perform this test by comparing the moments of the iterated function with the exact (in Landau approximation) moments.

Friedman (1949) has formulated a variation-iteration technique for the solution of equations such as (2.16). The Friedman method is discussed in Appendix III. This method has two major advantages: (1) the normalization of the iterated function is independent of the normalization of the trial function, (2) the error in the iterated function is of the order of magnitude of the *square* of the error in the trial function (this latter property is implied by the term "variation method").

Let  $v_0(E_0, E, \theta)$  be a trial solution of the integro-differential equation (2.16). The iterated solution in the Friedman scheme is then given by

$$v_1(E_0, E, \theta) = \frac{[v_0(E_0, E, \theta)]^2}{\int_E^{E_0} dE' \int_{-\infty}^{\infty} d\theta' [Fv_0(E_0, E', \theta') - (E_s/2E')^2 \partial^2 v_0 / \partial \theta'^2] v_0(E', E, \theta - \theta')} \dots \dots \dots (6.1)$$

It is easily seen from (2.16) that the choice  $v_0 = v$  (trial function = exact solution) leads to  $v_1 = v$  also. It is shown in Appendix III that the choice  $v_0 = v + E$  leads to  $v_1 = v + \text{Order}(E^2)$ ,  $E$  being the error of the trial function.

We have carried out the iteration (6.1) explicitly, using as our trial function the result of Section V; since the normalization of  $v_0$  in (6.1) is of no importance, we multiply (5.1) by the constant  $\mu E_s/2$ ; we also use the notation

$$x = 2\mu u = 2\mu E\theta/E_s. \dots \dots \dots (6.2)$$

We shall use  $x$  as our angle variable consistently. With this notation, the trial function becomes

$$v_0(E_0, E, x) = (E_0/E) \{ \exp(-|x|) - 2^{-\frac{1}{2}} \exp(-2^{\frac{1}{2}}|x|) \} + \frac{1}{2} E_0 \delta(E_0 - E) \exp(-E_0|x|/E), \} \\ v_0 = 0, \text{ when } E > E_0. \dots \dots \dots (6.3)$$

The operator  $F$ , (2.17), can be found explicitly from (1.1) and (2.1)-(2.3). We shall work with a finite but small value of  $\epsilon$  in (2.1), and let  $\epsilon$  approach 0 at the very end of the calculation. By interchanging orders of integration in the operator product  $BC$ , we get the following explicit form for the operator  $F$  acting on an arbitrary function  $f(E)$

$$Ff = \ln(E/\epsilon) f(E) - \int_{E+\epsilon}^{\infty} f(E')(E' - E)^{-1} dE' - 2 \int_E^{\infty} f(E')(E^{-1} - E'^{-1}) dE'. \dots \dots \dots (6.4)$$

We now apply this operator  $F$  to the function  $f=(E_0/E) \exp(-|x|)$ . Since  $x$  is defined by (6.2) so as to contain the energy variable  $E$ , we get

$$f(E')=(E_0/E') \exp(-E'|x|/E), \quad f=0 \text{ for } E'>E_0, \quad \dots (6.5)$$

and

$$Ff=(E_0/E) \left\{ \ln(E/\varepsilon) \exp(-|x|) - \int_{1+(\varepsilon/E)}^{E_0/E} \frac{\exp(-y|x|)}{y(y-1)} dy - 2 \int_1^{E_0/E} y^{-1}(1-y^{-1}) \exp(-y|x|) dy \right\} = (E_0/E)G(x, E_0/E). \quad \dots (6.6)$$

The function  $G$  defined by (6.6) can be expressed in closed form in terms of the exponential integral. We use the standard notation (Jahnke and Emde 1943)

$$-\text{Ei}(-x) = \int_x^\infty \frac{e^{-y}}{y} dy, \quad x > 0, \quad \dots (6.7)$$

and the Euler constant

$$\gamma = 1.7811, \quad \ln(\gamma) = C = 0.57722. \quad \dots (6.8)$$

We then get, after some manipulation,

$$G(x, y) = [\ln(\gamma|x|) + 2 - \text{Ei}(|x| - |xy|)]e^{-|x|} + (1 + 2|x|)[\text{Ei}(-|x|) - \text{Ei}(-|xy|)] - 2y^{-1}e^{-|xy|}. \quad \dots (6.9)$$

We introduce the notation  $g$  for the following function :

$$Fv_0 = (E_0/E)g(x, E_0/E), \quad \dots (6.10)$$

to get

$$g(x, y) = G(x, y) - 2^{-\frac{1}{2}}G(2^{\frac{1}{2}}x, y) + \frac{1}{2}\delta(y-1) \ln(E_0/\varepsilon)e^{-|xy|} + [y^{-1} - 1 - \frac{1}{2}(y-1)^{-1}]e^{-|xy|}. \quad \dots (6.11)$$

The term with  $\ln(E_0/\varepsilon)$  comes from the bremsstrahlung cross section divergence. Corresponding to the cut-off we have used for the bremsstrahlung cross section, we must assume that there are no electrons in the shower of energy between  $E_0$  and  $E_0 - \varepsilon$ . Thus the formulae written so far are not meant to apply within that region. In particular, this is important for the last term of (6.11), which becomes infinite at  $y=1$ , and would give a logarithmic singularity if integrated straightforwardly, in a later stage of this calculation (formulae (6.16) and (6.18)).

We also need the second term on the left-hand side of (2.16). We introduce the function  $h$  by

$$-(E_s/2E)^2 \partial^2 v_0 / \partial \theta^2 = -\mu^2 \partial^2 v_0 / \partial x^2 = +(\mu^2 E_0/E)h(x, E_0/E). \quad \dots (6.12)$$

By using relation (4.5) we find

$$h(x, y) = \delta(y-1)\delta(x) - \delta(y-1) \exp(-|xy|) - \exp(-|x|) + 2^{\frac{1}{2}} \exp(-2^{\frac{1}{2}}|x|). \quad \dots (6.13)$$

We now substitute these expressions into the denominator of (6.1). We introduce, instead of  $\theta'$ , the variable of integration

$$x' = 2\mu E \theta' / E_s, \quad \dots (6.14)$$

and we let  $D$  stand for the denominator of (6.1). We use the variable of integration  $y = E'/E$  to get the result

$$D = (E_s E_0 / 2 \mu E) \int_1^{E_0/E} dy \int_{-\infty}^{+\infty} dx' [g(xy, E_0/Ey) + \mu^2 h(xy, E_0/Ey)] \times [\exp(-|x-x'|) - 2^{-\frac{1}{2}} \exp(-2^{\frac{1}{2}}|x-x'|) + (1/2y)\delta(y-1) \exp(-|x-x'|)]. \dots\dots\dots (6.15)$$

It is very convenient to do the integration over energy (over  $y$ ) before the integration over the angle variable  $x'$ . We introduce the functions

$$\left. \begin{aligned} g_1(x,y) &= \int_1^y dy' g(xy', y/y'), \\ h_1(x,y) &= \int_1^y dy' h(xy', y/y'), \end{aligned} \right\} \dots\dots\dots (6.16)$$

to get the following expression for the denominator  $D$  :

$$D = (E_s E_0 / 2 \mu E) \int_{-\infty}^{+\infty} dx' [g(x', E_0/E) + \mu^2 h(x', E_0/E)] \frac{1}{2} \exp(-|x-x'|) + (E_s E_0 / 2 \mu E) \int_{-\infty}^{+\infty} dx' [g_1(x', E_0/E) + \mu^2 h_1(x', E_0/E)] [\exp(-|x-x'|) - 2^{-\frac{1}{2}} \exp(-2^{\frac{1}{2}}|x-x'|)]. \dots\dots\dots (6.17)$$

Explicit evaluation of the functions  $g_1$  and  $h_1$  defined by (6.16) is possible, and gives the following results :

$$g_1(x,y) = G_1(x,y) - 2^{-\frac{1}{2}} G_1(2^{\frac{1}{2}}x,y) + \frac{1}{2} e^{-|xy|} [y \ln(y/(y-1)) + 1 - y^{-1}], \dots\dots\dots (6.18)$$

$$G_1(x,y) = [x^{-1} \ln(\gamma x) - 1] e^{-x} - (x+1+x^{-1}) \text{Ei}(-x) - [x^{-1} \ln(\gamma xy) - y] e^{-xy} + (x+1+x^{-1}) \text{Ei}(-xy) + x^{-1} \ln[\gamma x(y-1)] e^{-xy} - x^{-1} e^{-x} \text{Ei}(x-xy) - (y-y^{-1}) e^{-xy}, \quad \text{for } x > 0, \dots\dots\dots (6.19)$$

$$G_1(-x,y) = G_1(x,y),$$

$$h_1(x,y) = |x|^{-1} [\exp(-|xy|) - \exp(-|x|) - \exp(-2^{\frac{1}{2}}|xy|) + \exp(-2^{\frac{1}{2}}|x|)] - \frac{1}{2} y \exp(-|xy|) + \delta(x). \dots\dots\dots (6.20)$$

We are now at the stage where we can make good use of the fact that we want the final expressions only in the limit  $y = E_0/E \rightarrow \infty$ . We therefore define limiting functions

$$\bar{g}(x) = \lim_{y \rightarrow \infty} g(x,y), \dots\dots\dots (6.21)$$

and similar expressions for all the other functions involved. Considerable care must be exercised in taking this limit. An expression such as  $y \exp(-|xy|)$

becomes zero in this limit for all non-zero values of  $x$ , but can not therefore be ignored. The fact that

$$\int_{-\infty}^{+\infty} y \exp(-|xy|) dx = 2$$

is independent of  $y$  indicates that such a term gives rise to a delta function contribution in the limiting process (6.21), in this particular case equal to  $2\delta(x)$ . The delta function contributions are very important. Inspection of (6.17) shows that  $\delta(x)$  terms in  $\bar{g}$  or  $\bar{h}$  give rise to terms proportional to  $\exp(-|x|)$ ; there are no such terms, actually; but a  $\delta(x)$  term in  $\bar{h}_1$ , which does occur, gives rise to a term proportional to the original trial structure function (5.4). If (5.4) had been an exact solution, the denominator  $D$  would be in its entirety proportional to (5.4). Thus, the more nearly the trial function is equal to the exact solution, the more nearly we have  $\bar{g} + \mu^2 \bar{h} = 0$  and  $\bar{g}_1 + \mu^2 \bar{h}_1$  equal to a constant times  $\delta(x)$ .

The terms proportional to  $\delta(y-1)$  in (6.11) and (6.13) can be ignored in the limit  $y \rightarrow \infty$ . There is one remark to be made, however. The  $\delta(y-1)$  term in  $g(x,y)$  (6.11) is multiplied by the constant  $\ln(E_0/\epsilon)$ ; if we go to the limit  $\epsilon \rightarrow 0$ , corresponding to the actual (supersimplified) bremsstrahlung cross section, this term becomes infinite. It contributes only at the initial energy  $E = E_0$  (i.e.  $y = 1$ ), and the fact that the denominator of (6.1) has an infinite contribution proportional to  $\delta(E_0 - E)$  indicates that the iterated function  $v_1(E_0, E, \theta)$ , unlike the trial function  $v_0$ , contains no term proportional to  $\delta(E_0 - E)$ . This is of course to be expected of the true solution  $v$ , and the fact that the very first iteration already gives this result is encouraging.

We now write down the barred functions :

$$\bar{g}(x) = \bar{G}(x) - 2^{-\frac{1}{2}} \bar{G}(2^{\frac{1}{2}}x), \dots \dots \dots (6.22)$$

$$\bar{G}(x) = [\ln(\gamma |x|) + 2] e^{-|x|} + (1 + 2|x|) \text{Ei}(-|x|), \dots \dots (6.23)$$

$$\bar{h}(x) = -\exp(-|x|) + 2^{\frac{1}{2}} \exp(-2^{\frac{1}{2}}|x|), \dots \dots \dots (6.24)$$

$$\bar{g}_1(x) = \bar{G}_1(x) - 2^{-\frac{1}{2}} \bar{G}_1(2^{\frac{1}{2}}x), \dots \dots \dots (6.25)$$

$$\bar{G}_1(x) = [|x|^{-1} \ln(\gamma |x|) - 1] e^{-|x|} - (|x| + 1 + |x|^{-1}) \text{Ei}(-|x|), \dots \dots \dots (6.26)$$

$$\bar{h}_1(x) = -|x|^{-1} [\exp(-|x|) - \exp(-2^{\frac{1}{2}}|x|)] + \ln(2) \delta(x). \dots (6.27)$$

It will be noticed that, although there are several "delta function terms" in  $\bar{G}_1(x,y)$  (6.19) these actually cancel out.\*

In order to exhibit the contribution of the delta function term in  $\bar{h}_1(x)$  more explicitly, we write

$$\left. \begin{aligned} \bar{h}_1(x) &= \bar{h}_2(x) + \ln(2) \delta(x), \\ \bar{h}_2(x) &= -|x|^{-1} [\exp(-|x|) - \exp(-2^{\frac{1}{2}}|x|)]. \end{aligned} \right\} \dots (6.27a)$$

\* The delta function terms were found by a more indirect method by Kalos (1952) and there appeared to be a very small delta function contribution in  $\bar{G}_1(x)$ . This was the result of round off errors in the numerical calculations. No significant change is introduced thereby into the final results; if anything, the iterated function is improved a little by this correction.

The denominator of (6.1) in the limit  $E \ll E_0$  is then given by (6.17) with the barred functions  $\bar{g}$  etc. instead of the original functions  $g$  etc. We now transform the integrals slightly, as follows :

$$\int_{-\infty}^{+\infty} dx' \bar{g}(x') \exp(-|x-x'|) = 2 \int_0^{\infty} dx' \bar{g}(x') K(x, x'), \quad \dots (6.28)$$

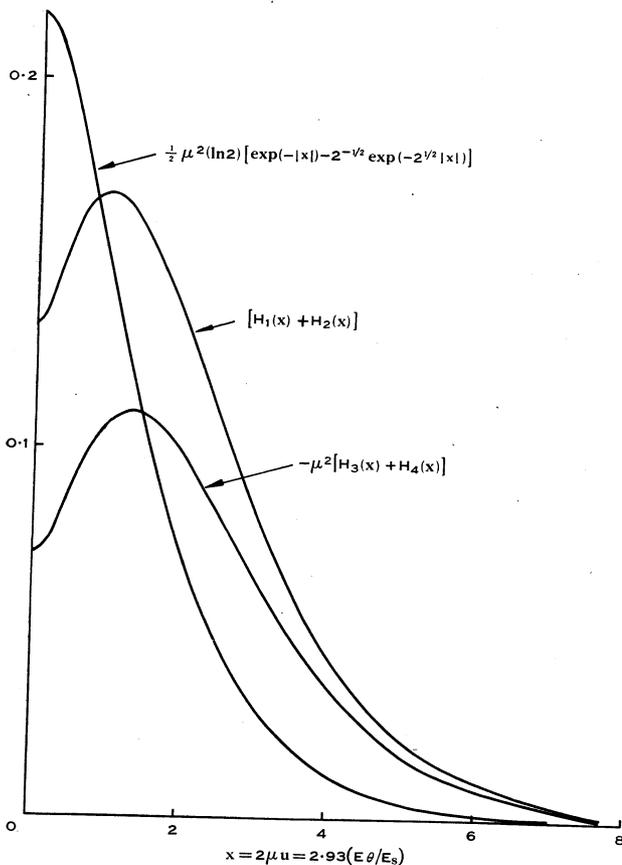


Fig. 1.—Intermediate quantities in the iteration of the track length angular distribution. The three quantities shown here add up to form the denominator of equation (6.35); notice the reversed sign on  $-\mu^2(H_3 + H_4)$ . The contribution of the “delta-function term”, i.e. the term proportional to the trial function, is very appreciable.

where the kernel  $K(x, x')$  is given by

$$\left. \begin{aligned} K(x, x') &= \exp(-x) \cosh x', & \text{if } x > x', \\ &= \exp(-x') \cosh x, & \text{if } x < x'. \end{aligned} \right\} \dots (6.29)$$

We then collect together the terms with the same kernel, to get

$$\begin{aligned} D &= (E_s E_0 / \mu E) [H_1(x) + H_2(x) + \mu^2 H_3(x) + \mu^2 H_4(x) \\ &\quad + \frac{1}{2} \mu^2 (\ln 2) \{ \exp(-|x|) - 2^{-1/2} \exp(-2^{1/2}|x|) \}], \quad \dots (6.30) \end{aligned}$$

where

$$H_1(x) = \int_0^\infty dx' [\frac{1}{2}\bar{g}(x') + \bar{g}_1(x')] K(x, x'), \quad \dots \quad (6.31)$$

$$H_2(x) = -2^{-\frac{1}{2}} \int_0^\infty dx' \bar{g}_1(x') K(2^{\frac{1}{2}}x, 2^{\frac{1}{2}}x'), \quad \dots \quad (6.32)$$

$$H_3(x) = \int_0^\infty dx' [\frac{1}{2}\bar{h}(x') + \bar{h}_2(x')] K(x, x'), \quad \dots \quad (6.33)$$

$$H_4(x) = -2^{-\frac{1}{2}} \int_0^\infty dx' \bar{h}_2(x') K(2^{\frac{1}{2}}x, 2^{\frac{1}{2}}x'). \quad \dots \quad (6.34)$$

TABLE I  
INTERMEDIATE QUANTITIES IN THE ANGULAR DISTRIBUTION ITERATION

$x$	$H_1(x)$	$H_2(x)$	$H_3(x) + H_4(x)$
0	0.53779	-0.40389	-0.03337
0.1	0.53182	-0.39599	-0.03393
0.2	0.52163	-0.38017	-0.03534
0.3	0.50818	-0.36057	-0.03727
0.4	0.49256	-0.33909	-0.03944
0.5	0.47548	-0.31688	-0.04166
0.6	0.45744	-0.29466	-0.04379
0.7	0.43881	-0.27290	-0.04573
0.8	0.41987	-0.25190	-0.04742
0.9	0.40085	-0.23187	-0.04880
1.0	0.38192	-0.21292	-0.04987
1.2	0.34486	-0.17847	-0.05105
1.4	0.30947	-0.14863	-0.05102
1.6	0.27623	-0.12316	-0.04996
1.8	0.24540	-0.10165	-0.04807
2.0	0.21710	-0.08363	-0.04557
3.0	0.11109	-0.03028	-0.02965
4.0	0.05394	-0.01059	-0.01623
5.0	0.02499	-0.00368	-0.00804
7.0	0.00485	-0.00042	-0.00166
9.0	0.00087	-0.00005	-0.00030

Combining equations (6.1), (6.3), (6.30), and (3.2), we get the following expression for the angular structure function of the shower at energies  $E \ll E_0$  (with  $x = 2\mu u = 2\mu E\theta/E_s$ )

$$f(E\theta/E_s) = \frac{\mu^3 \{ \exp(-|x|) - 2^{-\frac{1}{2}} \exp(-2^{\frac{1}{2}}|x|) \}^2}{\bar{H}_1(x) + \bar{H}_2(x) + \mu^2 [\bar{H}_3(x) + \bar{H}_4(x)] + \frac{1}{2} \mu^2 \ln 2 \{ \exp(-|x|) - 2^{-\frac{1}{2}} \exp(-2^{\frac{1}{2}}|x|) \}} \quad \dots \quad (6.35)$$

Before going into the evaluation of the integrals (6.31) to (6.34), we point out one general property of them, namely, the behaviour of these functions

near  $x=0$ . By differentiating  $H_1(x)$ , say, under the integration sign in (6.31), we can show immediately that the derivative of  $H_1(x)$  vanishes at  $x=0$ . The same holds for the other integrals. The  $\delta(x)$  contribution gives rise to a constant times the trial function (5.4), and this trial function itself has zero derivative at  $x=0$ . We therefore conclude that the whole denominator  $D$ , and hence also the iterated function  $v_1$  of (6.1), has zero derivative at  $x=0$ . There is no singularity

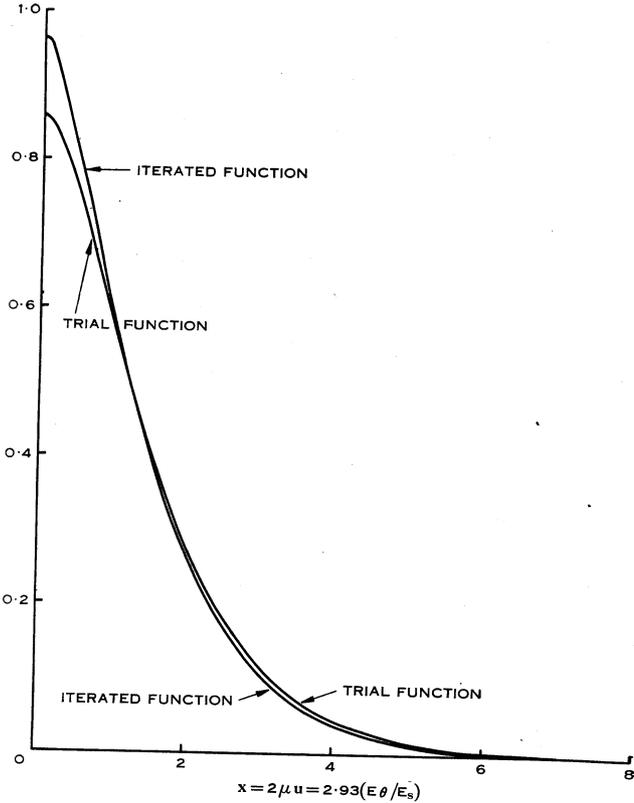


Fig. 2.—The trial function and the iterated function are rather close to each other over most of the range, indicating that the iterated function is likely to be very close to the true mathematical solution of the equations.

of the angular distribution function of monoenergetic electrons in the Landau approximation. This result is in complete agreement with the results of Belenky (1944) and of Molière (1946). The integrals  $H_3$  and  $H_4$  can be done analytically. The results are, for  $x > 0$ :

$$H_3(x) = \frac{1}{2}e^{-x}\left\{\frac{1}{2}(3-x) + \text{Ei}[(1-2^{\frac{1}{2}})x] - \ln(\frac{1}{2}\gamma x)\right\} + \frac{1}{2}e^{+x}\{\text{Ei}(-2x) - \text{Ei}[-(1+2^{\frac{1}{2}})x]\}, \dots \dots \dots (6.36)$$

$$H_4(x) = -8^{-\frac{1}{2}} \exp(-2^{\frac{1}{2}}x)\{\text{Ei}[(2^{\frac{1}{2}}-1)x] + \ln(8^{-\frac{1}{2}}\gamma x)\} + 8^{-\frac{1}{2}} \exp(+2^{\frac{1}{2}}x)\{\text{Ei}(-8^{\frac{1}{2}}x) + \text{Ei}[-(1+2^{\frac{1}{2}})x]\}. \dots (6.37)$$

The integrals (6.31) and (6.32) are best done numerically, and have been done in this fashion. Values of  $H_1(x)$ ,  $H_2(x)$ , and of the sum  $H_3+H_4$  are given in Table 1. These functions are plotted in Figure 1. There we have plotted also the contribution from the delta function term in  $\bar{h}_1$  so as to show the tremendous importance of this delta function contribution. It will also be noticed that the sum  $(H_1+H_2)$  nearly cancels  $\mu^2(H_3+H_4)$  for small values of  $x$ .

TABLE 2  
ANGULAR STRUCTURE FUNCTION: THE TRIAL FUNCTION AND THE  
ITERATED FUNCTION;  $x=2\mu E\theta/E_s=2\mu u$

$x$	$f_{\text{trial}}(x)$	$f_{\text{iter}}(x)$
0	0.85794	0.96235
0.1	0.85234	0.95220
0.2	0.83726	0.92283
0.3	0.81489	0.88584
0.4	0.78709	0.84446
0.5	0.75538	0.80065
0.6	0.72100	0.75584
0.7	0.68492	0.71082
0.8	0.64800	0.66646
0.9	0.61087	0.62295
1.0	0.57404	0.58089
1.2	0.50277	0.50116
1.4	0.43634	0.42993
1.6	0.37585	0.36603
1.8	0.32175	0.30997
2.0	0.27400	0.26117
3.0	0.11607	0.10437
4.0	0.046414	0.03882
5.0	0.017978	0.01373
7.0	0.0025671	0.00159
9.0	0.00035534	0.000171

The trial function and the iterated function are compared in Table 2 and in Figure 2. The change produced by the iteration is not very large except in the "tail" of the distribution function (large values of  $x$ ), where the iterated function is smaller than the trial function, as it should be.

We now examine the moments of the trial function, iterated function, and the "exact" moments. These moments are given in Table 3. Two sets of exact moments are given there, one for supersimplified cross sections, the other for conventional cross sections. We see immediately that the iterated function is a great improvement over the trial function. For example, the second moment of the trial function is in error by 17 per cent., the second moment of the iterated function by less than 3 per cent. The improvement is even bigger for the higher moments.

Next, let us compare the iterated function with the true function in the approximation we have been using (supersimplified cross sections) through their

moments. The second moment is too large by 3 per cent., the fourth moment by 7 per cent., the sixth moment by 14 per cent. The main contribution to the fourth moment comes from values of  $x$  near 4, that is, values of  $u$  near  $4/3$ , and to the sixth moment from  $x$  near 6 or  $u$  near 2. We conclude that *the iterated*

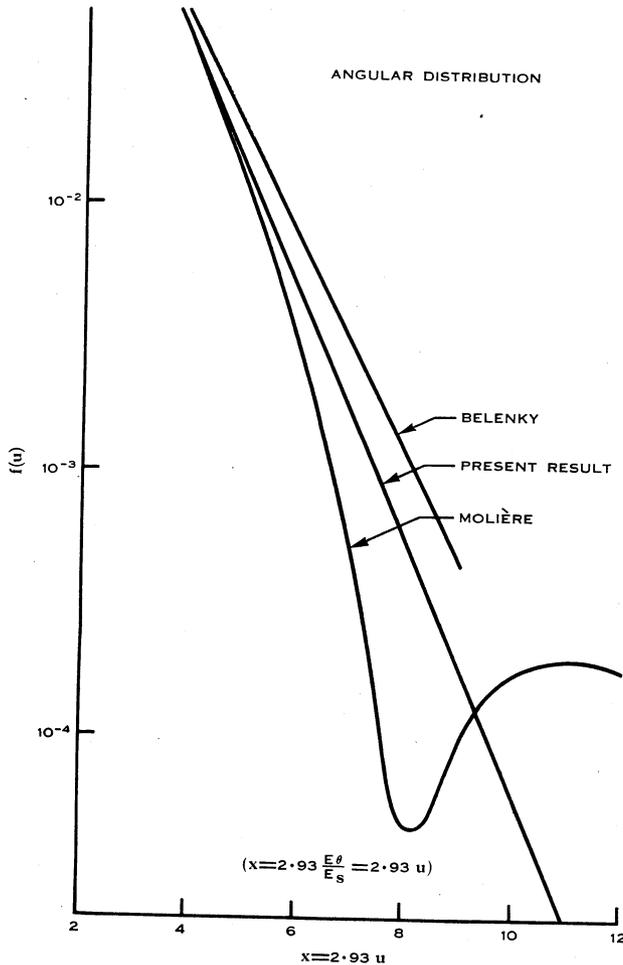


Fig. 3.—Comparison with other work. The results of Molière and Belenky differ from our iterated function out in the tail. A comparison with the exact moments indicates that our iterated function is the best of the three. No special significance attaches to this region of angles, since the basic approximations (especially the Landau approximation of pure multiple scattering) break down there.

*function is within 10 per cent. of the mathematically correct solution of our equations for values of  $u = E\theta/E_s < 1.5$ .*

Our equations contain several approximations within them. One of these is the use of supersimplified cross sections. When we compare the moments of

our iterated function with the exact moments using conventional cross sections, the agreement is of course not nearly as good. The second moment is too large by 8 per cent., the fourth moment by 21 per cent., and the sixth moment by 41 per cent. While it would be in principle possible to use the conventional rather than the supersimplified cross sections for the operator  $F$  in the denominator of the iteration equation (6.1), we feel that the additional work involved is hardly warranted by the present experimental data on the angular

TABLE 3  
MOMENTS OF THE ANGULAR DISTRIBUTION, IN LANDAU APPROXIMATION

$n$	$f_n = \langle (E\theta/E_s)^n \rangle_{ave}$			
	Trial	Iterated	Exact	
			Supersimplified Cross Sections	Conventional Cross Sections
0	1.00	1.01	1.00	1.00
2	0.350	0.309	0.300	0.285
4	0.571	0.433	0.406	0.360
6	2.138	1.362	1.191	0.970
8	14.416	7.654	5.940	4.89

distribution of monoenergetic electrons. The moments make it likely that the iterated function  $v_1$  is within 10 per cent. of the mathematically correct solution with conventional cross sections for values of  $u < 0.8$ .

Next, our function is in error by an unknown amount because of the use of the Landau approximation. We intend to perform the iteration once more, using the correct integral operator for Coulomb scattering rather than the operator  $(E_s/2E)^2 \partial^2 / \partial \theta^2$  in the denominator of (6.1). The discussion of the errors produced by the Landau approximation is reserved for a later publication.

Last, we compare our iterated function with the results of Belenky (1944) and Molière (1946). This is done in Figures 3 and 4. We see that the three functions agree very well indeed for small values of  $x$ . The methods of calculation used by these authors were different from each other and different from ours. Molière used supersimplified cross sections, and Belenky used the cross sections of the Tamm-Belenky theory, which are closely related to the supersimplified cross sections. Both authors also used the Landau approximation for the Coulomb scattering. Thus it is not surprising that their results should agree so well with ours. A comparison of the moments of the various functions with the exact moment shows that for larger values of  $x$ , where the three functions disagree more and more, our  $v_1$  is the closest approximation to the mathematically exact solution. However, no great stress should be laid on this point, since the Landau approximation has in any case no claim to validity for  $u > 1$  (for  $x > 3$ ). Molière's angular distribution function has a minimum at  $x=8$  ( $u=2.7$ ), and the corresponding distribution-in- $U$  ( $U=E\theta/E_s$ ) actually becomes negative there, but it is apparent from Figure 3 that this is way out in the "tail" of the

function, and hence the importance of this error in Molière's calculation has been overestimated in the past.\*

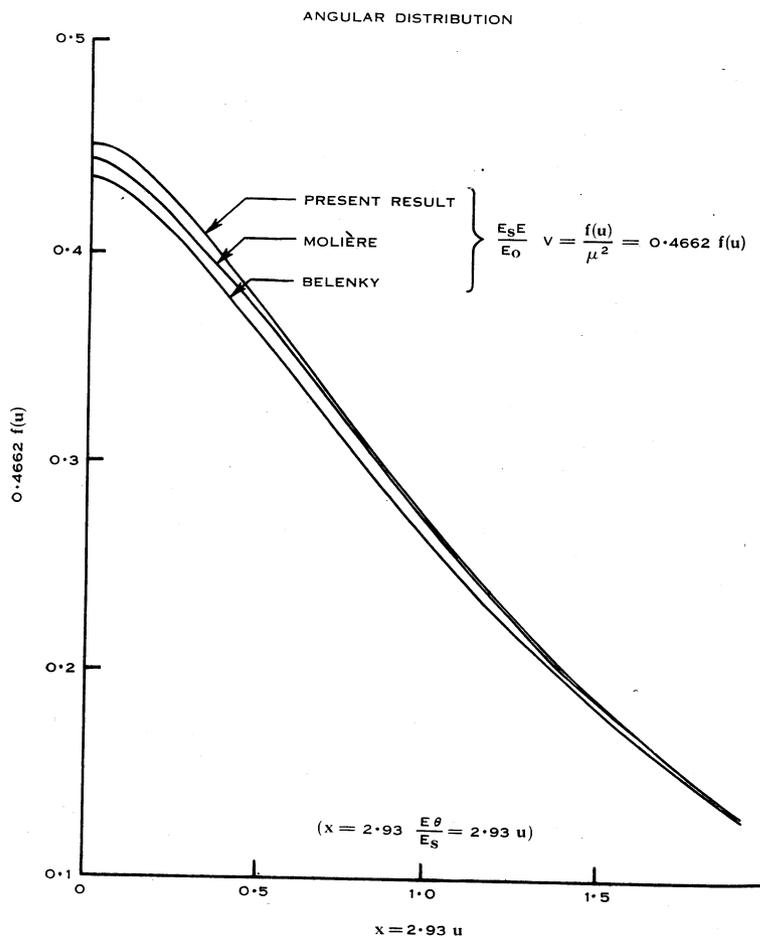


Fig. 4.—Comparison with other work. The results of Molière and Belenky agree closely with our iterated function in the significant region of angles. (Notice that the ordinate scale does not start at 0.) The dominant contribution to the second moment of  $f$  comes from the neighbourhood of  $x=2$ , at the extreme right of the figure. We conclude that it is impractical to deduce the behaviour of  $f(u)$  for small values of  $u$  from a study of the moments of  $f$ .

\* Blatt (1949). However, the doubts concerning the accuracy of Molière's *lateral* structure function are not decreased appreciably by the agreement of the angular structure function. The lateral structure function is very much harder to determine. The only improvements on Molière's work published to date have been concerned with the moments of the lateral distribution. These moments disagree with Molière's moments quite badly even in the Landau approximation, for example the sixth moment of Molière is too small by a factor 4.65, the 10th moment of Molière is too small by a factor 1000. These factors get worse if the Landau approximation is dropped.

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## APPENDIX I

*Moments of the Angular Distribution*

We start from the Nordheim recursion formula (2.12). We take the Mellin and Laplace transform on both sides, and we assume that the dependence of the transform  $\pi_n(E_0, s, \lambda)$  on the initial energy  $E_0$  is given by

$$\int_0^{E_0} dE E^s \int_0^\infty dt e^{\lambda t} \pi_n(E_0, E, t) = (E_0)^{s-n} M_n(s, \lambda). \quad \dots \quad (\text{A1})$$

We then get from (2.12), by inverting orders of integration,

$$M_n(s, \lambda) = n(n-1) \left(\frac{1}{2} E_s\right)^2 M_0(s-n, \lambda) M_{n-2}(s, \lambda). \quad \dots \quad (\text{A2})$$

The fact that  $E_0$  does not appear in (A2) shows that the assumption (A1) was correct. The solution of this recurrence relation is immediate, giving

$$M_n(s, \lambda) = n! \left(\frac{1}{2} E_s\right)^n M_0(s, \lambda) M_0(s-2, \lambda) M_0(s-4, \lambda) \dots M_0(s-n, \lambda). \quad \dots \quad (\text{A3})$$

The Laplace Mellin transform  $M_0(s, \lambda)$  refers to the zeroth moment of the angular problem, that is, to the longitudinal distribution function  $\pi_0 = \pi$  of Rossi and Greisen (1941). By comparison with equation (2.33) of Rossi and Greisen we see that

$$M_0(s, \lambda) = \frac{\sigma_0 + \lambda}{[A(s) + \lambda](\sigma_0 + \lambda) - B(s)C(s)}. \quad \dots \quad (\text{A4})$$

(A3) and (A4) constitute the complete solution to the moment problem in the Landau approximation. It will be noticed that the depth transform variable  $\lambda$  enters only in a trivial way here, that is, the calculation for arbitrary  $\lambda$  is in no way more difficult than the track length calculation ( $\lambda=0$ ). The inversion of the Laplace transform, to get explicit functions of the depth  $t$ , is straightforward although somewhat tedious. On the other hand, the inversion of the Mellin

transform for arbitrary  $t$  can be done only by the saddle point method. This is the reason why most published results do not give the depth dependent moments, even though they are in principle just as well known as the track length moments. The depth dependence of  $\langle \theta^2 \rangle_{\text{ave}}$  was given in a report by Osborne, Nordheim, and Blatt (unpublished, Echo Lake Conference 1949).

In this paper, we are concerned only with the track length values, so we set  $\lambda=0$  consistently. We can then invert the Mellin transform by using the theory of residues. For  $\lambda=0$ , the pole of  $M_0(s,0)$  with the largest real part of  $s$  occurs at the point  $s=1$ . Thus the dominant pole of (A3) occurs at  $s=n+1$ . The other poles give contributions involving lower powers of  $(E_0/E)$  and can therefore be ignored in the limit  $E \ll E_0$ . We define the  $n$ th moment of the angular structure function  $f(u)$ , (3.2) and (3.3), by

$$f_n = \left\langle \left( \frac{E\theta}{E_s} \right)^n \right\rangle_{\text{ave}} = \left( \frac{E}{E_s} \right)^n \frac{z_{\pi_n}(E_0, E)}{z_{\pi}(E_0, E)}, \quad \dots \dots \dots \text{(A5)}$$

where  $z_{\pi_n}$  is the track length  $n$ th angular moment, that is,

$$z_{\pi_n}(E_0, E) = \int_0^\infty \pi_n(E_0, E, t) dt = \int_{-\infty}^{+\infty} \theta^n v(E_0, E, \theta) d\theta. \quad \dots \text{(A6)}$$

This procedure leads to the result, for  $E \ll E_0$ ,

$$f_n = n! 2^{-n} M_0(3,0) M_0(5,0) M_0(7,0) \dots M_0(n+1,0). \quad \dots \text{(A7)}$$

Since our whole calculation has been based on supersimplified rather than conventional cross sections, we use the supersimplified cross sections here also, that is, we use expressions (2.17b) of Rossi and Greisen. This gives for integral values of  $k$

$$M_0(k,0) = \left[ 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k} - \frac{2}{k(k+1)} \right]^{-1} \dots \dots \dots \text{(A8)}$$

The "exact" moments given in Table 3 are determined from (A7) and (A8). The fact that  $f_n$  for  $E \ll E_0$  is independent of  $E$  and  $E_0$  shows that in this limit the angular structure function is indeed a function of  $(E\theta/E_s)$  only.

It is interesting to observe that these moments do *not* allow a practical determination of the distribution function  $v(E_0, E, \theta)$  for small  $\theta$ . If one takes our best function for  $v$ , that is, our  $v_1$  of Section VI, and computes the region of  $\theta$  which makes the most contribution to the  $n$ th moment, this turns out to be well in the "tail" of the function, even for the lowest usable  $n$ , that is,  $n=2$ .

APPENDIX II

*The Projected Distribution Function and its Inversion*

Let  $\Theta$  be the angle between the direction of motion of the particle and the shower axis, and let  $\theta$  be the angle between the projected motion (onto a plane containing the shower axis) and the shower axis. Let  $\theta'$  be the projected angle for a plane perpendicular to the first one, but also containing the shower axis. Then, in the small angle approximation, we have

$$\Theta = (\theta^2 + \theta'^2)^{\frac{1}{2}}. \quad \dots \dots \dots \text{(B1)}$$

Let  $f(\theta)$  be the distribution function in the projected angle, and  $F(\Theta)$  be the distribution function in the full angle, then clearly

$$f(\theta) = \int_{-\infty}^{+\infty} F[(\theta^2 + \theta'^2)^{\frac{1}{2}}] d\theta' = 2 \int_{\theta}^{\infty} \frac{uF(u)}{(u^2 - \theta^2)^{\frac{1}{2}}} du. \dots\dots (B2)$$

This is an integral equation for  $F(u)$  if  $f(\theta)$  is known. Since the kernel is a function of the ratio  $u/\theta$  only, this integral equation can be solved directly by means of Mellin transforms. The details are given by Kalos (1952). The result is

$$F(u) = -(\pi u^2)^{-1} \int_u^{\infty} \frac{x d/dx [xf(x)]}{(x^2 - u^2)^{\frac{1}{2}}} dx. \dots\dots\dots (B3)$$

If  $f(x)$  is an exponential  $e^{-x}$ , (B3) leads to the Bessel function  $(2/\pi)K_0(u)$ . The relevant integral representation of  $K_\nu(z)$  is formula (4), p. 172 of Watson (1948).

APPENDIX III

*The Friedman Variation Principle*

This variation principle is contained in a report by Friedman (1949). We would like to thank Dr. F. L. Friedman for permission to publish this slightly modified version of his proof.

The distribution functions used in this work can be considered to form a linear vector space. For example, consider two functions of energy and inclination,  $\varphi(E, \theta)$  and  $\psi(E, \theta)$ . We define their scalar product to be

$$(\varphi, \psi) = \int_0^{\infty} dE \int_{-\infty}^{\infty} d\theta \varphi(E, \theta) \psi(E, \theta). \dots\dots\dots (C1)$$

Functions of  $x$  can be treated in the same way; in defining the product for functions of  $t$  the integration is limited to the range 0 to  $\infty$ .

All linear operators used may be written in the form of integrals of the type

$$K\varphi(E, \theta) = \int_0^{\infty} dE' \int_{-\infty}^{\infty} d\theta' \varphi(E', \theta') k(E', E, \theta - \theta'). \dots\dots (C2)$$

As a rule the kernel  $k$  vanishes when  $E$  exceeds  $E'$ . The unit operator,  $I$ , defined by the equation  $I\varphi = \varphi$ , is associated with the kernel  $\delta(E - E')\delta(\theta - \theta')$ .

The effect of two operators of this type applied in turn to a function is given by

$$JK\varphi(E, \theta) = \int_0^{\infty} dE'' \int_{-\infty}^{\infty} d\theta'' \int_0^{\infty} dE' \int_{-\infty}^{\infty} d\theta' \varphi(E', \theta') k(E', E'', \theta'' - \theta') j(E'', E, \theta - \theta''), \dots\dots\dots (C3)$$

so that the product of the two operators  $J$  and  $K$  in that order is an integral operator whose kernel is  $Jk(E', E, \theta)$ .

Consider the problem of inverting a linear operator. That is, given the non-singular operator  $A$ , it is required to find  $B$  such that

$$AB = BA = I. \dots\dots\dots (C4)$$

We shall write a variational expression for the matrix elements of the operator  $B$ . Suppose that  $V$  is an operator, "close to" the inverse operator  $B$ , except perhaps for a constant multiplier  $\lambda$ , that is,

$$V = \lambda(B + \varepsilon R), \dots\dots\dots (C5)$$

where  $\varepsilon$  is by assumption a small quantity.

Let  $\varphi_1, \varphi_2, \dots$  be a complete set of base functions in the vector space in question. We define

$$F_{nm} = \frac{(\varphi_n, V \varphi_m)^2}{(\varphi_n, V A V \varphi_m)} \dots\dots\dots (C6)$$

The array  $F_{nm}$  defines an operator in the representation of the base functions  $\varphi_n$ . The operator  $F$  defined in this way depends on the choice of base. Since the operators  $V$  and  $A$  are linear, we may use (C5) to write the last equation as

$$\left. \begin{aligned} F_{nm} &= \frac{(\varphi_n, (B + \varepsilon R) \varphi_m)^2}{\{\varphi_n, (B + \varepsilon R) A (B + \varepsilon R) \varphi_m\}} \\ &= \frac{(\varphi_n, B \varphi_m)^2 + 2\varepsilon(\varphi_n, B \varphi_m)(\varphi_n, R \varphi_m) + (\varphi_n, R \varphi_m)^2 \varepsilon^2}{(\varphi_n, B A B \varphi_m) + \varepsilon(\varphi_n, B A R \varphi_m) + \varepsilon(\varphi_n, R A B \varphi_m) + \varepsilon^2(\varphi_n, R A R \varphi_m)}. \end{aligned} \right\} \dots\dots\dots (C6a)$$

With the use of (C4) the last equation may be written

$$\left. \begin{aligned} F_{nm} &= \frac{(\varphi_n, B \varphi_m)^2 + 2\varepsilon(\varphi_n, B \varphi_m)(\varphi_n, R \varphi_m) + \varepsilon^2(\varphi_n, R \varphi_m)^2}{(\varphi_n, B \varphi_m) + 2\varepsilon(\varphi_n, R \varphi_m) + \varepsilon^2(\varphi_n, R A R \varphi_m)} \\ &= (\varphi_n, B \varphi_m) + \text{terms of order } \varepsilon^2. \end{aligned} \right\} \dots\dots (C6b)$$

Thus the elements of the array  $F$  have stationary values when  $V$  is near  $\lambda B$ . Furthermore when  $V = \lambda B$ , the elements of  $F$  are the elements of  $B$  itself, in the representation in which the  $\varphi_n$  are base functions. By finding the stationary values of  $F_{nm}$  with respect to variations in  $V$ , we get  $B$ , that is, the inverse of  $A$ . Alternatively, we may use (C6) as an iteration scheme; if  $V$  is an approximation to the inverse operator, then the operator  $F$  is closer to the true inverse, provided the process converges.

In the special case of the vector space introduced above for the distribution function, an appropriate set of base functions is

$$\varphi_n = \delta(E - E_n) \delta(\theta - \theta_n), \dots\dots\dots (C7)$$

where  $n$  stands for the two continuous indices  $E_n$  and  $\theta_n$ . The element  $(\varphi_n, K \varphi_m)$  taken with the operator defined in (C2) and two different base functions of this kind is just the kernel  $k(E_m, E_n, \theta_n - \theta_m)$ . If we use this basis throughout the equation (C6) and apply (C3), we get a variational expression for the kernel  $b(E_0, E, \theta)$  of the integral operator  $B = A^{-1}$ . This expression is

$$f(E_0, E, \theta) = \frac{[v(E_0, E, \theta)]^2}{\int_0^\infty dE' \int_{-\infty}^\infty d\theta' [A v(E_0, E', \theta')] v(E', E, \theta - \theta')} \dots\dots (C8)$$

Here  $v$  is a trial function for  $b$  and  $f$  is the iterated function.