NETS COMPOSED OF PARTS OF CIRCLES FOR THE APPROXIMATE SOLUTION OF FIELD PROBLEMS

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Summary

The two-dimensional differential equation

$$\frac{\partial}{\partial x}\left(\sigma\frac{\partial\varphi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\sigma\frac{\partial\varphi}{\partial y}\right)+\tau=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

describes the current flow in a sheet of conductivity σ loaded by a transverse current density ($-\tau$), φ being the electrical potential. It is known that equation (**1**) can be solved approximately by a procedure in which the two-dimensional continuum is replaced by a net of straight-line bounded meshes, leading to an electrical network of conductances. The author shows that meshes bounded by " curvilinear rectangles " can be equally well dealt with and, on the basis of different conformal transformation functions for the individual meshes, derives the formulae required for a solution, if the mesh boundaries are circle arcs or circle arcs and straight lines. A good fit of the contours of the boundaries and equipotentials and their orthogonal trajectories can be obtained. This reduces the number of meshes without impairing the accuracy. Sharp corners at boundaries can be dealt with in a similar way. Formulae for a good accuracy computation of potential gradients and a method for changing the mesh size abruptly are given. Two examples using nets of only four meshes demonstrate the power of the method, the maximum errors being of the order of a few per cent.

I. Introduction

The problems dealt with in this paper are those governed by the two-dimensional differential equation

$$\frac{\partial}{\partial x}\left(\sigma\frac{\partial\varphi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\sigma\frac{\partial\varphi}{\partial y}\right)+\tau=0, \quad \ldots\ldots\ldots\ldots \quad (1)$$

in which φ is an unknown function of position and σ and τ are known functions of positions or functions of φ and its derivatives, or both. An important problem of this type is the electric conduction in a plane sheet. It will be used for all explanations in this paper. Then φ is the electric potential, σ the electric conductivity, and τ the current density of external currents entering the sheet. In electrostatic field problems φ is the electric potential, σ is given by $1/(4\pi)$-times the dielectric constant, and τ is the density of the space charge. Equation (**1**) covers also three-dimensional axially symmetrical arrangements if the distance from the axis (the radius) and the distance in the direction of the axis (the height) are dealt with as if they were Cartesian coordinates and the quantities substituted in equation (**1**) for σ and τ are the products of the radius and the actual values of σ and τ.

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The approximate numerical solution of (**1**) by a system of simultaneous linear equations that can be considered the network equations of a system of conductances (and are amenable to relaxation methods) can be described in the following way:

Step 1.—Select a sufficient number of points within and on the boundaries for which the values of φ are to be found. Let these points be called nodes.

Step 2.—Find a linear relation

$$F(\varphi_N, \varphi_R, \varphi_S, \varphi_T, \ldots)=\mathbf{R}_N \quad \ldots\ldots\ldots\ldots \quad (\mathbf{2})$$

between the value φ_N of φ at a node N and the values φ_R, φ_S, φ_T, . . . at neighbouring nodes R, S, T, . . . which when complied with for $\mathbf{R}_N=0$ secures that φ is an approximate solution of equation (**1**). Equation (**2**) takes the form of Kirchhoff's first rule with φ_N, φ_R, etc. denoting potentials. For an arrangement of the nodes as the corners of regular triangles, squares, and hexagons all conductances involved are equal, as Southwell (1946) has shown. The values of the conductances for an arrangement in which the nodes are the corners of irregular triangles can be found by formulae derived by the author (1949) and MacNeal (1953).

Step 3.—Solve the system of linear equations resulting from applying equation (**2**) to all nodes. A very convenient way of finding an approximate solution is Southwell's relaxation method (Motz and Worthy 1945; Southwell 1946; Tasny-Tschiassny 1949). The so-called residuals $\mathbf{R}_N$ are computed for an arbitrarily selected set of values φ. A significant residual $\mathbf{R}_N$, usually the largest residual, is either liquidated or adjusted to a suitable value by changing the value of φ_N by a certain amount. By this the residuals $\mathbf{R}_R$, $\mathbf{R}_S$, $\mathbf{R}_T$, . . . at the neighbouring nodes are altered too, but, in general, the changes are smaller than the change of the residual $\mathbf{R}_N$. Then another important residual is dealt with in the same way. The procedures converge fairly quickly and are continued until negligible values of all residuals are obtained. Instead of solving the system of linear equations numerically, analogues representing actual networks of conductances can be employed.

The boundaries require special artifices in the case of regular nets, because nodes need not necessarily lie on boundaries everywhere. If irregular nets are used, nodes can always be placed on the boundaries and no special problems arise. The errors in the values of φ, i.e. the differences between the values of φ, that comply with the system of linear equations (**2**) for $\mathbf{R}_N=0$ and the values of φ that comply with the differential equation (**1**), are greater for irregular than for regular nets. For this reason and because the number of straight lines simulating a sharply curved part of the boundary must be large, the number of nodes must also be large. This increases the labour in solving the simultaneous equations.

In the present paper we introduce the use of those curvilinear nets in which the mesh contours are parts of circles or parts of circles and straight lines. Basing our derivations on the conformal transformation of a curvilinear into a rectilinear mesh, in Section II methods are developed by which the interior of a "curvilinear rectangle" can be approximated by lumped conductances connected between its corners. This approximation permits the use of different transformation functions for the different meshes of a curvilinear net as long as these functions supply the same curve for the common boundary of adjacent meshes. If a suitable net is laid out and the interior of all meshes replaced by the conductances mentioned, an electrical network results in which the statement of Kirchhoff's first rule supplies the required equation (**2**).

The error occurring when using the described nets appropriately is much smaller than the error involved in nets with straight contours. The additional labour spent in laying out a curvilinear net may be often compensated for by the smaller number of nodes required for the same accuracy. Since it can always be arranged that nodes are on the boundaries, as in a net formed of irregular triangles, no special artifices are required for the boundaries. Contours used in engineering are often composed of parts of circles and straight lines; hence the shape of the boundary can generally be exactly adhered to. In certain types of problems, for instance, the problem of finding the maximum value of the voltage gradient occurring in a material, this is an advantage, because the maximum voltage gradient occurs usually at the boundaries.

II. The Replacement of the Interior of a "Curvilinear Rectangle" by Lumped Conductances

Let

$$w=u+\mathrm{j}v=w(z)=w(x+\mathrm{j}y) \qquad \ldots\ldots\ldots\ldots \quad (\mathbf{3})$$

be an analytical function. Then u and v comply with the Cauchy-Riemann differential equations

$$\left.\begin{aligned} \frac{\partial u}{\partial a}&=\frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y}&=-\frac{\partial v}{\partial x}. \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{4})$$

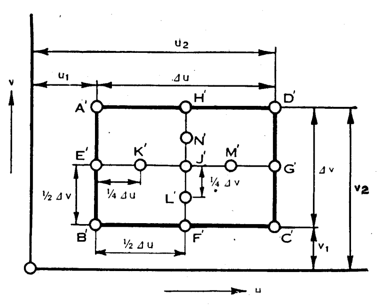

Fig. 1.—Rectilinear rectangle in the w-plane.

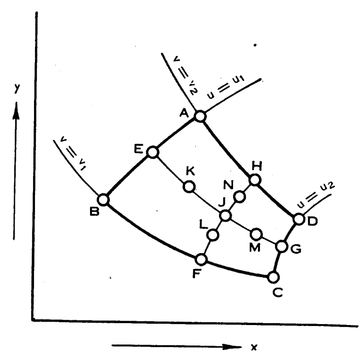

Fig. 2.—Curvilinear rectangle in the z-plane.

A rectangle $A'B'C'D'$ in the w-plane (Fig. 1) with the mid points E', F', G', H' of its sides, its centre J', and the mid points K', L', M', and N' between J' and E', F', G', and H' respectively is the result of a transformation by $w(z)$ of a "curvilinear rectangle" $ABCD$ in the z-plane (Fig. 2) marked correspondingly without primes. Let x and y be the variables appearing in equation (**1**) and let the curvilinear rectangle $ABCD$ (Fig. 2) be a mesh of a curvilinear net with the nodes A, B, C, and D. To replace the interior of this mesh by a network of conductances we proceed in the following way. First we express the difference in potential $(\varphi_A - \varphi_B)$ between the points A and B as an integral taken along the contour BEA, i.e. along the contour defined by $u = u_1$.

$$\varphi_A - \varphi_B = \int_B^A \frac{\partial \varphi}{\partial v} \mathrm{d}v. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

If $\partial\varphi/\partial v$ is expressed by Taylor's expansion about the point E and the integration carried out we obtain

$$\varphi_A - \varphi_B = \left(\frac{\partial \varphi}{\partial v}\right)_E . \Delta v + O(\Delta v^3), \quad \ldots\ldots\ldots\ldots \quad (6)$$

where the subscript E denotes the value at the point E and Δv is given by

$$\Delta v = v_2 - v_1. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

The term $O(\Delta v^3)$ contains $(\partial^3\varphi/\partial v^3)_E$ and higher derivatives of φ.

In the approximation by lumped conductances the current passing within the conducting sheet through the line $EKJLF$ is to be made equal to the current collected at the point B. The current I_{EJ} through the line EKJ is given by

$$I_{EJ} = \int_E^J \sigma\left(\frac{\partial \varphi}{\partial y}\mathrm{d}x - \frac{\partial \varphi}{\partial x}\mathrm{d}y\right). \quad \ldots\ldots\ldots\ldots \quad (8)$$

If $\partial\varphi/\partial x$ and $\partial\varphi/\partial y$ are expressed in terms of $\partial\varphi/\partial u$, $\partial\varphi/\partial v$, $\partial u/\partial x$, $\partial v/\partial x$, $\partial u/\partial y$, and $\partial v/\partial y$, and equations (**4**) are used, expressions for the total differentials $\mathrm{d}u$ and $\mathrm{d}v$ result. The formula

$$I_{EJ} = \int_E^J \sigma\left(\frac{\partial \varphi}{\partial v}\mathrm{d}u - \frac{\partial \varphi}{\partial u}\mathrm{d}v\right) \quad \ldots\ldots\ldots\ldots\ldots \quad (9)$$

is obtained. For the contour EKJ $\mathrm{d}v$ is zero and the second term in the bracket vanishes. With the aid of various expansions according to Taylor's theorem this integral can be approximated on the basis of σ_K, $(\partial\varphi/\partial v)_E$, and $(\partial^2\varphi/\partial u\partial v)_J$, where the subscripts denote the values at the appropriate points. If, further, $(\partial^2\varphi/\partial u\partial v)_J$ is approximated by

$$\left(\frac{\partial^2 \varphi}{\partial u \partial v}\right)_J = \frac{\varphi_B + \varphi_D - \varphi_A - \varphi_C}{\Delta u \Delta v} + O(\Delta v^2) + O(\Delta u^2), \quad \ldots\ldots \quad (10)$$

and the resulting expression for I_{EJ} divided by equation (**6**) we obtain

$$\frac{I_{EJ}}{\varphi_A - \varphi_B} = \sigma_K \frac{\Delta u}{2\Delta v}\left[1 + \tfrac{1}{4}\frac{\varphi_B + \varphi_D - \varphi_A - \varphi_C}{\varphi_A - \varphi_B} + O(\Delta u^2) + O(\Delta v^2)\right]. \quad \ldots \quad (11)$$

The terms $O(\Delta u^2)$ and $O(\Delta v^2)$ contain expressions in σ and φ obtained by at least three differentiations, with respect to u or v. The term $\tfrac{1}{4}(\varphi_B + \varphi_D - \varphi_A - \varphi_C)/(\varphi_A - \varphi_B)$ is $O(\Delta u)$, as can be seen from equations (**6**) and (**10**).

Expressions similar to equation (**11**) can be obtained for $I_{FJ}/(\varphi_C-\varphi_B)$, $I_{GJ}/(\varphi_D-\varphi_C)$, and $I_{HJ}/(\varphi_D-\varphi_A)$. Closer scrutinizing, i.e. computing $(I_{EJ}+I_{FJ})$, $(I_{FJ}+I_{GJ})$, $(I_{GJ}+I_{HJ})$, and $(I_{HJ}+I_{EJ})$, results in two alternative networks of conductances replacing the interior of the curvilinear rectangle *ABCD*. The first alternative (see Fig. 3) neglects errors $O(\Delta u)$ and $O(\Delta v)$. This means that

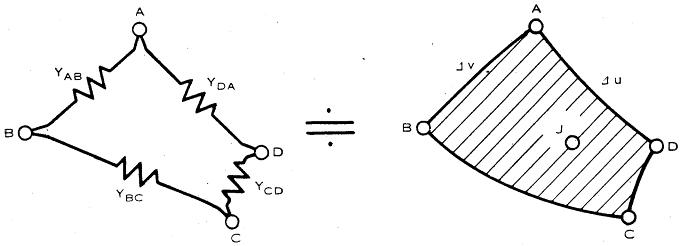

Fig. 3.—Approximate representation of a mesh with a relative error of the order of Δu and Δv.

$$Y_{AB}=Y_{CD}=\tfrac{1}{2}\sigma_J\left|\frac{\Delta u}{\Delta v}\right|,$$

$$Y_{DA}=Y_{BC}=\tfrac{1}{2}\sigma_J\left|\frac{\Delta v}{\Delta u}\right|.$$

terms like $\frac{1}{4}(\varphi_B+\varphi_D-\varphi_A-\varphi_C)/(\varphi_A-\varphi_B)$ in equation (**11**) are neglected and that σ_K, σ_L, σ_M, and σ_N can be replaced by σ_J. Figure 3 gives the details of the network. In the second alternative (Fig. 4) the error is $O(\Delta u^2)$ plus $O(\Delta v^2)$. Terms like $\frac{1}{4}(\varphi_B+\varphi_D-\varphi_A-\varphi_C)/(\varphi_A-\varphi_B)$ in equation (**11**) are retained, but it is

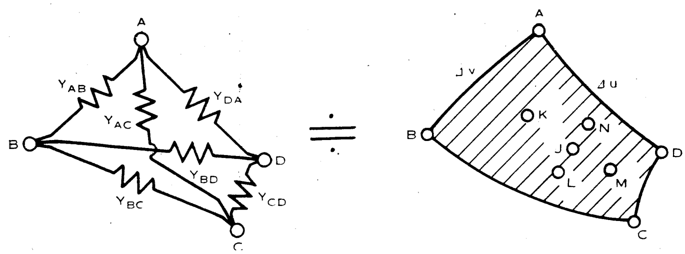

Fig. 4.—Approximate representation of a mesh with a relative error of the order of Δu^2 and Δv^2.

$$Y_{AB}=\tfrac{1}{2}\sigma_K\left|\frac{\Delta u}{\Delta v}\right|-Y_{AC};\quad Y_{CD}=\tfrac{1}{2}\sigma_M\left|\frac{\Delta u}{\Delta v}\right|-Y_{AC};$$

$$Y_{BC}=\tfrac{1}{2}\sigma_L\left|\frac{\Delta v}{\Delta u}\right|-Y_{AC};\quad Y_{DA}=\tfrac{1}{2}\sigma_N\left|\frac{\Delta v}{\Delta u}\right|-Y_{AC};$$

$$Y_{AC}=Y_{BD}=\tfrac{1}{8}\sigma_J\left(\left|\frac{\Delta u}{\Delta v}\right|+\left|\frac{\Delta v}{\Delta u}\right|\right).$$

admissible to replace the multiplying factors σ_K, σ_L, σ_M, and σ_N by σ_J as far as these terms are concerned. In practice one will usually replace σ_K, σ_L, σ_M, and σ_N by σ_J throughout or by values pertaining to one of the corners of the curvilinear square *ABCD*, because the variation of σ with position will not be rapid.

For σ=constant the error vanishes if the lines of constant φ in the w-plane are straight, because then all derivatives of φ with respect to u and v higher than the first vanish and these higher derivatives are multiplying factors in the terms $O(\Delta u^2)$ and $O(\Delta v^2)$ of equations (**10**) and (**11**). In particular this is the case if two opposite sides of the curvilinear rectangle $ABCD$ coincide with lines of constant potential.

The method described requires that a loading of the curvilinear rectangle $BFJE$ by external currents is lumped as an external current applied at the node B. Similarly the loadings of the rectangles $FCGJ$, $JGDH$, and $EJHA$ are concentrated at the nodes C, D, and A respectively. The magnitudes of the concentrated currents can be computed approximately as the products of the areas of the rectangles concerned and mean values of the specific loading $(-\tau)$.

III. Nets in which the Mesh Contours are Parts of Circles and Straight Lines

The results of Section II show that there is no objection to employing different analytical functions for different meshes of the net as long as adjacent contours coincide. In this section it will be shown how curvilinear rectangles bounded by parts of circles or straight lines can be conveniently dealt with.

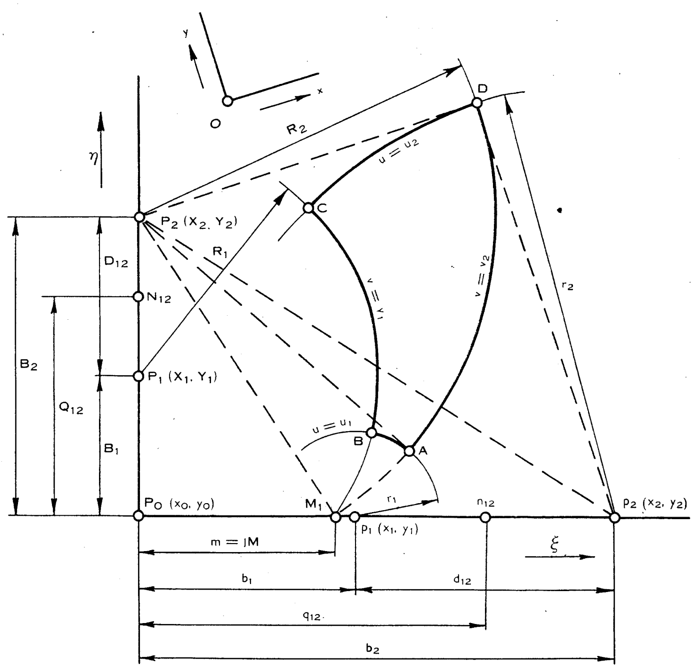

Fig. 5.—Curvilinear rectangle composed of arcs of circles.

(*a*) *The Field Produced by One Source and One Sink*

In a system of Cartesian coordinates ξ, η (Fig. 5) that does not usually coincide with the system of coordinates x, y used in equation (**1**), let the point $M_1(m, 0)$ be a sink and the point $M_2(-m, 0)$, not shown in the diagram, be a source of current, both of the same intensity. The ξ-axis will be called the source axis, the η-axis the sourceless axis, and the origin P_0 the geometric centre. If the intensity of source and sink is appropriately chosen, the analytical function

$$w=\ln\frac{\zeta-m}{\zeta+m} \qquad \text{(12)}$$

of $\zeta=\xi+j\eta$ supplies in its real part

$$u=\tfrac{1}{2}\ln\frac{(\xi-m)^2+\eta^2}{(\xi+m)^2+\eta^2} \qquad \text{(13)}$$

the family of equipotentials (u is the parameter of the family), and in its imaginary part

$$v=\tan^{-1}\frac{\eta}{\xi-m}-\tan^{-1}\frac{\eta}{\xi+m} \qquad \text{(14)}$$

the family of flow functions (v is the parameter of the family) peculiar to this arrangement. By appropriate manipulations on equations (**13**) and (**14**) or by straight-out verification it can be shown that a circle with centre $(b_1, 0)$ and radius r_1 where

$$r_1^2=b_1^2-m^2 \qquad \text{(15)}$$

is an equipotential for a value of the potential

$$u_1=\sinh^{-1}\left(\frac{m}{r_1}\right) \qquad \text{(16)}$$

(and similarly for other subscripts) and that a circle with centre (O, B_1) and radius R_1 where

$$R_1^2=B_1^2+m^2 \qquad \text{(17)}$$

is a flow line for the value of the flow function

$$v_1=\sin^{-1}\left(\frac{m}{R_1}\right) \qquad \text{(18)}$$

(and similarly for other subscripts). If the quantity m, henceforth called the parameter, is given and two pairs of values u_1, u_2 and v_1, v_2 are selected, four circles result from equations (**15**)-(**18**) and determine a curvilinear rectangle *ABCD* (Fig. 5). This rectangle can be replaced by a network of conductances (see Figs. 3 and 4). The quantities Δu and Δv are the absolute values of the differences (u_1-u_2) and (v_1-v_2) respectively.

Equations (**15**) and (**17**) on the one hand and equations (**16**) and (**18**) on the other become identical, if fictitious quantities

$$\left.\begin{aligned} U&=jv,\\ M&=jm \end{aligned}\right\} \qquad \text{(19)}$$

are introduced and in writing down the equations *either* capital letters *or* small letters are used. This symmetry with small letters referring to the u-circles and capital letters referring to the v-circles is useful.

(*b*) *Basic Relations for a Curvilinear Rectangle*

In Figure 5 besides the axes of coordinates ξ and η with the origin (the geometric centre) P_0 the axes of coordinates x and y, referring to equation (**1**) are shown. The coordinates added in parentheses to the individual points refer to the frame (x, y). In the curvilinear rectangle $ABCD$ p_1, p_2 are the centres of the u-circles $u=u_1$, $u=u_2$, and P_1, P_2 the centres of the v-circles $v=v_1$, $v=v_2$. The points n_{12} and N_{12} are the mid points between the respective circle centres. From Pythagoras's theorem applied to the triangles P_2p_2D and $P_2p_2P_0$ we obtain

$$\overline{P_2D}^2=\overline{P_2P_0^2}+\left(q_{12}+\frac{d_{12}}{2}\right)^2-r_2^2, \quad \ldots\ldots\ldots\ldots\ldots \quad (\mathbf{20})$$

where d_{12} is the distance between the centres of the two u-circles and q_{12} the distance between n_{12} and P_0. Similarly we obtain

$$\overline{P_2A}^2=\overline{P_2P_0^2}+\left(q_{12}-\frac{d_{12}}{2}\right)^2-r_1^2. \quad \ldots\ldots\ldots\ldots\ldots \quad (\mathbf{21})$$

Since $\overline{P_2D}=\overline{P_2A}$ we obtain from equations (**20**) and (**21**)

$$q_{12}=\frac{r_2^2-r_1^2}{2d_{12}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{22})$$

It follows from the triangle $M_1P_0P_2$ that

$$\overline{P_2M_1^2}=P_2P_0^2+m^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{23})$$

Since $\overline{P_2D}=\overline{P_2A}=\overline{P_2M_1}$ and

$$\left.\begin{aligned} b_1&=q_{12}-\frac{d_{12}}{2}, \\ b_2&=q_{12}+\frac{d_{12}}{2}, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{24})$$

we obtain from equations (**20**), (**21**), and (**23**) the conditions for orthogonality (see equation (**15**))

$$\left.\begin{aligned} m^2&=b_2^2-r_2^2, \\ m^2&=b_1^2-r_1^2, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{25})$$

and

$$m^2=q_{12}^2+\left(\frac{d_{12}}{2}\right)^2-\frac{r_2^2+r_1^2}{2}. \quad \ldots\ldots\ldots\ldots\ldots \quad (\mathbf{26})$$

Equations (**22**), (**24**), (**25**), and (**26**) refer to the u-circles. It can be easily shown that these equations hold good for the v-circles, if the quantities r_1, r_2, q_{12}, d_{12}, b_1, b_2, and m are replaced by the corresponding capital letter quantities (see Fig. 5 and equation (**19**)).

In Section IV it will be discussed in detail how these relations can be utilized to solve the problems connected with the layout of a curvilinear net. At present it should be pointed out only that the frame (ξ, η) and the value of m are unequivocally determined if two u-circles or two v-circles are given.

(c) *General Points Regarding the Layout of a Net*

It is advisable to work with the same frame (x, y) for the whole net. Figure 6 shows a convenient way of recording in a single figure for the whole net: the centres (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) of four contour circles, their radii r_1, r_2, r_3, r_4, the geometric centre (x_0, y_0), and the parameter m of a mesh $ABCD$ and, with the aid of the arrowed lines starting at the value of m, which circles are the v-circles. If a circle degenerates into a straight line (e.g., the line AF, Fig. 6), the direction tangent a_6 of the line and a point (x_6, y_6) through which it passes are indicated, instead of the position of the centre and the length of the radius which are infinite.

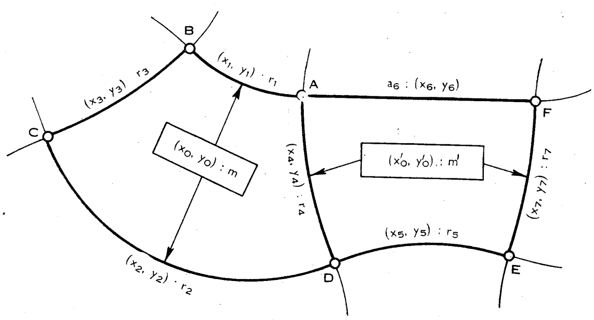

Fig. 6.—Method of recording the characteristic data in a diagram.

If a net consisting entirely of curvilinear rectangles is to be laid out, one tries to follow approximately the direction of the equipotentials and their orthogonal trajectories. If curvilinear meshes are to be employed near the boundaries only, one tries to avoid too abrupt changes of the angle between the directions of the equipotentials and the mesh boundaries. These procedures ensure that the lines of constant φ in the w-plane are only slightly curved (see end of Section II). After forming an idea of the mesh sizes in the individual parts of the field one starts at a boundary and proceeds from mesh to mesh. Thereby problems 1, 2, and 3, to be dealt with in Section IV (*a*), are to be solved in succession. Meshes in which simultaneously two v-circles degenerate into straight lines are dealt with in Section IV (*b*). If the mesh size is to be changed, the method described in Section VI is used. Sharp corners occurring at the boundaries or at the surfaces between dielectrics of different dielectric constants can be included by a procedure described in Section V.

IV. Detailed Procedures for the Layout of a Net

If the accuracy requirements are not very stringent it will suffice for most of the meshes to rely on the drawing and to measure the required dimensions. For some meshes, or if greater accuracy is required, for a considerable number of them computations must replace measurements. It is recommended that computations be carried out in Cartesian coordinates common to all meshes, as mentioned before. The procedures most suitable for use with Cartesian coordinates are described below.

(*a*) *At least One of the* u-*Circles and One of the* v-*Circles do not degenerate into a Straight Line*

Problem 1.—Two u-circles or two v-circles are given. It is not necessarily known whether they are u- or v-circles. Find the geometric centre and the value of the parameter.

Solution.—Refer to Figure 5. Since it is not known whether the circles are u- or v-circles, small letters will be used for the symbols, but the procedure is similar for capital letter symbols.

(1) Find the mid point n_{12} between the two centres of the circles.
(2) Using equation (**22**) find the length q_{12}.
(3) On the line joining the centres of the two circles transfer the length q_{12} from n_{12} to that side on which the centre of the smaller circle lies. This determines the geometric centre P_0.
(4) Using equation (**24**) compute b_1 or b_2 and using equation (**25**) compute m. Alternatively, m can be found directly from equation (**26**). If m is real, the two given circles are u-circles, if m is imaginary, they are v-circles.

If one of the given circles degenerates into a straight line, the geometric centre is found as the intersection of this line with the perpendicular to it through the centre of the non-degenerate circle. The value of the parameter is given by one of the two equations (**25**).

Problem 2.—The geometric centre, the source and sourceless axes, the value of the parameter, and a point are given. Find the u-circle and the v-circle passing through the given point.

Solution.—Refer to Figure 5. Let D be the given point. If either p_2, the centre of the u-circle through D, or P_2, the centre of the v-circle through D are given, draw a perpendicular to the line p_2D (or P_2D) through D and intersect with the axis on which $p_2(P_2)$ does not lie. The point of intersection is the centre of the circle not given. If neither circle through D is given, find the centre of the v-circle through D as the intersection of the bisector of DM_1 (or DM_2) and the sourceless axis.

Problem 3.—One u- and one v-circle, the geometric centre, the two axes, and the value of the parameter are given. Find the two points of intersection, one of which will be used.

Solution.—Refer to Figure 5. Let the given circles be the u_2- and v_2-circle with the centres p_2 and P_2 respectively. The graphical solution is straightforward. Analytically the point D can, often with less labour, be found as the point of intersection of the straight lines p_2D and P_2D. If a is the direction tangent of the line p_2P_2, the direction tangents of the lines p_2D and P_2D equal $\tan[\tan^{-1} a \pm \tan^{-1}(R_2/r_2)]$ and $\tan[\tan^{-1} a \mp \tan^{-1}(r_2/R_2)]$ respectively.

If one of the two circles degenerates into a straight line, let it be called the circle 1. Then the distance from the geometric centre P_0 of the points of intersection between the other circle 2 and this straight line is equal to $(b_2 \pm r_2)$ or $(B_2 \pm R_2)$, as the case may be, positive values being on the side of P_0 on which p_2 or P_2 lies.

b_2 or B_2 is to be computed from equation (**25**).

(*b*) *Both* v-*Circles Degenerate into Straight Lines*

In this case the bundle of v-circles degenerates into a pencil of straight lines and the u-circles are concentric circles with their centre in the point of intersection of the v-lines. If the direction tangents of two v-lines are a_1 and a_2 and the radii of two u-circles R_1 and R_2, we obtain

$$\Delta u = u_2 - u_1 = \ln \frac{R_2}{R_1} = 2 \cdot 30259 \log_{10} \left(\frac{R_2}{R_1}\right), \quad \ldots\ldots\ldots \quad (\mathbf{27})$$

$$\Delta v = v_2 - v_1 = \tan^{-1} a_2 - \tan^{-1} a_1 = \tan^{-1} \frac{a_2 - a_1}{1 + a_1 a_2}. \quad \ldots \quad (\mathbf{28})$$

(*c*) *Notes Regarding the Computation of* Δu *and* Δv

When using equations (**16**) and (**18**) for the computation of u_1, u_2, v_1, v_2 care must be exercised—because both $\sinh^{-1}$ and $\sin^{-1}$ are multivalued functions. The following rules eliminate any possibility of an error in the computation of $|\Delta u/\Delta v|$ and $|\Delta v/\Delta u|$, that is, the quantities required for the computation of the conductances in Figures 3 and 4.

Rule for the Computation of Δu

To find $|\Delta u|$ take the difference of $|u_1|$ and $|u_2|$ if the sourceless line is outside the curvilinear square, and add $|u_1|$ and $|u_2|$ if it passes through it.

Rules for the Computation of Δv

(1) *Definition of " small " and " great " arcs.*—Let that part of the v-circle that lies between the sources M_1 and M_2 and contains the arc considered be drawn (or thought to be drawn). If this part of the circle is greater than a half-circle, viz. if the centre of the circle is within the area defined by the part of the v-circle drawn and the straight line connecting the sources M_1 and M_2, the arc shall be called a " great " arc. If this is not the case the arc shall be called a " small " arc.

(2) If $\mathrm{Sin}^{-1}(m/R_1)$ is the value of $\sin^{-1}(m/R)$ that is between 0 and $\frac{1}{2}\pi$, then

$$v_1 = \sin^{-1}\left(\frac{m}{R_1}\right) = \mathrm{Sin}^{-1}\left(\frac{m}{R_1}\right), \qquad \text{for " small " arcs,}$$

$$v_1 = \sin^{-1}\left(\frac{m}{R_1}\right) = \pi - \mathrm{Sin}^{-1}\left(\frac{m}{R_1}\right), \qquad \text{for " great " arcs.}$$

(3) To find $|\Delta v|$, take the difference of $|v_1|$ and $|v_2|$ if the source line is outside the curvilinear square, and add $|v_1|$ and $|v_2|$ if it passes through it.

V. SHARP CORNERS

Sharp corners may occur at the electrodes and at the border lines of different dielectrics. Usually a sharp corner is formed by two straight lines. If this is not the case it can for a certain distance be approximated by a corner of this type to make the following treatment possible.

In Figure 7 let aOc be the sharp corner of the aperture

$$\alpha = p\pi, \qquad (29)$$

and let Ob be the bisector of α. Let the distance $OA = OB = OC = t$ be conveniently chosen. Let three circles of equal radius R with their centres on the lines Oa, Ob, and Oc be drawn in such a way that they intersect at right angles

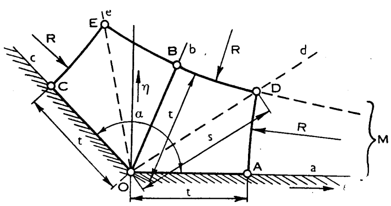

Fig. 7.—Sharp corner in the ζ-plane.

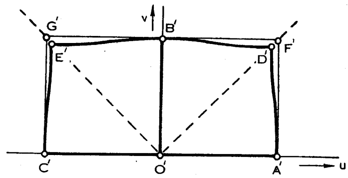

Fig. 8.—Sharp corner in the w-plane.

at the points D and E that lie on the bisectors Od and Oe of the angles aOb and bOc respectively. The analysis of the triangle ODM shows that the radii R of these circles and the distances s of the points D and E from the corner O are given by

$$R = \frac{t \,.\, \sin(p\pi/4)}{1/\sqrt{2} - \sin(p\pi/4)}, \qquad (30)$$

$$s = \frac{t \,.\, \sin[(1-p)\pi/4]}{1/\sqrt{2} - \sin(p\pi/4)}. \qquad (31)$$

The ratios (R/t) and (s/t) for a few typical angles α are contained in Table 1. If a system of Cartesian coordinates (ξ, η) with the origin O and the direction of the ξ-axis coinciding with the direction Oa is chosen (Fig. 7), the configuration of Figure 7 can be conformally transformed into the configuration of Figure 8 (Schwarz-Christoffel transformation) and the transforming function is

$$\zeta = \xi + \mathrm{j}\eta = t\left(\frac{w}{t}\right)^p = t\left(\frac{u+\mathrm{j}v}{t}\right)^p. \qquad (32)$$

The curvilinear square $O'A'D'B'$ deviates only slightly from the rectilinear square $O'A'F'B'$. This is evident from Table 1, in which the ratios

$$\frac{O'D'}{O'F'} = \left(\frac{s}{t}\right)^{1/p} \cdot \frac{1}{\sqrt{2}} \qquad (33)$$

are tabulated for various angles α.

TABLE 1

RATIOS (R/t), (s/t), and $(O'D'/O'F')$ FOR TYPICAL ANGLES α

α	45°	90°	135°	180°	225°	270°	315°
p	0·250	0·500	0·750	1·000	1·250	1·500	1·750
R/t ..	0·381	1·180	3·667	∞	—6·684	—4·262	—3·597
s/t ..	1·085	1·178	1·288	1·414	1·570	1·762	2·035
$O'D'/O'F'$..	0·980	0·982	0·991	1·000	1·014	1·031	1·061

An argument based on the subdivision of the figures $OADB$ and $OBEC$ into (n^2-1) parts which are nearly curvilinear squares (n is an integer) and one figure which is geometrically similar to the original figure, shows that the figures $OADB$ and $OBEC$ can in very good approximation be replaced by the networks of conductances shown in Figures 3 and 4, as if they were curvilinear squares $(\Delta u=\Delta v=t)$ with the corners at O, A, D, B and O, B, E, C respectively.

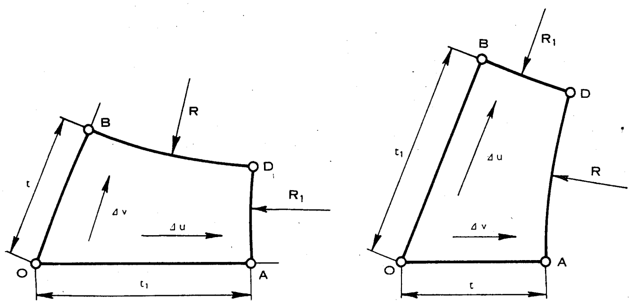

Fig. 9.—Curvilinear rectangle with a sharp corner, $t_1>t$.

It is sometimes convenient to make the lengths OA and OB slightly different. In this case Figures 3 and 4 can still be used, if the following specifications referring to Figure 9 for $t_1>t$ are adhered to, in which the directions of Δu and Δv are indicated.

R is given by equation (**30**).

$$R_1=\frac{(t-t_1)[(t-t_1)/\sqrt{2}+(t+t_1)\sin(p\pi/4)]+2\sqrt{2}\,t\,t_1\sin^2(p\pi/4)}{2[1/\sqrt{2}-\sin(p\pi/4)]\,.\,[t\,.\,\sqrt{2}\,.\,\sin(p\pi/4)+(t-t_1)]}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{34})$$

$$\left.\begin{aligned}\frac{\Delta u}{\Delta v}&=\left(\frac{t_1}{t}\right)^{1/p},\\ \frac{\Delta v}{\Delta u}&=\left(\frac{t}{t_1}\right)^{1/p}.\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (\mathbf{35})$$

Equation (**34**) is derived from the condition that for given values of t and t_1 and for R given by equation (**30**), the circles of radii R and R_1 intersect at right angles. Equations (**35**) are the consequence of the transformation equation (**32**).

VI. Change of the Mesh Size

In parts of the field in which the field gradient is smaller and does not change rapidly an increase of the mesh size reduces the labour considerably without affecting the degree of accuracy. If in Figure 10 the circle which passes through the points A, B, and C forms the boundary AC in the curvilinear rectangle $ACDE$ and the boundary CB in the curvilinear rectangle $CBFD$—which can be arranged for in the layout of the net—the node C can be eliminated in the

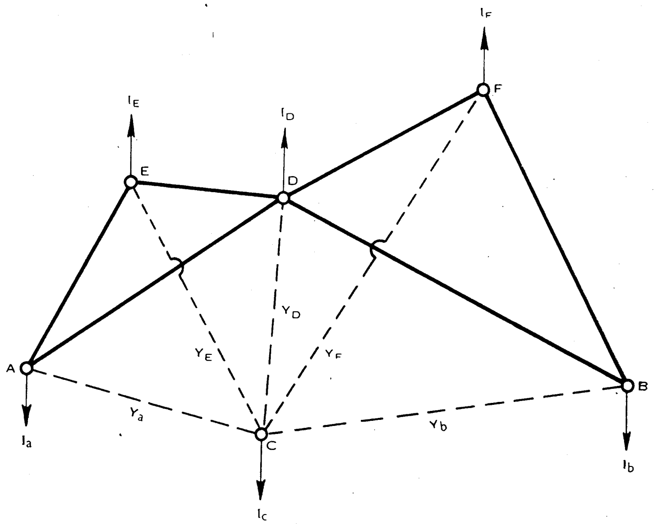

Fig. 10.—Part of a net with a node C not yet eliminated.

following way. Prerequisites are that, in the neighbourhood of C, σ does not vary very much and the equipotentials and field lines in the w-plane are nearly straight. Then we can assume that approximately

$$\varphi_c = k_a\varphi_a + k_b\varphi_b, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (36)$$

where

$$\left.\begin{aligned} k_a &= \frac{Y_a}{Y_a+Y_b}, \\ k_b &= \frac{Y_b}{Y_a+Y_b}, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (37)$$

and where Y_a and Y_b are the conductances connecting the nodes CA and CB respectively. Let us split each of the conductances Y_E, Y_D, and Y_F that connect the nodes E, D, and F respectively and C (Fig. 10) into two parallel conductances (k_aY_E), (k_aY_D), (k_aY_F) leading to node A, and (k_bY_E), (k_bY_D), (k_bY_F) leading to node B respectively (Fig. 11). Let us, further, connect the nodes A and B by a conductance equal to the series combination of Y_a and Y_b. The broken lines in Figures 10 and 11 are the conductances that are affected by this procedure and the full lines are those that are not. If the potentials φ_A, φ_B, φ_D, φ_E, φ_F of the nodes A, B, D, E, and F, and the external node currents I_A, I_B, I_D, I_E, and I_F at these nodes (Figs. 10 and 11) respectively are assumed to be equal in pairs, the analyses of Figures 10 and 11 show that the current I_C appearing in Figure 10 is split into two parts $(k_a I_C)$ and $(k_b I_C)$ loading additionally the nodes A and B as shown in Figure 11. Since this way of accounting for the current I_C is reasonable, the given method for the elimination of the node C is sound.

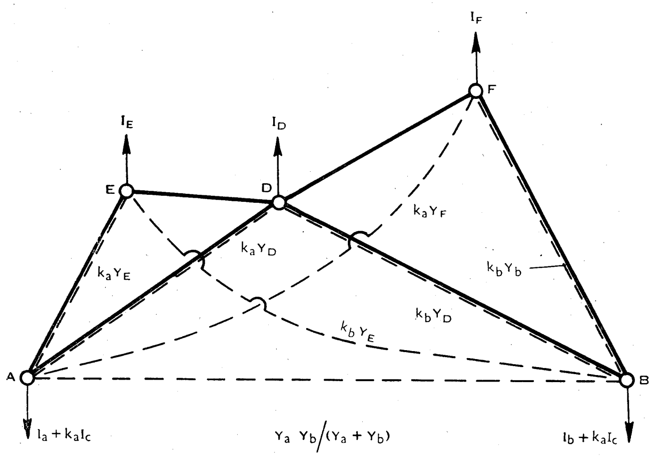

Fig. 11.—Part of the net of Figure 10 with the node C eliminated.

VII. The Computation of the Potential Gradients

(a) *Curvilinear Squares*

If the values φ_A, φ_B, φ_C, and φ_D of the potentials at the four corners of a curvilinear rectangle $ABCD$ (Fig. 2) are given, the potential gradient at any point that is not outside this rectangle can be computed with the aid of the formulae

$$\left.\begin{aligned} g_u&=\frac{\partial\varphi}{\partial u}\cdot\left|\frac{\mathrm{d}w}{\mathrm{d}\zeta}\right|,\\ g_v&=\frac{\partial\varphi}{\partial v}\cdot\left|\frac{\mathrm{d}w}{\mathrm{d}\zeta}\right|. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (38)$$

In equations (**38**) g_u and g_v are the potential gradients in the directions of the orthogonal lines $v=$constant and $u=$constant respectively, so that the magnitude g of the gradient is given by

$$g^2=g_u^2+g_v^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (39)$$

If the field produced by φ in the w-plane is nearly uniform, the value $\partial\varphi/\partial u$ can be interpolated from the values $(\partial\varphi/\partial u)_{AD}$ and $(\partial\varphi/\partial u)_{BC}$ where $(\partial\varphi/\partial u)_{AD}$ can be approximated by

$$\left(\frac{\partial\varphi}{\partial u}\right)_{AD}=\frac{\varphi_D-\varphi_A}{\Delta u}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (40)$$

and similarly for $(\partial\varphi/\partial u)_{BC}$ and $\partial\varphi/\partial v$. For $|\,\mathrm{d}w/\mathrm{d}\zeta\,|$ we find by differentiation of (**12**) after some manipulations

$$\left|\frac{\mathrm{d}w}{\mathrm{d}\zeta}\right|=\frac{2m}{\sqrt{\{(\rho^2-m^2)^2+4m^2\eta^2\}}}, \quad \ldots\ldots\ldots\ldots \quad (41)$$

where $\rho=|\,\zeta\,|$ is the distance of the point considered from the geometric centre, η its distance from the source line, and m the parameter.

The use of equation (**40**) leads to errors, if the field of φ in the w-plane is considerably curved. For practical work the case of importance is that one for which one of the two lines of constant φ, say the line $\varphi=\varphi_D$, is a straight line of constant u or v in the w-plane, corresponding to a boundary equipotential in the z-plane, and the other line of constant φ, that is, the line $\varphi=\varphi_A$, can be approximated by a circle. This is shown in Figures 12 (a) and (b) with the notations that correspond to the boundary equipotential in the z-plane being a line of constant v.

If coordinates ξ and η are introduced equalling u and v (Fig. 12 (a)) or $(-v)$ and u (Fig. 12 (b)) respectively, the field in the w-plane corresponding to Figure 12 (a) or 12 (b) can be assumed to be produced by a source and sink of equal intensities as was explained in Section III (a). If the intensity of the source and sink is C, then φ is given for Figure 12 (a) by the right-hand side of equation (**13**) multiplied by C, and for Figure 12 (b) by the right-hand side of

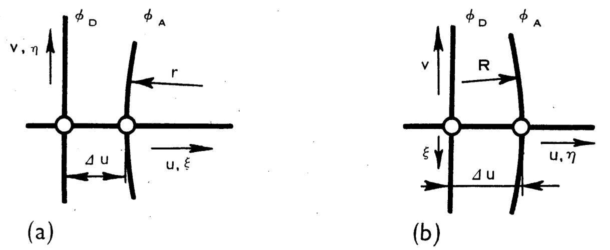

Fig. 12.—Equipotentials in the w-plane. (a) Case 1 ; (b) case 2.

equation (**14**) multiplied by C. A pair of these C-multiplied equations (**13**) or (**14**) written down for two pairs of values C, m and C', m' and equated for $\xi=\Delta u$, $\eta=0$ (Fig. 12 (a)) or $\xi=0$, $\eta=\Delta u$ (Fig. 12 (b)) gives a relation between C, m, C', and m' for a fixed distance Δu and a fixed potential φ_A, but the equipotential lines φ_A are of different curvatures. If the C-multiplied right-hand side of equation (**13**) is differentiated with respect to ξ and ξ made equal to zero, and the C-multiplied right-hand side of equation (**14**) differentiated with respect to η and η made equal to zero, expressions for $\partial\varphi/\partial u$ along the equipotential φ_D result. The ratio c of $\partial\varphi/\partial u$ for a given value of m, to $(\partial\varphi/\partial u)'$ for $m'\to\infty$ can be computed and determines the increase or decrease of $\partial\varphi/\partial u$ compared with the case of a uniform field. If C is eliminated by using the relation between C, m, C', and m', C' cancels out and we obtain, after expressing m in terms of the radius of curvature r or R with the aid of equation (**15**) or (**17**) and an obvious relation eliminating b or B with the aid of Δu :

For Figure 12 (a)

$$c=\frac{\partial\varphi}{\partial u}\Bigg/\left(\frac{\partial\varphi}{\partial u}\right)'=\frac{m^2}{m^2+\eta^2}\cdot c_0, \quad \ldots\ldots\ldots\ldots \quad (42)$$

$$c_0=\frac{\Delta u/m}{\tanh^{-1}(\Delta u/m)}, \quad \ldots\ldots\ldots \quad (43)$$

$$\frac{\Delta u}{m}=\sqrt{\left(\frac{\Delta u}{2r+\Delta u}\right)}. \quad \ldots\ldots\ldots\ldots \quad (44)$$

For Figure 12 (*b*)

$$c = \frac{\partial\varphi}{\partial u} \bigg/ \left(\frac{\partial\varphi}{\partial u}\right)' = \frac{m^2}{m^2 - \xi^2} \cdot c_0, \quad \ldots\ldots\ldots\ldots \quad (45)$$

$$c_0 = \frac{\Delta u/m}{\tan^{-1}(\Delta u/m)}, \quad \ldots\ldots\ldots\ldots \quad (46)$$

$$\frac{\Delta u}{m} = \sqrt{\left(\frac{\Delta u}{2R - \Delta u}\right)}. \quad \ldots\ldots\ldots\ldots \quad (47)$$

c_0 is the ratio $(\partial\varphi/\partial u)/(\partial\varphi/\partial u)'$ at the line of symmetry of Figures 12 (*a*) and 12 (*b*) and is the value sought. $(\partial\varphi/\partial u)'$ is the value given by equation (**40**).

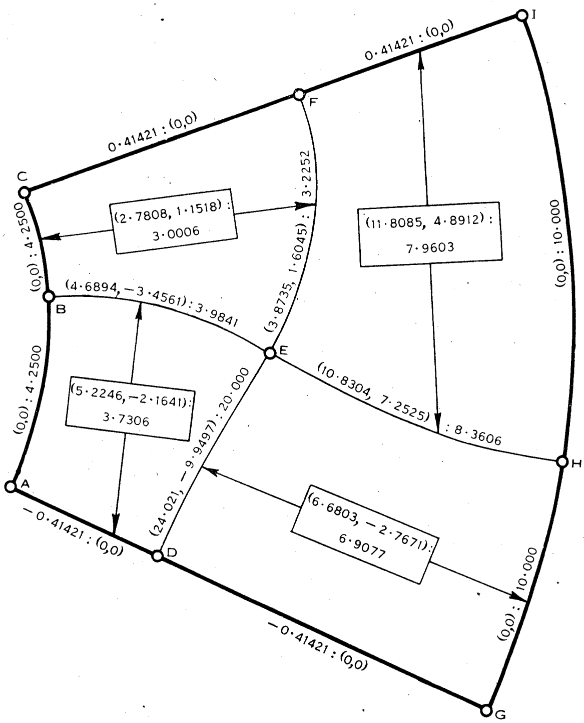

Fig. 13.—Example VIII (*a*). Field between two concentric circles. Coordinates of points :

A	(3·9265,	—1·6264)
B	(4·2205,	0·50026)
C	(3·9265,	1·6264)
D	(5·5433,	—2·2961)
E	(6·6713,	0)
F	(6·8532,	2·8387)
G	(9·2388,	—3·8268)
H	(9·9437,	—1·0609)
I	(9·2388,	3·8268)

(*b*) *Squares with a Sharp Corner*

For $\alpha > \pi$ the voltage gradient at the corner itself is infinite, but, as Cohn and Vogel (1953) have emphasized, its value at a short distance from the corner has significance in high voltage engineering. The voltage gradient at the corner for $\alpha < \pi$ is zero and its value in the neighbourhood of the corner is of little interest.

To compute the voltage gradient for a point at the distance ρ from the corner equations (**38**) and (**39**) are applied again, but $|\,dw/d\zeta\,|$ is given by

$$\left|\frac{dw}{d\zeta}\right| = \frac{1}{p} \cdot \left(\frac{t}{\rho}\right)^{(p-1)/p} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{48})$$

Equation (**48**) is obtained from equation (**32**) by differentiation. For Δu and Δv the values t_1 and t respectively are taken (Fig. 9). The maximum gradient at the distance ρ from a corner of conducting material occurs for $\alpha > \pi$ on the bisector of the angle α and is directed from the corner. Since this is the only value of interest, no additional work regarding the directions of the lines u=constant and v=constant arises in this case.

VIII. Examples

Two simple examples without current loading ($\tau=0$) will demonstrate the power of the use of curvilinear nets. For the conductivity σ the value $\sigma=1$ is assumed. All numerical values given were computed with a computational accuracy to five digits. This accuracy is unnecessary, unless wanted for the purpose of comparison.

(*a*) *Field between Two Concentric Circles*

Figure 13 shows a portion of a sector of 45° aperture bounded by two concentric circle arcs *ABC* and *GHI* of radii 4·25 and 10 respectively as equipotentials and by two radii *AG* and *CI* as flow lines. This arrangement can easily be analysed by well-known formulae. We start arbitrarily at the point *D* at a distance 1·75 from the point *A* and, to simulate unfavourable conditions, select as the mesh boundary *DE* a circle arc of radius 20 with its centre on the line *AD* produced beyond the point *D*. The point *E* where the mesh boundary ends is the point of its intersection with the bisector of the sector. These assumptions determine a net of four meshes unequivocally. The computed characteristic data of the net are contained in Figure 13, the origin of the Cartesian frame used being the centre of the sector and the x-axis its bisector through *E*.

The values of u_1, u_2, Δu, v_1, v_2, Δv, $\Delta u/\Delta v$, and $\Delta v/\Delta u$ resulting for the individual meshes are shown in Table 2.

Table 2

Values of u_1, u_2, Δu, v_1, v_2, Δv, $\Delta u/\Delta v$, and $\Delta v/\Delta u$ for the meshes of Figure 13

Mesh	*ABED*	*BCFE*	*DEHG*	*EFIH*
u_1	0·7923	0	0	1·6353
u_2	—0·1855	0·6956	0·7530	0·7296
Δu	0·9778	0·6956	0·7530	0·9057
v_1	0	0·7839	0·3526	0
v_2	1·2121	1·9463	—0·7626	1·2601
Δv	1·2121	1·1624	1·1152	1·2601
$\Delta u/\Delta v$	0·8067	0·5984	0·6752	0·7188
$\Delta v/\Delta u$	1·240	1·6720	1·4810	1·3912

If the scheme of Figure 4 is taken as the basis, and if the individual conductances are computed and all parallel conductances between two nodes are lumped, the network of Figure 14 results. The potentials φ_D, φ_E, and φ_F of the points D, E, and F were taken as unknowns and the potential of the node (A-B-C) was set equal to 1 and that of the node (G-H-I) set equal to zero. The results of the computations are given below and in the brackets are added the theoretically correct values and the per cent. deviations from them. Solving

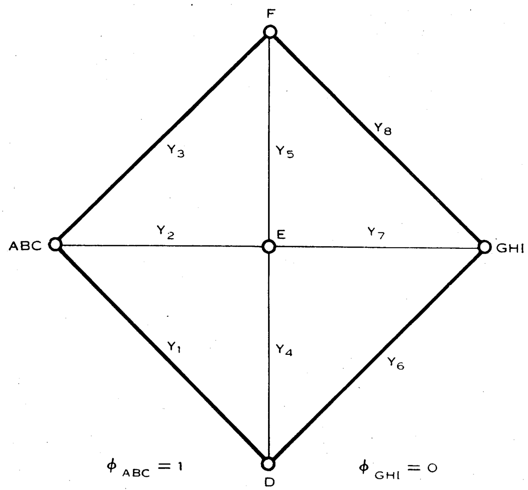

Fig. 14.—Values of the final conductances in examples VIII (*a*) and (*b*).

Conductance	Example VIII (*a*)	Example VIII (*b*)
Y_1	0·6200	0·6036
Y_2	0·9192	1·1036
Y_3	0·2992	0·5000
Y_4	0·6185	0·0550
Y_5	0·6478	0·5000
Y_6	0·3376	1·1004
Y_7	1·0332	1·6004
Y_8	0·6956	0·5000

three simultaneous linear equations expressing Kirchhoff's first rule leads to $\varphi_D=0{\cdot}5781$ $(0{\cdot}5970,\ -3{\cdot}2\%)$, $\varphi_E=0{\cdot}4707$ $(0{\cdot}4730,\ -0{\cdot}57\%)$, $\varphi_F=0{\cdot}3678$ $(0{\cdot}3491,\ +5{\cdot}4\%)$. It is somewhat misleading to consider the per cent. deviations from the actual values of the potentials, the value $1-\varphi_F=0{\cdot}6322$ $(0{\cdot}6509,\ -2{\cdot}9\%)$ is more important, for instance, than φ_F. If the deviations are referred to the potential difference between the two outer electrodes the percentages are much smaller. The total conductance between the arcs ABC and GHI has the value 0·9373 (0·9179, +2·1%). Hence the error in the computation of the capacity would be +2·1%. With the aid of the formulae (**38**), (**40**), and (**41**) the approximate values of the gradients g_A at A and g_C at C were computed. The results are: g_A=0·2696 (0·2750, −2·0%) and g_C=0·3097 (0·2750, +12·6%). Correction factors c_0 can be computed, for the node A with the

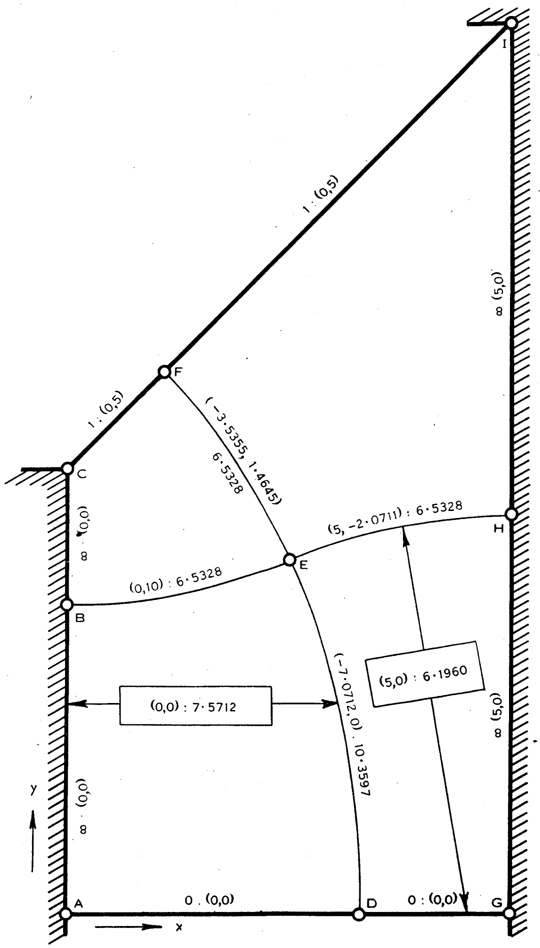

Fig. 15.—Example VIII (*b*). Field in a coaxial square cable. Coordinates of points:

A (0, 0)	F (1·0839, 6·0839)
B (0, 3·4672)	G (5, 0)
C (0, 5)	H (5, 4·4617)
D (3·2885, 0)	I (5, 10)
E (2·5, 3·9645)	

aid of equations (**46**) and (**47**), and for the node C with the aid of equations (**43**) and (**44**). The radius R required in equation (**47**) was ascertained by finding by linear interpolation in the w-plane the point of potential φ_D on the line $B'E'$. For the radius r required for equation (**44**) the point of potential

φ_E was similarly found on the line CF. We obtain $R=3 \cdot 802$ and $r=1 \cdot 3733$, $c_{0A}=1 \cdot 0474$, $c_{0C}=0 \cdot 9057$ and for the corrected values of the gradients $c_{0A} \cdot g_A=0 \cdot 2824$ $(0 \cdot 2750, +2 \cdot 7\%)$ and $c_{0C} \cdot g_C=0 \cdot 2805$ $(0 \cdot 2750, +2 \cdot 0\%)$.

Considering that the net consists of four meshes only and that the mesh boundaries deviate appreciably from the equipotentials and flow lines, all deviations from the correct values are surprisingly small.

(b) *Field in a Square Coaxial Cable*

A square conductor of side length 10 is surrounded by a square conducting sheath of side length 20 in an arrangement in which four axes of symmetry exist. This numerical problem has been dealt with by Woods (1953) and others. Figure 15 shows an eighth of the arrangement, AC being the considered portion of the inner equipotential, GI that of the outer equipotential, and AG and CI being flow lines. C is a sharp corner of 270° aperture and I one of 90° aperture. A is the origin of the selected Cartesian frame and AG its x-axis.

The condition that a net of four meshes be used determines the meshes unequivocally, the point E being the intersection of the (not drawn) bisectors of the angles ACI and CIG. Figure 15 contains the characteristic data of the net and Table 3 the values characteristic of the meshes.

Table 3

Values of u_1, u_2, Δu, v_1, v_2, Δv, $\Delta u/\Delta v$ and $\Delta v/\Delta u$ for the meshes of Figure 15

Mesh	*ABED*	*BCFE*	*DEHG*	*EFIH*
u_1	0	0	0	0
$u_2=\Delta u$	0·9894	1	0·5672	1
v_1	0	0	0	0
$v_2=\Delta v$	0·8195	1	1·2483	1
$\Delta u/\Delta v$	1·2073	1	0·4544	1
$\Delta v/\Delta u$	0·8283	1	2·2008	1

The final resulting network of conductances is shown in Figure 14. The result of the analysis together with the figures ascertained by Woods (1953) or found from them by interpolation and the deviations from these figures that can be considered correct are given in the following:

$$\varphi_D=0 \cdot 3562 \quad (0 \cdot 3326, \ +7 \cdot 0\%),$$
$$\varphi_E=0 \cdot 4171 \quad (0 \cdot 4330, \ -3 \cdot 7\%),$$
$$\varphi_F=0 \cdot 4724 \quad (0 \cdot 4950, \ -4 \cdot 6\%).$$

The total conductance is $1 \cdot 2957$ $(1 \cdot 2791, +1 \cdot 6\%)$.

As in example VIII (*a*) the results are surprisingly good.

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