SHARPENING OF OBSERVATIONAL DATA IN TWO DIMENSIONS

By E. J. Burr*

[*Manuscript received August* 20, 1954]

Summary

Distributions in two dimensions as measured are always blurred or smoothed by limitations in the observing technique. Recovery of the true distribution involves the solution of an integral equation of the form

$$g(x', y')=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x, y)f(x'-x, y'-y)\mathrm{d}x\,\mathrm{d}y,$$

where the functions g, h are known from observation.

In this paper the uniqueness and stability of the solution are discussed, and the validity and usefulness of several methods of solution are examined. A new technique for the application of polynomial solutions is presented. It is suggested that in some cases the use of Fourier transforms or Fourier-Bessel transforms may be practicable, and tables of selected transforms are supplied. Finally, conditions for convergence of certain sequences of approximate solutions are given. The effect of random errors in the observational data is taken into account wherever possible.

I. Introduction

Astronomical observations are sometimes concerned with distributions of surface intensity or statistical frequency distributions in two dimensions. Since it is not possible to measure a surface intensity or a frequency at a mathematical point, the limitations of the observational method always produce a smoothing or averaging effect on the true distribution, lowering and broadening the peaks, and reducing the amplitude of the fluctuations. The problem of recovering the true distribution from the smoothed distribution which we observe may be called " sharpening " of the observational data, and involves the solution of a linear integral equation.

The corresponding problem in one dimension has been frequently discussed, and it is recognized that the methods of solving this problem can in principle be extended to two dimensions, but this has rarely been done in practice. A solution in the form of a double series has been given by Coutrez (1949), but this does not appear to have been used. Kapteyn (1920) has found a special solution for use in the statistics of stellar total proper motions. Kreisel (1949) has described a method of partial sharpening which has been applied to a problem in gravity survey. Bolton and Westfold (1950) have solved a problem in radio astronomy with the aid of an iterative method described in one dimension by Burger and van Cittert (1932, 1933). A formal solution in terms of Fourier

* Commonwealth Observatory, Canberra, A.C.T.; present address: 23 Ruskin Street, Taringa, Brisbane, Qld.

transforms can be written down, but does not appear to have been used. Solutions by " trial and error " lead to uncertain results because of the indeterminacy caused by random errors of observation.

The object of this paper is to examine some of the methods of solving the integral equation in two dimensions, and devise means of applying them to numerical problems with the minimum of computational labour. The effect of random errors in the data is taken into account wherever possible.

II. The Integral Equation

We denote the true distribution by $f(x, y)$ where (x, y) are rectangular Cartesian coordinates. The observed value at any point (x', y') is a weighted mean of the values of $f(x, y)$ in some neighbourhood of (x', y'), and we denote it by $g(x', y')$. Now suppose that, when the true distribution consists only of a point concentration of unit strength at (x_0, y_0), the observed distribution is $h(x'-x_0, y'-y_0)$. Then the functions f, g, h are connected by the integral equation

$$g(x', y') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x'-x, y'-y)f(x, y)\mathrm{d}x\mathrm{d}y,$$

or

$$g(x', y') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x, y)f(x'-x, y'-y)\mathrm{d}x\mathrm{d}y. \qquad \ldots\ (\mathbf{2.1})$$

We shall call $h(x, y)$ the kernel, but in various contexts it may also be called the instrumental function, distribution of errors, or polar diagram. Equation (**2.1**) may be expressed in words by saying that $g(x, y)$ is the convolution, resultant, Faltung, obliteration, or smooth of $f(x, y)$ and $h(x, y)$. The process of recovering $f(x, y)$ from a knowledge of $g(x, y)$ and $h(x, y)$ may be called sharpening, restoration, or correction of the observed distribution $g(x, y)$.

We say that a function is L, or belongs to the class L, when its (Lebesgue) integral over the infinite plane converges absolutely. It is assumed throughout this paper that the kernel $h(x, y)$ is L. Then if $f(x, y)$ is bounded, and integrable in every finite region, the integral in (**2.1**) converges absolutely and uniformly, and $g(x', y')$ is bounded and continuous.

It is usual to *normalize* the kernel by multiplying it by a constant factor in order to make its total surface integral equal to unity. When this has been done, the total surface integrals of f and g are equal, provided that either of these functions is L. Loosely speaking, the two distributions f, g then differ in shape, but not in scale.

Equation (**2.1**) has the useful property that it is invariant under a homogeneous linear transformation of unit modulus. That is, if we write

$$\begin{aligned} x &= aX + bY, & ad-bc &= 1, \\ x' &= aX' + bY', & f(x, y) &\equiv f(aX+bY, cX+dY) \equiv \bar{f}(X, Y), \\ y &= cX + dY, & g(x, y) &\equiv \bar{g}(X, Y), \\ y' &= cX' + dY', & h(x, y) &\equiv \bar{h}(X, Y), \end{aligned}$$

then equation (**2.1**) reduces to

$$\bar{g}(X', Y') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{h}(X, Y)\bar{f}(X'-X, Y'-Y)\mathrm{d}X\mathrm{d}Y.$$

This property enables particular solutions to be generalized, and certain problems to be simplified. For example, a problem in which one of the functions is a function of $(x^2/a^2+y^2/b^2)$ only can be reduced to a problem in which this function has circular symmetry.

In astronomical applications, kernels of several different types occur in the following ways.

(i) In photoelectric photometry, an extended source of light may be scanned by a photocell behind a circular aperture of radius a in the focal plane of a telescope. f, g are the true and observed distributions of surface brightness, and the normalized kernel is

$$\left.\begin{aligned} h(x, y) &= 1/\pi a^2, \quad x^2+y^2<a^2, \\ &= 0, \quad x^2+y^2>a^2, \end{aligned}\right\} \quad \ldots\ldots\ldots \text{(2.2)}$$

provided that a is large compared with the effective resolving power of the telescope.

(ii) In photographic photometry, f is as in (i), and g is deduced from microphotometric tracings of a photograph of the source. The kernel is obtained from a stellar image by the same process. Many factors contribute to the smoothing process, the three most important being atmospheric turbulence, imperfect guiding of the telescope, and the finite size of the scanning " spot " in the microphotometer. As an example, measurements by de Vaucouleurs (1948) at the Cassegrain focus of a 32-in. reflector gave a kernel which is reasonably well represented by

$$\left.\begin{aligned} h(x, y) &= A\mathrm{e}^{-\frac{1}{2}(x^2/a^2+y^2/b^2)} \simeq A\mathrm{e}^{-\frac{1}{2}r^2/c^2}, \quad r<2\cdot5, \\ &= 5\cdot2A/r^4, \quad r>2\cdot5, \end{aligned}\right\} \quad \ldots\ldots \text{(2.3)}$$

where $r^2=x^2+y^2$, the unit of measurement is 1 sec of arc, $a=1\cdot15$, $b=1\cdot35$, $c=1\cdot25$, and $A=0\cdot0904$. Equation (**2.1**) is strictly applicable only when stars in different parts of the field lead to the same kernel.

(iii) In radio astronomy, f, g are the true and observed distributions of surface intensity of an extended radio source. The kernel may be found by scanning a point source with the radio telescope, but this is often not practicable, and the theoretical diffraction pattern (polar diagram) of the telescope is then used. For example, for a circular aperture of diameter D, the diffraction pattern is

$$h(x, y) = \frac{A^2}{4\pi}\left[\frac{2}{Ar}J_1(Ar)\right]^2, \quad A=\frac{\pi D}{\lambda}, \quad \ldots\ldots\ldots \text{(2.4)}$$

where λ is the wavelength of the radiation, $r^2=x^2+y^2$, and r is measured in radians. (This formula is only accurate for $r \ll 1$.) This kernel, in common with all diffraction patterns due to apertures of finite dimensions, has the important property that its Fourier transform vanishes outside a finite region.

(iv) In statistics, we take as a typical example the statistics of stellar proper motions. The number of stars in a given region whose true components of annual proper motion lie in the range $(x, x+\mathrm{d}x\,;\ y, y+\mathrm{d}y)$ is $f(x, y)\mathrm{d}x\mathrm{d}y$. The measures of (x, y) for individual stars are subject to random errors whose frequency distribution may be taken as

$$h_1(x, y)=(2\pi a^2)^{-1}\mathrm{e}^{-\frac{1}{2}r^2/a^2}. \quad \ldots\ldots\ldots\ldots \quad (2.5)$$

The observed frequency distribution is smoothed by (**2.5**) and also by the necessity for taking samples over finite regions of the (x, y) plane. If $g(x', y')$ is taken as the number of stars with observed proper motions (x, y) in the range $|\,x-x'\,|\leqslant b$, $|\,y-y'\,|\leqslant c$, then $g(x', y')$ is given by (**2.1**) with a kernel obtained from the convolution of (**2.5**) with

$$\left.\begin{aligned} h_2(x, y)&=(4bc)^{-1}, && |\,x\,|<b,\ |\,y\,|<c\\ &=0, && \text{elsewhere.}\end{aligned}\right\} \quad \ldots\ldots \quad (2.6)$$

We may also mention a problem in gravity survey (Kreisel 1949) in which the kernel is the normal gravitational field over a horizontal plane due to unit mass at depth a below the origin, which is

$$h(x, y)=Ga(a^2+r^2)^{-3/2}. \quad \ldots\ldots\ldots\ldots\ldots \quad (2.7)$$

In all these applications the distributions are, strictly speaking, over a sphere, not a plane. But in nearly every case the angular diameter of the region with which we have to deal is so small that the region may be regarded as plane without sensible error.

III. The Formal Solution

Any function $f(x, y)$ of L has a double Fourier transform (Bochner 1932)

$$F(u, v)=(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\mathrm{e}^{\mathrm{i}(ux+vy)}\mathrm{d}x\mathrm{d}y, \quad \ldots\ldots \quad (3.1)$$

which is bounded and continuous and tends to zero as $u^2+v^2\to\infty$. The inverse relation

$$f(x, y)=(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u, v)\mathrm{e}^{-\mathrm{i}(ux+vy)}\mathrm{d}u\mathrm{d}v \quad \ldots\ldots \quad (3.2)$$

holds wherever $f(x, y)$ is continuous; provided that, if the integral in (**3.2**) does not converge absolutely, we interpret it as the limit as $p, q\to\infty$ of the integral over $|\,u\,|<p$, $|\,v\,|<q$. If two functions belonging to L have the same Fourier transform, they are equal almost everywhere, so that, if they represent distributions of a physical quantity, these distributions are identical.

Now let $f(x, y)$, $h(x, y)$ be L; then $g(x, y)$ of equation (**2.1**) is also L, and, if we denote Fourier transforms by capital letters, the Faltung theorem (Bochner 1932, §43.6) gives

$$G(u, v)=2\pi H(u, v)F(u, v). \quad \ldots\ldots\ldots\ldots \quad (3.3)$$

This leads at once to a formal solution in the form (**3.2**), in which

$$F(u, v)=\frac{G(u, v)}{2\pi H(u, v)}. \quad \ldots\ldots\ldots\ldots\ldots \quad (3.4)$$

Now suppose that g, h are known from observation, and belong to L. Then we can (in principle at least) compute their Fourier transforms and form the quotient (**3.4**). If $|H(u, v)|$ is everywhere positive, as is the case with kernels such as (**2.5**) and (**2.7**), then (**3.4**) is defined everywhere, and, if this function has a Fourier transform belonging to L, the solution exists and is unique. But if $H(u, v)$ vanishes at certain real points, then $G(u, v)$ ought also to vanish at these points and (**3.4**) is undefined at such points. (Note that we are concerned only with real values of (u, v), although F, G, H are in general complex functions of these variables.) Denote by R the region or point set in which $H(u, v)=0$. We may arbitrarily set $F(u, v)=0$ in R, and, if $F(u, v)$ then has a Fourier transform, this transform is a possible solution of the integral equation, which Bracewell and Roberts (1954) call the *principal solution*. Other solutions may be obtained by adding to the principal solution functions of the form

$$\iint_R \varphi(u, v)\mathrm{e}^{-\mathrm{i}(ux+vy)}\mathrm{d}u\mathrm{d}v, \quad \ldots\ldots\ldots\ldots \quad (\mathbf{3.5})$$

where $\varphi(u, v)$ is an arbitrary function. If R is a set of zero measure, as is the case with kernels such as (**2.2**) and (**2.6**), then (**3.5**) vanishes unless $\varphi(u, v)$ is an improper function, and in many applications the indeterminacy is then somewhat trivial. But if R has positive measure the indeterminacy is much more serious. This occurs with kernels like (**2.4**) which are of special importance in radio astronomy, and the problems raised by this indeterminacy have been fully discussed in the one-dimensional case by Bracewell and Roberts (1954). Their discussion can easily be generalized to two dimensions.

Fourier transforms may be expressed in polar coordinates by setting

$$\begin{aligned} x &= r\cos\theta, & u &= s\cos\varphi, \\ y &= r\sin\theta, & v &= s\sin\varphi. \end{aligned}$$

Then (**3.1**), (**3.2**) take the form

$$\left.\begin{aligned} F(s, \varphi) &= (2\pi)^{-1}\int_0^\infty \int_{-\pi}^{\pi} f(r, \theta)\mathrm{e}^{\mathrm{i}sr\cos(\theta-\varphi)}\, r\mathrm{d}r\mathrm{d}\theta, \\ f(r, \theta) &= (2\pi)^{-1}\int_0^\infty \int_{-\pi}^{\pi} F(s, \varphi)\mathrm{e}^{-\mathrm{i}sr\cos(\theta-\varphi)}s\mathrm{d}s\mathrm{d}\varphi. \end{aligned}\right\} \quad . \quad (\mathbf{3.6})$$

In particular, when $f(r, \theta)$ is independent of θ, it is clear that $F(s, \varphi)$ is independent of φ, and these functions may be written as $f(r)$, $F(s)$. Using Parseval's integral for the Bessel function (Watson 1944), the relations (**3.6**) then become

$$F(s) = \int_0^\infty f(r)J_0(sr)r\mathrm{d}r, \quad \ldots\ldots\ldots\ldots \quad (\mathbf{3.7})$$

$$f(r) = \int_0^\infty F(s)J_0(sr)s\mathrm{d}s. \quad \ldots\ldots\ldots\ldots \quad (\mathbf{3.8})$$

The functions $f(r)$, $F(s)$ are called Fourier-Bessel transforms of one another.

If the functions f, g, h of equation (**2.1**) all have circular symmetry, they may be written $f(r)$, $g(r)$, $h(r)$, and (**3.3**) becomes

$$G(s)=2\pi H(s)F(s),$$

so that the formal solution has the form (**3.8**), in which

$$F(s)=\frac{G(s)}{2\pi H(s)}.$$

IV. Instability of the Solution

If g, h are specified exactly, as in a theoretical problem, there may be a unique and exact solution. But if the values of $g(x, y)$ are uncertain due to observational errors, however small these errors may be, the solution is always completely indeterminate even with the restriction that $f(x, y)\to 0$ as $x^2+y^2\to\infty$. For let any solution be given an (additive) increment of the form

$$\Delta f(x, y)=A\mathrm{e}^{-\mathrm{i}(ux+vy)}w(x, y),$$

where

$$\begin{aligned} w(x, y)&=(1-|\,x\,|/a)(1-|\,y\,|/b), \qquad |\,x\,|<a,\ |\,y\,|<b,\\ &=0 \quad \text{elsewhere,}\end{aligned}$$

and A, a, b are arbitrary. Then it may be shown that the corresponding increment of g can be made arbitrarily small by taking u^2+v^2 sufficiently large. This property may be expressed by saying that $f(x, y)$ is not continuously determined by $g(x, y)$. (Compare Kreisel 1949.)

In other words, oscillating increments of arbitrarily large amplitude in $f(x, y)$ correspond to unobservably small increments in $g(x, y)$ provided that the wavelength of the oscillations is sufficiently small. Such oscillations appear when the computer tries to determine $f(x, y)$ with greater precision than is warranted by the accuracy of the observations.

The solution must therefore be stabilized by excluding functions which have oscillations of large amplitude and short wavelength. This means that complete sharpening can never be achieved except in theoretical problems. Following a suggestion by Fellgett and Schmeidler (1952), if we are given the autocorrelation function of the errors in $g(x, y)$, we can use the Wiener-Kolmogaroff smoothing theory (Bode and Shannon 1950) to determine a solution giving the best possible compromise between errors due to magnification of the errors in $g(x, y)$ and errors due to incomplete sharpening. But the difficulty of measuring the autocorrelation function would seem to make this method of little practical value. It is therefore necessary to use more arbitrary methods of stabilization. Unfortunately this means that the solution adopted is to some extent subjective, that is, depending on the computer's judgment. But, if the statement of the solution is accompanied by a statement as to which method of sharpening was used, the result so expressed is purely objective, and is therefore to be preferred to a solution reached by " trial and error ".

When the highest possible accuracy is required, it appears that the best procedure is to find a sequence of approximate solutions of (theoretically)

increasing accuracy, taking the observational data at their face value ; and then to terminate the sequence just before physically improbable undulations begin to appear.

V. Polynomial Solutions

If $f(x, y)$ is the polynomial

$$f(x, y)=\sum_{m=0}^{A}\sum_{n=0}^{B} f_{mn}x^m y^n/m!n!, \quad \ldots\ldots\ldots\ldots \quad (5.1)$$

then the integral in (**2.1**) converges if $h(x, y)$ possesses moments of all requisite orders. We define these moments by

$$M_{pq}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x^p y^q}{p!q!}h(x, y)\mathrm{d}x\mathrm{d}y. \quad \ldots\ldots\ldots\ldots \quad (5.2)$$

The integral in (**2.1**) can then be evaluated either by term-by-term binomial expansion of $f(x'-x, y'-y)$, or by Taylor expansion of this function in powers of x and y. Thus $g(x', y')$ may be expressed in either of the two forms

$$g(x', y')=\sum_{m=0}^{A}\sum_{n=0}^{B} f_{mn}\sum_{p=0}^{m}\sum_{q=0}^{n}(-1)^{m+n-p-q}M_{m-p,\ n-q}x'^p y'^q/p!q! \quad \ldots \quad (5.3)$$

$$=\sum_{m=0}^{A}\sum_{n=0}^{B}(-1)^{m+n}M_{mn}\frac{\partial^{m+n}}{\partial x'^m\partial y'^n}f(x', y'). \quad \ldots\ldots\ldots\ldots \quad (5.4)$$

Similarly, if $g(x, y)$ is the polynomial

$$g(x, y)=\sum_{m=0}^{A}\sum_{n=0}^{B} g_{mn}x^m y^n/m!n!, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5.5)$$

it may be verified by substitution that (**2.1**) is satisfied by

$$f(x, y)=\sum_{m=0}^{A}\sum_{n=0}^{B} g_{mn}\sum_{p=0}^{m}\sum_{q=0}^{n}(-1)^{m+n-p-q}\mu_{m-p,\ n-q}x^p y^q/p!q! \quad \ldots\ldots \quad (5.6)$$

$$=\sum_{m=0}^{A}\sum_{n=0}^{B}(-1)^{m+n}\mu_{mn}\frac{\partial^{m+n}}{\partial x^m\partial y^n}g(x, y), \quad \ldots\ldots\ldots\ldots \quad (5.7)$$

where the inverse moments μ_{pq} are defined by

$$\left.\begin{aligned}\mu_{00}M_{00}&=1,\\ \sum_{p=0}^{m}\sum_{q=0}^{n}\mu_{m-p,\ n-q}M_{pq}&=0, \quad (m+n>0).\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (5.8)$$

Thus the solution (**5.6**) is valid provided that $M_{00}\neq 0$, and it is easy to show that it is the only polynomial solution. A result equivalent to (**5.7**) has been obtained by Coutrez (1949), and the one-dimensional form of (**5.6**) has been given by van de Hulst (1946). (The moments as defined by these authors correspond to $p!q!M_{pq}$ in our notation.)

If the integrals (**5.2**) converge absolutely for all non-negative integers p, q, then the integral defining $2\pi H(u, v)$ may be differentiated any number of times

under the integral sign. We then find that $2\pi H(u, v)$ and all its derivatives are bounded and continuous for all (u, v), so that the Maclaurin expansion

$$2\pi H(u, v) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} M_{mn}(iu)^m(iv)^n \quad \ldots\ldots\ldots \quad (5.9)$$

converges. From (**5.8**) and (**5.9**),

$$1/2\pi H(u, v) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \mu_{mn}(iu)^m(iv)^n. \quad \ldots\ldots\ldots \quad (5.10)$$

The relations (**5.9**) and (**5.10**) facilitate computation of the moments and inverse moments when $H(u, v)$ is expressible in terms of elementary functions. Note that, for a normalized kernel, $2\pi H(0, 0) = M_{00} = \mu_{00} = 1$.

The results (**5.3**)–(**5.7**) remain true when the polynomials are extended to infinite series provided that the series for $h(x, y)f(x'-x, y'-y)$ converges for all (x, y) and may be integrated term by term over the infinite (x, y) plane. However, we are here concerned primarily with distributions given by observational data, and the need for infinite series does not arise.

The results (**5.4**) and (**5.7**) may be formally applied to functions which are not polynomials. In (**5.4**) it is usually possible to write down a remainder term setting an upper limit to the error caused by neglecting higher derivatives, but in (**5.7**) there is no simple way of estimating this error. However, if $f(x, y)$ and all its moments exist, it may be shown that the moments of the right-hand member of (**5.7**) are the same as the moments of the true solution $f(x, y)$ up to and including moments of order (A, B), so that in this narrow sense every term of (**5.7**) effects an improvement in the approximation to $f(x, y)$ as a whole. (Compare Eddington's treatment (1940) of the one-dimensional problem with Gaussian kernel.) But it does not necessarily follow that the series converges at any particular point (x, y).

VI. Polynomial Solutions with Circular Symmetry

If f, g, h are functions of r only, where $r^2 = x^2 + y^2$, we may write them as $f(r), g(r), h(r)$ and the integral equation (**2.1**) becomes

$$g(r') = \int_0^{\infty} \int_{-\pi}^{\pi} h(r) f[\sqrt{(r'^2 + r^2 - 2rr' \cos\theta)}] r\,dr\,d\theta. \quad \ldots\ldots \quad (6.1)$$

If $f(r)$ is a polynomial of degree A in r^2,

$$f(r) = \sum_{n=0}^{A} f_n r^{2n}/n!^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6.2)$$

then the integral in (**6.1**) converges if $h(r)$ possesses *circular moments* of orders up to A. We define these circular moments by

$$M_p = 2\pi \int_0^{\infty} \frac{r^{2p}}{p!^2} h(r) r\,dr. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6.3)$$

Observe that, by multiplying the binomial expansions of $(r'-re^{i\theta})^n/n!$ and $(r'-re^{-i\theta})^n/n!$, we obtain

$$\frac{(r'^2+r^2-2rr'\cos\theta)^n}{n!^2}=\sum_{p=0}^{n}\sum_{q=0}^{n}\frac{r'^{p+q}}{p!q!}\frac{(-r)^{2n-p-q}}{(n-p)!(n-q)!}\cos(q-p)\theta,$$

so that

$$\int_{-\pi}^{\pi}\frac{(r'^2+r^2-2rr'\cos\theta)^n}{n!^2}d\theta=2\pi\sum_{p=0}^{n}\frac{r'^{2p}}{p!^2}\frac{r^{2n-2p}}{(n-p)!^2}. \quad \ldots\ldots\ldots\ldots (6.4)$$

Then on substituting from (**6.2**) in (**6.1**) and carrying out the integrations with the aid of (**6.4**) and (**6.3**), we obtain

$$g(r')=\sum_{n=0}^{A}f_n\sum_{p=0}^{n}M_{n-p}r'^{2p}/p!^2. \quad \ldots\ldots\ldots\ldots (6.5)$$

Similarly, if $g(r)$ is the polynomial

$$g(r)=\sum_{n=0}^{A}g_nr^{2n}/n!^2, \quad \ldots\ldots\ldots\ldots (6.6)$$

it may be verified by substitution that (**6.1**) is satisfied by

$$f(r)=\sum_{n=0}^{A}g_n\sum_{p=0}^{n}\mu_{n-p}r^{2p}/p!^2, \quad \ldots\ldots\ldots\ldots (6.7)$$

where the *inverse circular moments* μ_p are defined by

$$\left.\begin{aligned}&\mu_0M_0=1,\\&\sum_{p=0}^{n}\mu_{n-p}M_p=0, \quad n>0.\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (6.8)$$

Thus the solution (**6.7**) is valid provided that $M_0\neq0$, and it is easy to show that it is the only polynomial solution.

We may express (**6.5**) and (**6.7**) in terms of the derivatives of f, g, by using the differential operator D defined by

$$D\varphi(r)=\frac{d}{d(r^2)}\left[r^2\frac{d\varphi}{d(r^2)}\right]=\frac{1}{4r}\frac{d}{dr}\left(r\frac{d\varphi}{dr}\right)=\frac{1}{4}\nabla^2\varphi.$$

Equations (**6.5**) and (**6.7**) then become

$$g(r')=\sum_{n=0}^{A}M_nD^nf(r'), \quad \ldots\ldots\ldots\ldots (6.9)$$

$$f(r)=\sum_{n=0}^{A}\mu_nD^ng(r). \quad \ldots\ldots\ldots\ldots (6.10)$$

If the integrals (**6.3**) converge absolutely for all p, then the integral defining $2\pi H(s)$ may be differentiated any number of times under the integral sign, and we find that

$$2\pi H(s)=\sum_{n=0}^{\infty}M_n(-s^2/4)^n, \quad \ldots\ldots\ldots\ldots (6.11)$$

$$1/2\pi H(s)=\sum_{n=0}^{\infty}\mu_n(-s^2/4)^n. \quad \ldots\ldots\ldots\ldots (6.12)$$

In this case the kernel also possesses all Cartesian moments M_{mn}, and these can be expressed in terms of the circular moments M_n. For if in (**6.11**) we write $s^{2n}=(u^2+v^2)^n$, expand each term binomially, and identify the result with (**5.9**), we obtain

$$\left.\begin{aligned} M_{2m+1,\,n}&=M_{m,\,2n+1}=0,\\ M_{2m,\,2n}&=\frac{(m+n)!\,M_{m+n}}{m!\,n!\,4^{m+n}}. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (\mathbf{6.13})$$

Similarly from (**6.12**) and (**5.10**) we obtain

$$\left.\begin{aligned} \mu_{2m+1,\,n}&=\mu_{m,\,2n+1}=0,\\ \mu_{2m,\,2n}&=\frac{(m+n)!\,\mu_{m+n}}{m!\,n!\,4^{m+n}}. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (\mathbf{6.14})$$

VII. Application of Polynomial Solutions

Let $h(x, y)$ be given and possess moments of at least the first few orders, which are computed from (**5.2**). The inverse moments are then found from (**5.8**). Let $g(x, y)$ be given by a table of its values at an array of points (pa, qb) where p, q are any integers and a, b are constants. To compute an approximate value of f at any one of these points, we may temporarily regard this point as the origin of coordinates without loss of generality, so that, from (**5.6**),

$$f(0,\,0)=\Sigma\Sigma(-1)^{m+n}g_{mn}\mu_{mn}, \quad \ldots\ldots\ldots\ldots \quad (\mathbf{7.1})$$

where the coefficients g_{mn} are determined so that the polynomial $\Sigma\Sigma g_{mn}x^m y^n/m!\,n!$ coincides with the tabulated g-values at a set of points (pa, qb) in the neighbourhood of the (temporary) origin. The number of points in this set equals the number of coefficients g_{mn}, and it is clear that this point set should extend over a region at least as large as the region in which $h(x, y)$ differs appreciably from zero. Then it is found that (**7.1**) is a homogeneous linear function of the g-values at these points, whose coefficients depend only on the inverse moments and on the geometry of the set of points. For a particular kernel and point set, these coefficients constitute a set of weights which need be numerically evaluated only once and can be used repeatedly to correct different observed distributions or different parts of the same observed distribution.

To illustrate the method, we consider in more detail kernels which have two orthogonal axes of symmetry. Taking these as coordinate axes, we find that $M_{mn}=0$, $\mu_{mn}=0$ whenever m or n is odd. Then (**7.1**) becomes

$$f(0,\,0)=\Sigma\Sigma g_{2m,\,2n}\mu_{2m,\,2n}.$$

Evidently those terms of the polynomial which contain an odd power of x or of y contribute nothing to $f(0, 0)$, so that the correction of $g(x, y)$ is equivalent to the correction of the function

$$\bar{g}(x, y)=\tfrac{1}{4}[g(x, y)+g(-x, y)+g(x, -y)+g(-x, -y)].$$

Now $\bar{g}(x, y)$ is fully determined by its values in the first quadrant of the (x, y) plane, so that if the point set is symmetrical with respect to both axes, then the

number of points within or bordering the first quadrant must be equal to the number of terms in the polynomial for $\bar{g}(x, y)$. We take this polynomial to be complete and of degree N in (x^2, y^2), that is, it contains all terms of the form $g_{2m,2n}x^{2m}y^{2n}$ for $0 \leqslant m+n \leqslant N$. The number of terms in this polynomial is $\frac{1}{2}(N+1)(N+2)$.

We take for our point set all points (pa, qb) for which $|p|+|q| \leqslant N$; that is, the points within or bordering a rhombus whose vertices are $(\pm Na, 0)$, $(0, \pm Nb)$. There are $2N^2+2N+1$ such points, of which $\frac{1}{2}(N+1)(N+2)$ are within or bordering the first quadrant, as required. By generalizing Lagrange's interpolation formula, the polynomial for $\bar{g}(x, y)$ may be written

$$\bar{g}(x, y) = \sum_{p=0}^{N} \sum_{q=0}^{N-p} \bar{g}(pa, qb)\varphi_{pq}(x, y)/\varphi_{pq}(pa, qb), \quad \ldots\ldots\ldots\ldots \quad (7.2)$$

where

$$\varphi_{pq}(x, y) = (x^2-p^2a^2)^{-1}(y^2-q^2b^2)^{-1} \prod_{m=0}^{N} \prod_{n=0}^{N-m} (x^2-m^2a^2)(y^2-n^2b^2).$$

To complete the computation of the weights for a particular N, we have now only to expand (7.2) in polynomial form and then replace $x^{2m}y^{2n}/(2m)!(2n)!$ by $\mu_{2m,2n}$. Writing the inverse moments in the dimensionless form

$$m_{2p,2q} = \mu_{2p,2q}/a^{2p}b^{2q},$$

the result for $N=1$, involving five points, is

$$f(0, 0) = g(0, 0)(m_{00}-2m_{20}-2m_{02}) + \bar{g}(a, 0)2m_{20} + \bar{g}(0, b)2m_{02}, \quad \ldots\ldots \quad (7.3)$$

and for $N=2$, involving 13 points,

$$\begin{aligned} 6f(0, 0) = {} & g(0, 0)(6m_{00}-15m_{20}+36m_{40}+24m_{22}+36m_{04}) \\ & + \bar{g}(a, 0)(16m_{20}-48m_{40}-24m_{22}) + \bar{g}(0, b)(16m_{02}-48m_{04}-24m_{22}) \\ & + \bar{g}(2a, 0)(-m_{20}+12m_{40}) + \bar{g}(0, 2b)(-m_{02}+12m_{04}) + \bar{g}(a, b)24m_{22} \\ & \qquad\qquad\qquad\qquad\qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7.4) \end{aligned}$$

The formula for $N=3$, involving 25 points, is so unwieldy that it is not likely to be of much practical value.

Expressions for $g(0, 0)$ in terms of the values of f at a rhomboidal array of points can also be derived in the same way. They may be obtained immediately from (7.3) and (7.4) by interchanging the symbols f and g, and substituting $M_{2p,2q}$ for $\mu_{2p,2q}$.

When the kernel has circular symmetry and $a=b$, the formulae may be simplified by expressing the moments in terms of the circular moments according to (6.14). Writing $m_p = \mu_p/a^{2p}$, the formulae (7.3) and (7.4) then become

$$f(0, 0) = g(0, 0)(m_0-m_1) + \bar{g}(a, 0)\tfrac{1}{2}m_1 + \bar{g}(0, a)\tfrac{1}{2}m_1, \quad \ldots\ldots\ldots\ldots \quad (7.5)$$

$$\begin{aligned} 24f(0, 0) = {} & g(0, 0)(24m_0-30m_1+30m_2) + [\bar{g}(a, 0)+\bar{g}(0, a)](16m_1-24m_2) \\ & + [\bar{g}(2a, 0)+\bar{g}(0, 2a)](-m_1+3m_2) + \bar{g}(a, a)12m_2. \quad \ldots\ldots\ldots \quad (7.6) \end{aligned}$$

The formulae (**7.3**), (**7.5**) are exact if $g(x, y)$ is a polynomial of degree 3 in (x, y) and (**7.4**), (**7.6**) are exact if $g(x, y)$ is a polynomial of degree 5 in (x, y). If the observed data $g(pa, qb)$ contain random (additive) errors of mean zero and mean square ε^2, then the weighted sums will contain errors of mean zero and mean square $\lambda^2\varepsilon^2$, where λ^2 is the sum of the squares of the $(2N^2+2N+1)$ weights in the expression for $f(0, 0)$. If we increase N without limit, keeping aN, bN constant, we find that λ^2 increases without limit. This is a manifestation of the instability discussed in Section IV. In practice we implicitly exclude rapidly undulating functions by approximating to f, g by polynomials of low degree, that is, by keeping N small. Thus we obtain stable solutions, but the sharpening is incomplete when $g(x, y)$ is not a polynomial, or is a polynomial whose degree exceeds $2N+1$ in (x, y).

For the Gaussian kernel

$$h(r)=(2\pi c^2)^{-1}\mathrm{e}^{-\frac{1}{2}r^2/c^2}, \quad \ldots\ldots\ldots\ldots\ldots \quad (7.7)$$

the circular moments are $M_n=(2c^2)^n/n!$, $\mu_n=(-2c^2)^n/n!$, and weights for the correction $f-g$, computed from (**7.5**) and (**7.6**), are given in Table 1. The meaning of the weights for $N=1$, written in full, is

$$f(x, y)-g(x, y)=(c^2/2a^2)[4g(x, y)-g(x+a, y)-g(x-a, y)-g(x, y+a) \\ -g(x, y-a)],$$

from which the meaning of the other weights should be clear.

For the kernel representing a circular aperture

$$\left.\begin{aligned} h(r)&=1/\pi c^2 \quad (r<c) \\ &=0 \quad\;\; (r>c). \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (7.8)$$

the circular moments are $M_n=c^{2n}/n!(n+1)!$, $\mu_0=1$, $\mu_1=-\frac{1}{2}c^2$, $\mu_2=c^4/6$, $\mu_3=-7c^6/144$, $\mu_4=13c^8/960$, . . ., and sets of weights for the correction $f-g$ are given in the lower half of Table 1.

When f, g, h all have circular symmetry, a similar method may be used to find formulae involving only g-values at suitably spaced points along a diameter. We choose $2N+1$ equally spaced points extending from $r=(n-N)a$ to $r=(n+N)a$, $n\geqslant N$, and fit a polynomial of degree $2N$ in r^2 at these points. Lagrange's interpolation formula then gives

$$g(r)=\sum_{p=-N}^{N} g(na+pa)\varphi_p(r)/\varphi_p(na+pa), \quad \ldots\ldots\ldots \quad (7.9)$$

where

$$\varphi_p(r)=[r^2-(n+p)^2a^2]^{-1}\prod_{j=-N}^{N}[r^2-(n+j)^2a^2].$$

We now expand (**7.9**) in polynomial form and then replace $r^{2m}/m!^2$ by $\sum_{i=0}^{m}\mu_{m-i}r^{2i}/i!^2$ to obtain $f(r)$. To obtain $f(na)$, $n\geqslant N$, we set $r=na$ in the result. The expression for $f(r)$ derived from the points $0, a, 2a, \ldots, 2Na$ may be taken as the best approximation to $f(r)$ for all $r\leqslant Na$. Thus we obtain sets of weights for $f(0), f(a), f(2a), \ldots$ Unfortunately these sets are all different, because the

TABLE 1

WEIGHTS FOR COMPUTING $f(x, y)-g(x, y)$

Weights with the kernel (7.7)

$N=1$	$N=2,\ a=2c$	$N=2,\ a=c\sqrt{2}$
	7	5
−1	6 −88 6	6 −56 6
−1 4 −1	7 −88 300 −88 7	5 −56 180 −56 5
−1	6 −88 6	6 −56 6
	7	5
Multiply by $c^2/2a^2$ If $a=2c$, $\lambda=1\cdot52$	Multiply by 1/384 $\lambda=1\cdot84$	Multiply by 1/96 $\lambda=3\cdot11$

Weights with the kernel (7.8)

$N=1$	$N=2,\ a=c$	$N=2,\ a=c/\sqrt{2}$
	1	3
−1	1 −12 1	4 −32 4
−1 4 −1	1 −12 40 −12 1	3 −32 100 −32 3
−1	1 −12 1	4 −32 4
	1	3
Multiply by $c^2/8a^2$ If $a=c$, $\lambda=1\cdot52$	Multiply by 1/48 $\lambda=1\cdot90$	Multiply by 1/48 $\lambda=3\cdot29$

origin can no longer be chosen arbitrarily. For $N=1$, involving three points, the general formula is

$$\begin{aligned}4n(4n^2-1)[f(r)-m_0g(r)]=&(2n+1)g(\overline{n-1}a)[4m_2-m_1(2n^2+2n+1-4r^2/a^2)]\\&-4n\,g(na)[4m_2-m_1(2n^2+2-4r^2/a^2)]\\&+(2n-1)g(\overline{n+1}a)[4m_2-m_1(2n^2-2n+1-4r^2/a^2)],\end{aligned}$$

where $m_p=\mu_p/a^{2p}$. For the particular kernel (7.8), with $a=c$, the first few formulae for $N=1$ are:

$$\begin{aligned}f(0)-g(0) &= +0\cdot792g(0) -0\cdot889g(a) +0\cdot097g(2a),\\ f(a)-g(a) &= +0\cdot292g(0) -0\cdot222g(a) -0\cdot069g(2a),\\ f(2a)-g(2a) &= -0\cdot035g(a) +0\cdot156g(2a) -0\cdot121g(3a),\\ f(3a)-g(3a) &= -0\cdot081g(2a) +0\cdot210g(3a) -0\cdot129g(4a),\\ f(4a)-g(4a) &= -0\cdot097g(3a) +0\cdot228g(4a) -0\cdot131g(5a),\\ f(5a)-g(5a) &= -0\cdot105g(4a) +0\cdot236g(5a) -0\cdot131g(6a),\end{aligned}$$

where the coefficients are correct to three decimal places. For the same kernel with $a=c$ and $N=2$, we find:

$$\begin{aligned}
f(0)-g(0) &= +1\cdot132g(0)-1\cdot414g(a)+0\cdot331g(2a)-0\cdot053g(3a)+0\cdot004g(4a),\\
f(a)-g(a) &= +0\cdot086g(0)+0\cdot095g(a)-0\cdot210g(2a)+0\cdot031g(3a)-0\cdot002g(4a),\\
f(2a)-g(2a) &= -0\cdot576g(0)+0\cdot808g(a)-0\cdot147g(2a)-0\cdot091g(3a)+0\cdot006g(4a),\\
f(3a)-g(3a) &= -0\cdot084g(a)+0\cdot058g(2a)+0\cdot169g(3a)-0\cdot155g(4a)+0\cdot013g(5a).
\end{aligned}$$

For comparison, we give three formulae for correction of one-dimensional distributions when the kernel represents a slit of width $2c$, that is,

$$\begin{aligned}
h(x) &= 1/2c \qquad (|x|<c)\\
&= 0 \qquad\quad (|x|>c).
\end{aligned}$$

The three-point formula is then

$$f(x)-g(x)=(c^2/6a^2)[-g(x-a)+2g(x)-g(x+a)].$$

When $a=c$, the five-point formula is

$$30[f(x)-g(x)]=g(x-2a)-9g(x-a)+16g(x)-9g(x+a)+g(x+2a),$$

and when $a=c/\sqrt{2}$,

$$180[f(x)-g(x)]=19g(x-2a)-136g(x-a)+234g(x)-136g(x+a)+19g(x+2a).$$

It should be stressed that the formulae of this section usually give accurate results only when the corrections are small. This is partly because the observational data cannot usually be well represented by polynomials of low degree, and partly because we have characterized the kernel by only a few parameters.

VIII. The Use of Fourier Transforms

The formal solution by Fourier transforms, given in Section III, is hardly suitable for direct numerical computations. But when $g(x, y)$ and $h(x, y)$ can be closely approximated by elementary functions whose transforms are known, this form of solution may be useful. Also, in radio astronomy, measurements with a variable-spacing interferometer give $|F(u, v)|^2$ directly over a finite region, from which hypotheses concerning $f(x, y)$ may be tested.

Some Fourier transforms likely to prove useful for these purposes are listed in Table 2. The two columns of this table may be interchanged by interchanging (u, v) with (x, y) and writing $-\mathrm{i}$ for i. In lines (1) and (5), F_1, F_2 are the Fourier transforms of f_1, f_2. Lines (1)–(7) give rules for extending known transforms. Lines (8)–(11) enable us to find the transforms of peak functions and of periodic functions. All the results in Table 2 are well known or have obvious proofs, possibly excepting line (14). By using the relations

$$(\mathrm{d}/\mathrm{d}x)H_n(x)=nH_{n-1}(x)=xH_n(x)-H_{n+1}(x)$$

we can prove by induction that the (simple) Fourier transform of $H_n(a_1x)\mathrm{e}^{-\frac{1}{2}x^2}$ is $(\mathrm{i}\alpha)^nH_n(a_1u/\alpha)\mathrm{e}^{-\frac{1}{2}u^2}$, where $\alpha^2=a_1^2-1$; from which line (14) follows. We give here the first few of the Hermite polynomials:

$$\begin{aligned}
&H_0(x)=1,\ H_1(x)=x,\ H_2(x)=x^2-1,\ H_3(x)=x^3-3x,\\
&H_4(x)=x^4-6x^2+3,\ H_5(x)=x^5-10x^3+15x,\\
&H_6(x)=x^6-15x^4+45x^2-15,\ H_n(x)=n!\sum_p(-1)^px^{n-2p}/2^pp!(n-2p)!
\end{aligned}$$

For brevity, we write

$$\varphi(a^2, b^2)=(2\pi ab)^{-1}\mathrm{e}^{-\frac{1}{2}(x^2/a^2+y^2/b^2)}. \quad \ldots\ldots\ldots \quad (8.1)$$

Then, if $h(x, y)=\varphi(a^2, b^2)$, the solution of the integral equation (**2.1**) can be written down whenever $g(x, y)$ is expressible as one of the functions on lines (12)–(15) of Table 2, or as a sum of such functions. For example, if

$$g(x, y)=\Sigma\Sigma c_{mn}\varphi(a_m^2, b_n^2), \quad \ldots\ldots\ldots\ldots\ldots \quad (8.2)$$

TABLE 2

FOURIER TRANSFORMS

The functions in the right-hand column are Fourier transforms of those in the left-hand column. The Hermite polynomials are defined by $(\mathrm{d}/\mathrm{d}x)^n\mathrm{e}^{-\frac{1}{2}x^2}=(-1)^nH_n(x)\mathrm{e}^{-\frac{1}{2}x^2}$

$f(x, y)\equiv (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u, v)\mathrm{e}^{-\mathrm{i}(ux+vy)}\mathrm{d}u\mathrm{d}v$	$F(u, v)\equiv (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x, y)\mathrm{e}^{\mathrm{i}(ux+vy)}\mathrm{d}x\mathrm{d}y$
(1) $af_1(x, y)+bf_2(x, y)$	$aF_1(u, v)+bF_2(u, v)$
(2) $f(x-a, y-b)$	$F(u, v)\mathrm{e}^{\mathrm{i}(au+bv)}$
(3) $f(x/a, y/b)$	$abF(au, bv)$
(4) $f(ax+by, cx+dy),\ ad-bc\equiv\Delta$	$\Delta^{-1}F\{(du-cv)/\Delta,\ (-bu+av)/\Delta\}$
(5) $2\pi f_1(x, y)f_2(x, y)$	$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_1(U, V)F_2(u-U, v-V)\mathrm{d}U\mathrm{d}V$
(6) $(\partial/\partial x)^m(\partial/\partial y)^nf(x, y)$	$(-\mathrm{i}u)^m(-\mathrm{i}v)^nF(u, v)$
(7) $x^my^nf(x, y)$	$(-\mathrm{i})^{m+n}(\partial/\partial u)^m(\partial/\partial v)^nF(u, v)$
(8) $2\pi\delta(x, y)$	1
(9) $2\pi\delta(x-a, y-b)$	$\mathrm{e}^{\mathrm{i}(au+bv)}$
(10) $\pi\delta(x-a, y-b)+\pi\delta(x+a, y+b)$	$\cos(au+bv)$
(11) $\pi\delta(x-a, y-b)-\pi\delta(x+a, y+b)$	$\mathrm{i}\sin(au+bv)$
(12) $\mathrm{e}^{-\frac{1}{2}(x^2/a^2+y^2/b^2)}$	$ab\mathrm{e}^{-\frac{1}{2}(a^2u^2+b^2v^2)}$
(13) $(x/a)^m(y/b)^n\mathrm{e}^{-\frac{1}{2}(x^2/a^2+y^2/b^2)}$	$\mathrm{i}^{m+n}H_m(au)H_n(bv)ab\mathrm{e}^{-\frac{1}{2}(a^2u^2+b^2v^2)}$
(14) $(\alpha/a)^m(\beta/b)^nH_m(a_1x/\alpha a)H_n(b_1y/\beta b)\times\mathrm{e}^{-\frac{1}{2}(x^2/a^2+y^2/b^2)}$	$\mathrm{i}^{m+n}H_m(a_1u)H_n(b_1v)ab\mathrm{e}^{-\frac{1}{2}(a^2u^2+b^2v^2)}$ where $a_1^2=a^2+\alpha^2,\ b_1^2=b^2+\beta^2.$
(15) $\mathrm{e}^{-\frac{1}{2}[(\alpha x+\beta y)^2/a^2+(\gamma x+\delta y)^2/b^2]}$, where $\alpha\delta-\beta\gamma=1$	$ab\mathrm{e}^{-\frac{1}{2}[a^2(\delta u-\gamma v)^2+b^2(-\beta u+\alpha v)^2]}$
(16) $(1+x^2/a^2+y^2/b^2)^{-3/2}$	$ab\mathrm{e}^{-\lvert(a^2u^2+b^2v^2)^{\frac{1}{2}}\rvert}$
(17) $1,\ \lvert x\rvert<a,\ \lvert y\rvert<b$ $0,$ elsewhere	$\frac{4ab}{2\pi}\frac{\sin au}{au}\frac{\sin bv}{bv}$
(18) $(1-\lvert x\rvert/2a)(1-\lvert y\rvert/2b),\ \lvert x\rvert<2a,\ \lvert y\rvert<2b$ $0,$ elsewhere	$\frac{4ab}{2\pi}\left[\frac{\sin au}{au}\frac{\sin bv}{bv}\right]^2$

then

$$f(x, y)=\Sigma\Sigma c_{mn}\varphi(a_m^2-a^2, b_n^2-b^2).$$

Again, if

$$g(x, y)=\varphi(a_1^2, b_1^2)\Sigma\Sigma c_{mn}(A_1x)^m(B_1y)^n, \quad \ldots\ldots\ldots \quad (8.3)$$

then

$$f(x, y)=\varphi(a_2^2, b_2^2)\Sigma\Sigma c_{mn}H_m(A_2x)H_n(B_2y),$$

where $a_2^2=a_1^2-a^2$, $b_2^2=b_1^2-b^2$, $A_1=a_2/a_1a$, $B_1=b_2/b_1b$, $A_2=a_1/a_2a$, $B_2=b_1/b_2b$. The last result can also be verified by direct integration of equation (**2.1**); variants of it have been discussed by Malmquist (1943), van de Hulst (1946), and Lyttkens (1949). Finally, if

$$g(x, y)=(ab)^{-1}\mathrm{e}^{-\frac{1}{2}(x'^2/a_1^2+y'^2/b_1^2)}, \quad \ldots\ldots\ldots\ldots \quad (8.4)$$

$$x'=x\cos\theta+y\sin\theta,$$
$$y'=-x\sin\theta+y\cos\theta,$$

then

$$f(x, y)=(AB)^{-1}\mathrm{e}^{-\frac{1}{2}(X'^2/A^2+Y'^2/B^2)},$$
$$X'=x\cos\Theta+y\sin\Theta,$$
$$Y'=-x\sin\Theta+y\cos\Theta,$$

where A, B, Θ are given by

$$\cot 2\Theta=\cot 2\theta-\operatorname{cosec} 2\theta(a^2-b^2)/(a_1^2-b_1^2),$$
$$2A^2=a_1^2+b_1^2-a^2-b^2+(a_1^2-b_1^2)\sin 2\theta\operatorname{cosec} 2\Theta,$$
$$2B^2=a_1^2+b_1^2-a^2-b^2+(b_1^2-a_1^2)\sin 2\theta\operatorname{cosec} 2\Theta.$$

We take the value of Θ which lies in the same quadrant as θ, and this value is always real and uniquely determined. But if the expression for A^2 or B^2 is negative, there is no solution.

When $g(x, y)$ is not expressible with sufficient accuracy in one of the forms (**8.2**), (**8.3**), (**8.4**), it may still be possible to express it as a sum of one of these functions and a residual, and the two components can be sharpened separately. If the residual and its gradient are numerically small, the sharpening of the residual by approximate methods is a much simpler task than the sharpening of the distribution $g(x, y)$.

Reverting to an arbitrary kernel, if $g(x, y)$ is a doubly periodic function whose Fourier series is

$$g(x, y)=\sum_m\sum_n[g_{mn}\cos(max+nby)+g'_{mn}\sin(max+nby)],$$

it is found by using Table 2 that

$$f(x, y)=\sum_m\sum_n[f_{mn}\cos(max+nby)+f'_{mn}\sin(max+nby)],$$

where

$$f_{mn}+\mathrm{i}f'_{mn}=(g_{mn}+\mathrm{i}g'_{mn})/2\pi H(ma, nb).$$

When f, g, h all have circular symmetry, their Fourier transforms become Fourier-Bessel transforms, by the use of which several particular solutions of equation (**2.1**) may be obtained. Some Fourier-Bessel transforms which will be found useful for this purpose are listed in Table 3. The two columns of this table may be interchanged by interchanging r and s. Some of these transforms were taken from Watson (1944) and others were derived from them by using the rules on lines (1)–(6) of the table. The polynomials $Q_n(x)$ may be defined by

$$Q_n(x)\mathrm{e}^{-x}=[(\mathrm{d}/\mathrm{d}x)(x\mathrm{d}/\mathrm{d}x)]^n\mathrm{e}^{-x},$$

TABLE 3

FOURIER-BESSEL TRANSFORMS

J_0, J_1, K_0, K_1 denote Bessel functions in the notation of Watson (1944). ∇^2 is the Laplacian operator $s^{-1}(\mathrm{d}/\mathrm{d}s)(s\mathrm{d}/\mathrm{d}s)$, and Q_n, R_n are the polynomials

$$Q_n(x)=n!^2\sum_{p=0}^{n}(-1)^{n-p}x^p/p!^2(n-p)!,$$

$$R_n(x)=2^{-n}\sum_{p=0}^{n}(2n-p)!(2x)^p/p!(n-p)!$$

$f(r)\equiv\int_0^\infty F(s)J_0(rs)s\mathrm{d}s$	$F(s)\equiv\int_0^\infty f(r)J_0(rs)r\mathrm{d}r$
(1) $af_1(r)+bf_2(r)$	$aF_1(s)+bF_2(s)$
(2) $f(r/a)$	$a^2F(as)$
(3) $2\pi f_1(r)f_2(r)$	$\int_0^\infty\int_{-\pi}^{\pi}F_1(s')F_2(S)s'\mathrm{d}s'd\theta$, where $S^2=s^2+s'^2-2ss'\cos\theta$
(4) $\int_r^\infty xf(x)\mathrm{d}x$	$-s^{-1}F'(s)$
(5) $(\mathrm{d}/\mathrm{d}r)\{rf(r)\}$	$-(\mathrm{d}/\mathrm{d}s)\{sF(s)\}$
(6) $r^{2n}f(r)$	$(-1)^n\nabla^{2n}F(s)$
(7) $2\pi\delta(r)$	1
(8) $2\pi\delta(r-a)$	$J_0(as)$
(9) $\mathrm{e}^{-\frac{1}{2}r^2/a^2}$	$a^2\mathrm{e}^{-\frac{1}{2}a^2s^2}$
(10) $(\frac{1}{2}r^2/a^2)^n\mathrm{e}^{-\frac{1}{2}r^2/a^2}$	$(-1)^nQ_n(\frac{1}{2}a^2s^2)a^2\mathrm{e}^{-\frac{1}{2}a^2s^2}$
(11) $(\alpha/a)^{2n}Q_n(\frac{1}{2}a_1^2r^2/\alpha^2a^2)\mathrm{e}^{-\frac{1}{2}r^2/a^2}$	$(-1)^nQ_n(\frac{1}{2}a_1^2s^2)a^2\mathrm{e}^{-\frac{1}{2}a^2s^2}$, where $a_1^2=a^2+\alpha^2$
(12) $(a^2+r^2)^{-\frac{1}{2}}$	$s^{-1}\mathrm{e}^{-as}$
(13) $(a^2+r^2)^{-3/2}$	$a^{-1}\mathrm{e}^{-as}$
(14) $1.3.5\ldots(2n+1)(a^2+r^2)^{-\frac{1}{2}(2n+3)}$	$R_n(as)a^{-2n-1}\mathrm{e}^{-as}$
(15) $(a^2+r^2)^{-1}$	$K_0(as)$
(16) $2a^2(a^2+r^2)^{-2}$	$asK_1(as)$
(17) $4a^4(a^2+r^2)^{-3}$	$asK_1(as)+\frac{1}{2}a^2s^2K_0(as)$
(18) $(\pi a^2)^{-1}$, $r<a$; 0, $r>a$	$\frac{1}{2\pi}\left[\frac{2}{as}J_1(as)\right]$
(19) $2\pi^{-2}a^{-2}\{\arccos(r/2a)-(r/2a)\sqrt{1-(r/2a)^2}\}$, $r<2a$; 0, $r>2a$	$\frac{1}{2\pi}\left[\frac{2}{as}J_1(as)\right]^2$

and the first five of them are

$$Q_0(x)=1,\quad Q_1(x)=x-1,\quad Q_2(x)=x^2-4x+2,$$
$$Q_3(x)=x^3-9x^2+18x-6,\quad Q_4(x)=x^4-16x^3+72x^2-96x+24.$$

The first five of the polynomials $R_n(x)$ are

$$R_0(x)=1,\quad R_1(x)=x+1,\quad R_2(x)=x^2+3x+3,$$
$$R_3(x)=x^3+6x^2+15x+15,\quad R_4(x)=x^4+10x^3+45x^2+105x+105.$$

Also

$$R_n(x)=(2n-1)R_{n-1}(x)+x^2R_{n-2}(x).$$

For brevity, we write

$$\varphi(a^2)=(2\pi a^2)^{-1}\mathrm{e}^{-\frac{1}{2}r^2/a^2},$$
$$\psi(a)=(a/2\pi)(a^2+r^2)^{-3/2}.$$

Then, if $h(r)=\varphi(a^2)$ and $g(r)=\Sigma c_n\varphi(a_n^2)$, we find

$$f(r)=\Sigma c_n\varphi(a_n^2-a^2).$$

This form of solution was known to Kapteyn (1920), who used it to compute true frequency distributions of stellar proper motions. With the same kernel, if

$$g(r)=\varphi(a_1^2)\Sigma c_n(\tfrac{1}{2}A_1^2r^2)^n,$$

then

$$f(r)=\varphi(a_2^2)\Sigma c_nQ_n(\tfrac{1}{2}A_2^2r^2)^n,$$

where $a_2^2=a_1^2-a^2$, $A_1=a_2/a_1a$, $A_2=a_1/a_2a$.

If $h(r)=\psi(a)$ and $g(r)=\Sigma c_n\psi(a_n)$, we find

$$f(r)=\Sigma c_n\psi(a_n-a).$$

The convolution of $\varphi(a^2)$ and $\psi(b)$ is the analogue of the Voigt function in one dimension.

Other particular solutions may be found from Table 3 when the kernel is of the form $(2\pi a^2)^{-1}\mathrm{e}^{-r/a}$ or of the form $(4\pi a^3)^{-1}rK_1(r/a)$.

Fourier-Bessel transforms may also be computed by numerical integration. For best results the major variations of the function should be taken out by using the functions of Table 3, and the residual function transformed numerically. This process is tedious, but has been used successfully to sharpen the observed light curves of external galaxies with circular symmetry. Numerical integration of Fourier transforms has been used similarly in one dimension by Stokes (1948) and by Fellgett and Schmeidler (1952).

Instability of the kind described in Section IV arises through the uncertainty of $F(u, v)$ for large (u, v). In practice, plausible solutions can be obtained by extrapolating $F(u, v)$ outside the region in which it is well determined, on the assumption that it falls " smoothly " to zero.

IX. Approximate Inverse Kernels

In this section, following a method introduced by Kreisel (1949), we overcome the instability of the solution by seeking only partially sharpened solutions $f_n(x, y)$, given by the convolution of the true solution $f(x, y)$ with pre-assigned smoothing functions $j_n(x, y)$ which are, loosely speaking, broad enough to smooth out spurious undulations of short wavelength in $f(x, y)$, but not so broad as to distort seriously the significant features of $f(x, y)$.

For brevity, we now write $2\pi H(u, v)=\overline{H}(u, v)$, and similarly for other functions of (u, v). The arguments (x, y) and (u, v) following functional symbols

will be omitted when there is no risk of confusion, and the convolution of any two functions f_1, f_2 will be denoted by $f_1 * f_2$. Equation (**2.1**) is thus written

$$g = h * f. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{9.1})$$

As always, we assume that h is L. If f is L also, then

$$\bar{G} = \bar{H}\bar{F}.$$

We define $\bar{K}(u, v)$ by $\bar{K}\bar{H} = 1$, so that

$$\bar{F} = \bar{K}\bar{G}.$$

(In this section, the symbols J, K do not refer to Bessel functions.) If $K(u, v)$ were the transform of a function $k(x, y)$ of L, the solution of (**9.1**) would therefore be $f = k * g$, and $k(x, y)$ could be called the inverse kernel. But if $h(x, y)$ is a proper function, we know that $K(u, v) \to \infty$ with $u^2 + v^2$, so that $k(x, y)$ does not exist.

However, in many cases there exists a sequence of approximate inverse kernels $k_n(x, y)$ of L, and associated smoothing functions $j_n \equiv k_n * h$, such that the approximate solution $f_n \equiv k_n * g$ is the smooth of f with j_n, is stable, and under certain conditions the sequence $f_n(x, y)$ converges uniformly to $f(x, y)$. Observe that, when $g(x, y)$ is specified by its values at a finite set of points together with an interpolation formula, there is in general no exact solution $f(x, y)$, and the best we can do is to find an approximate solution f_n such that $g_n \equiv h * f_n$ agrees with the observational data within acceptable limits of error. It will be shown that, whether $f(x, y)$ exists or not, the sequence $g_n(x, y)$ converges uniformly to $g(x, y)$; provided that $g(x, y)$ is bounded and of bounded gradient everywhere, that is,

$$| g(x', y') - g(x, y) | \leqslant m\varphi(x' - x, y' - y), \quad \ldots\ldots\ldots \quad (\mathbf{9.2})$$

$$\varphi(x, y) = | (x^2 + y^2)^{\frac{1}{2}} |, \quad x^2 + y^2 \leqslant M^2/m^2,$$
$$= M/m, \qquad x^2 + y^2 > M^2/m^2;$$

and provided also that, for $\varepsilon > 0$, there exists an integer N such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} | j_n(x, y) | \varphi(x, y) \mathrm{d}x\mathrm{d}y < \varepsilon/m \quad \ldots\ldots\ldots\ldots \quad (\mathbf{9.3})$$

for every $n \geqslant N$. We suppose also that the $j_n(x, y)$ are normalized:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} j_n(x, y) \mathrm{d}x\mathrm{d}y = 1, \qquad n = 1, 2, 3, \ldots \quad \ldots\ldots \quad (\mathbf{9.4})$$

We now give a formal statement and proof of these results.

Statement.—If

(i) g is bounded, and integrable in every finite region,
(ii) g is of bounded gradient everywhere, as in (**9.2**),
(iii) functions k_n and $j_n \equiv k_n * h$ can be chosen such that (**9.3**) and (**9.4**) hold, and h, k_n are L,
(iv) $f_n = k_n * g$;

then

(v) f_n is bounded, and integrable in every finite region, so that $g_n=h*f_n$ exists,

(vi) j_n is L, and $g_n=j_n*g$,

(vii) f_n is continuously determined by g,

(viii) $|\,g_n-g\,|\to 0$ everywhere uniformly as $n\to\infty$.

If also

(ix) f exists such that $g=h*f$, and f is bounded, and integrable in every finite region;

then

(x) $f_n=j_n*f$.

If also

(xi) f is of bounded gradient everywhere;

then

(xii) $|\,f_n-f\,|\to 0$ everywhere uniformly as $n\to\infty$.

Proofs

(v) Since $f_n=k_n*g$, and k_n is L, (v) follows from (i).

(vi) Since h, k_n are L and $j_n=k_n*h$, therefore j_n is L. Also, by (v), (iv), and (iii), $g_n=h*(k_n*g)=(h*k_n)*g=j_n*g$, the change in order of integration being justified by absolute convergence.

(vii) Let $\Delta g(x, y)$ be a variation of $g(x, y)$, such that $|\,\Delta g(x, y)\,|<\varepsilon$ for all (x, y). Let $\Delta f_n(x, y)$ be the corresponding variation of $f_n(x, y)$. Then, since k_n is L,

$$|\,\Delta f_n\,|=|\,k_n*\Delta g\,|\leqslant|\,k_n\,|*|\,\Delta g\,|<\varepsilon\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\,k_n(x, y)\,|\,\mathrm{d}x\mathrm{d}y=\varepsilon C_n,$$

where C_n is a positive number depending only on n. Therefore $f_n(x, y)$ is continuously determined by $g(x, y)$, that is, it is stable. (But note that this continuity is not uniform with respect to n, for $C_n\to\infty$ with n.)

(viii) From (vi) and (**9.4**),

$$\begin{aligned}|\,g_n(x', y')-g(x', y')\,|&=\Big|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}j_n(x'-x, y'-y)[g(x, y)-g(x', y')]\mathrm{d}x\mathrm{d}y\,\Big|\\&\leqslant m\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\,j_n(x'-x, y'-y)\,|\,\varphi(x'-x, y'-y)\mathrm{d}x\mathrm{d}y\\&<\varepsilon,\ \text{for } n\geqslant N,\end{aligned}$$

by (**9.2**) and (**9.3**).

(x) Since h, k_n are L, we have from (iv) and (ix),

$$f_n=k_n*(h*f)=(k_n*h)*f=j_n*f.$$

(xii) Proof is similar to the proof of (viii).

The treatment above owes much to Kreisel's "Theorem A" (1949). In our notation, Kreisel assumes (ii), (iii), (ix), and (x) and deduces (iv), (vii), and in effect (viii), though without explicitly using the idea of a sequence. To facilitate comparison of the two treatments, we give a conversion table for notation :

Burr :	f	f_n	g	g_n	h	H	j_n	J_n	k_n	K_n
Kreisel :	ψ	$\overline{\psi}$	k	$\overline{k}$	L	Λ	μ	M	λ	M/Λ

The conditions (iii) may be relaxed sufficiently to admit improper functions of the form

$$k_n(x, y)=C_n\delta(x, y)+l_n(x, y), \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9.5)$$

where l_n belongs to L ; and the theorem remains true, with slight modifications in the proofs.

In another variant, (**9.3**) is replaced by the condition that

$$\overline{J}_n(u, v)\to 1 \text{ uniformly in every finite region,} \quad \ldots\ldots \quad (9.6)$$

and (ii) is replaced by the condition that g is L and continuous, with the remaining conditions unchanged. It follows that g_n is L and continuous, and $\overline{G}_n=\overline{J}_n\overline{G}$, and proposition (viii) is replaced by

$$\begin{aligned}\overline{G}-\overline{G}_n&=(1-\overline{J}_n)\overline{G}\\ &\to 0 \text{ as } n\to\infty, \quad \ldots\ldots\ldots\ldots \quad (9.6a)\end{aligned}$$

by (**9.6**). In this variant, the writer has been unable to prove that $g-g_n\to 0$ except in the rather restrictive case in which g, $\overline{G}$ both belong to L. However, most computers will accept (**9.6a**) as sufficient justification for using the process with any observational data. Similar remarks apply to f, $\overline{F}$ when these functions exist. The proofs of (v), (vi), (vii), and (x) are unchanged.

We now give two forms for the sequence of smoothing functions which have been used in numerical problems.

Kreisel (1949) chooses smoothing functions of the form

$$j_n(x, y)=(2\pi a_n b_n)^{-1}\mathrm{e}^{-\frac{1}{2}(x^2/a_n^2+y^2/b_n^2)}, \quad \ldots\ldots\ldots \quad (9.7)$$

where a_n, $b_n\to 0$, as $n\to\infty$, so that (**9.3**) and (**9.4**) are satisfied. The approximate inverse kernels $k_n(x, y)$ have then to be computed through their Fourier transforms :

$$\overline{K}_n(u, v)=[\overline{H}(u, v)]^{-1}\mathrm{e}^{-\frac{1}{2}(a_n^2u^2+b_n^2v^2)}. \quad \ldots\ldots\ldots \quad (9.8)$$

The method is applicable only if the $k_n(x, y)$ exist and belong to L. It is valid for algebraic kernels such as (**2.7**), for which some functions $k_n(r)$ have been tabulated by Bullard and Cooper (1948). But, if $H(u, v)$ has zeros or if the kernel is Gaussian, the form (**9.7**) is not applicable.

The second form for the sequence is given by

$$\overline{J}_n=1-(1-\overline{H})^n, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9.9.)$$

$$\begin{aligned}\overline{K}_n&=1+(1-\overline{H})+(1-\overline{H})^2+\ldots+(1-\overline{H})^{n-1}\\ &=n-{}_nc_2\overline{H}+{}_nc_3\overline{H}^2-\ldots+(-\overline{H})^{n-1}. \quad \ldots\ldots\ldots \quad (9.10)\end{aligned}$$

Since $\bar{H}^2, \bar{H}^3, \ldots$ are the Fourier transforms of $2\pi h*h, 2\pi h*h*h, \ldots$, which belong to L, it follows that j_n is L, and that

$$k_n(x, y)=n\delta(x, y)-l_n(x, y), \quad \ldots\ldots\ldots\ldots \quad (9.11)$$

where $l_n(x, y)$ is L. Also, from (**9.9**), since $\bar{H}(u, v)$ is continuous, we see that (**9.6**) is satisfied provided that

$$|1-\bar{H}(u, v)|<1 \quad \ldots\ldots\ldots\ldots \quad (9.12)$$

everywhere. We conclude that when the kernel satisfies (**9.12**) the method is applicable under the conditions discussed under equation (**9.6a**). Approximate solutions of this form have been discussed or applied by several authors, for example Burger and van Cittert (1932, 1933) and van de Hulst (1941).

It is easily verified from (**9.9**) and (**9.10**) that the sequence f_n satisfies the recurrence relations

$$f_{n+1}=g+f_n-h*f_n, \qquad n=0, 1, 2, \ldots, \ldots (9.13)$$

$$f_{n+1}-f_n=(f_n-f_{n-1})-h*(f_n-f_{n-1}), \qquad n=1, 2, 3, \ldots, \ldots (9.14)$$

with $f_0=0, f_1=g$. From (**9.13**) we can compute $f_2, f_3, f_4, \ldots$ in turn, one integration being required at each step. (By one integration we mean one for each point (x, y).) Note that an error $e(x, y)$ introduced at any step will be reduced after a further m integrations to an error whose Fourier transform is $[1-\bar{H}(u, v)]^m E(u, v)$, which tends to zero as $m\to\infty$. Equation (**9.13**) may also be written $f_{n+1}-f_n=g-g_n$, so that the process may be stopped when $f_{n+1}-f_n$ is smaller than the errors of observation in g. The numerical integration is easier if (**9.14**) is used instead of (**9.13**), because f_n-f_{n-1} is in general numerically smaller and of smaller gradient than f_n. We have taken for our first approximation $f_1=g$, but *any* function could have been taken. Thus the computer can use his judgement in selecting a trial solution for f_1, compute f_2 from (**9.13**) and $f_3, f_4, \ldots$ from (**9.13**) or (**9.14**). In numerical examples with circular symmetry, it was found that graphs of the sequence $(f_{n+1}-f_n)$ along a diameter formed a "smooth" sequence of curves, and after plotting four or five such curves it was possible to extrapolate the sequence with tolerable accuracy and sketch the form of the next two or three curves. The sum of the ordinates of all these curves gave the distribution finally adopted for $f(x, y)$.

Alternatively we may compute the approximate inverse kernel k_n in the form (**9.11**) for some particular n, say $n=5$, and then obtain f_n in a single integration:

$$f_n=k_n*g\equiv ng-l_n*g. \quad \ldots\ldots\ldots\ldots \quad (9.15)$$

We then form $g_n=h*f_n$, and, if $g-g_n$ is not negligible, we can compute $f_{2n}, f_{3n}, \ldots$ in turn using the recurrence relation

$$f_{(m+1)n}-f_{mn}=k_n*(g-g_{mn}), \quad m=0, 1, 2, \ldots \quad \ldots\ldots \quad (9.16)$$

in which we have taken $f_0=0$, although it is possible to choose f_0 arbitrarily, setting $g_0=h*f_0$. The computation of $l_n(x, y)$ is tedious, but is worth while if

several observed distributions with the same kernel have to be sharpened. It can be computed by $n-2$ integrations,

$$l_n = {}_nc_2 h - {}_nc_3 h*h + {}_nc_4 h*h*h - \ldots,$$

or by taking two Fourier transforms:

$$\bar{L}_n = n - \frac{1-(1-\bar{H})^n}{\bar{H}},$$

though this may not be practicable except in cases with circular symmetry. If the kernel is expressible as a sum of Gaussian terms or as a sum of terms of the form $a(b^2+r^2)^{-3/2}$, then $l_n(r)$ can be expressed in the same form. For example, if

$$h(x, y) = \varphi(a^2, b^2) \equiv (2\pi ab)^{-1} e^{-\frac{1}{2}(x^2/a^2 + y^2/b^2)},$$

then

$$l_5(x, y) = 10\varphi(a^2, b^2) - 10\varphi(2a^2, 2b^2) + 5\varphi(3a^2, 3b^2) - \varphi(4a^2, 4b^2).$$

This method has been used to correct light curves of the same external galaxies as those mentioned in the preceding section. The function $l_{10}(r)$ corresponding to the kernel (**2.3**) was computed, and it was found possible to integrate $l_{10}*g$ with sufficient accuracy to determine $f_{10}(r)$ within about 2 per cent. The agreement of the results obtained by the two methods was very satisfying.

From (**9.15**), it appears that random errors in the data $g(x, y)$ will be magnified at least n times in $f_n(x, y)$. Thus the solution becomes increasingly unstable as n is increased. Significant errors may also be introduced by repeated numerical integrations. But by using 50 or more cells on the transparent grid representing $h(x, y)$ or $l_n(x, y)$, these errors may be kept reasonably small. This has been verified by a test in which $h(r) = (2\pi)^{-1} e^{-\frac{1}{2}r^2}$, $f(r) = 200e^{-\frac{1}{2}r^2}$, $g(r) = 100e^{-r^2/4}$. The approximations f_2, f_3, f_4, f_5 were computed analytically, and also by four numerical integrations using (**9.14**) with a grid of 55 cells representing $h(r)$. The errors due to numerical integration in f_2, f_3, f_4, f_5 were found to be not greater than 0·6, 1·0, 1·2, and 1·5 respectively. $f_5(r)$ was also computed directly from (**9.15**) with a grid of 79 cells representing $l_5(r)$. The errors in this case were found to be less than 1·0. These errors are regarded as reasonably small, because they are not larger than the errors of observation to be expected in $g(r)$, which are usually at least 1 or 2 per cent. of $g(0)$.

X. Conclusions

It is not possible to lay down fixed rules for deciding which method of solution is best for a particular problem, but the following may serve as a rough guide.

(*a*) If the corrections are small and the kernel has the requisite moments, use a polynomial solution.

(*b*) If the corrections are large, or the moments do not exist, or both, whilst $|H(u, v)| > 0$ everywhere, then Kreisel's method or the solutions (**9.13**)–(**9.16**) (or both) may be applicable.

(*c*) If $H(u, v)=0$ in some region but $|1-\overline{H}(u, v)|<1$ elsewhere, relations (**9.13**)–(**9.16**) will give approximations to the principal solution, provided that the unwanted Fourier components are filtered out as described by Bracewell and Roberts (1954).

(*d*) If (*a*), (*b*), and (*c*) are inapplicable, the solution must be obtained by Fourier transforms or by trial and error. In problems with circular symmetry, the use of Fourier-Bessel transforms is sometimes preferable to the methods in (*b*) and (*c*).

XI. Acknowledgments

The author is indebted to Dr. S. Gascoigne and Dr. G. de Vaucouleurs for encouragement and helpful criticism.

XII. References

Bochner, S. (1932).—"Vorlesungen über Fouriersche Integrale." (Akademische Verlagsgesellschaft: Leipzig.)
Bode, H. W., and Shannon, C. E. (1950).—*Proc. Inst. Radio Engrs., N.Y.* **38**: 417.
Bolton, J. G., and Westfold, K. C. (1950).—*Aust. J. Sci. Res.* A **3**: 25.
Bracewell, R. N., and Roberts, J. A. (1954).—*Aust. J. Phys.* **7**: 615.
Bullard, E. C., and Cooper, R. I. B. (1948).—*Proc. Roy. Soc.* A **194**: 332.
Burger, H. C., and van Cittert, P. H. (1932).—*Z. Phys.* **79**: 722.
Burger, H. C., and van Cittert, P. H. (1933).—*Z. Phys.* **81**: 428.
Coutrez, R. (1949).—Commun. Obs. Belg. No. 11.
Eddington, A. S. (1940).—*Mon. Not. R. Astr. Soc.* **100**: 354.
Fellgett, P. B., and Schmeidler, F. B. (1952).—*Mon. Not. R. Astr. Soc.* **112**: 445.
van de Hulst, H. C. (1941).—*B.A.N.* **9**: 225.
van de Hulst, H. C. (1946).—*B.A.N.* **10**: 75.
Kapteyn, J. C. (1920).—*Publ. Astr. Lab. (Obs.) Groningen* **30**: 43.
Kreisel, G. (1949).—*Proc. Roy. Soc.* A **197**: 160.
Lyttkens, E. (1949).—Ark. Astr. **1**, Nos. 3, 4, 5, 7.
Malmquist, K. G. (1943).—Ark. Mat. Astr. Fys. **29B**, No. 8.
Stokes, A. R. (1948).—*Proc. Phys. Soc. Lond.* **61**: 382.
de Vaucouleurs, G. (1948).—*Ann. Astrophys.* **11**: 247.
Watson, G. N. (1944).—"Theory of Bessel Functions." 2nd Ed. (Cambridge Univ. Press.)