THE DETERMINATION OF ELECTRON TRAJECTORIES IN THE PRESENCE OF SPACE CHARGE

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Summary

The analogy between space charge in an electrostatic field and current sources in an electrolytic tank is considered and methods of simulating clouds of distributed charge by discrete current sources are developed.

The field patterns of charge regions of simple geometrical shapes are compared with those of sources and from these results rules are deduced giving the dimensions and arrangement of a source array suitable for any particular problem.

Experimental results show these principles applied to a plane diode, an axially symmetric electron beam, and a general asymmetric beam. It is concluded that in most problems space charge can be represented adequately in electrolytic models by a comparatively small number of sources.

I. INTRODUCTION

Although several different electron trajectory tracers have been described by Gabor (1937), Langmuir (1937, 1950), Marvaud (1948, 1952), and Sander, Oatley, and Yates (1949), comparatively few machines of this kind are in use. At the same time, problems requiring a knowledge of the paths of electrons or other particles arise frequently and are rarely in a form amenable to direct calculation.

One of the reasons why more use is not made of path tracers is that the effect of space charge is neglected. In many problems of practical interest this leads to appreciable errors. For example, a converging electron stream which is traced to a fine focus may be found to broaden excessively when the current is raised to a useful value, while deflected beams may be changed in position and shape by the influence of their charge. Frequently these effects limit performance, so that the tracings are least accurate when they would otherwise be of the greatest value.

In an attempt to overcome this limitation Musson-Genon (1947) devised an electrolytic tank in which the depth of the liquid was varied locally by an amount depending on the space charge density in that region. Recently Alma, Diemer, and Groendijk (1953) have solved space charge problems by adding distributed forces to models based on the rubber membrane analogy.

In the present work purely electrical methods have been developed, the space charge being represented by currents injected from sources spaced at intervals in an electrolytic model. Methods have been found of spreading the currents over an appreciable volume of the liquid in order to represent distributed charge and of finding the strengths and positions of the sources.

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II. Theoretical Basis

(a) *An Outline of the Method*

In the majority of space charge problems the charge distribution is unknown initially. As a first step, a family of trajectories is drawn for the given boundary conditions but neglecting space charge. A semi-automatic electron path tracer attached to an electrolytic tank is used, the paths being traced, without computation, in a few minutes. Sufficient paths are drawn to subdivide the beam into " tubes of flow " small enough to be considered to have uniform current density across any section. A pattern of current sources is then set up, using the principles to be described later, to correspond to the charge distribution predicted from the flow lines and the measured potential. The gradients from these sources modify the trajectories and the source currents are readjusted to suit the new flow lines. The successive approximations usually converge rapidly.

The use of current injection raises the problems of representing, by means of concentrated sources, charge distributed over an appreciable volume, as the sources are surrounded by strong local gradients which would cause distortion of the trajectories. It would be possible to reduce these by greatly increasing the number of current injection points, but as each of the currents has to be set individually, this becomes impossibly complicated. A solution which reduces to a minimum the number of injection points has been found from a consideration of the source field patterns and the results have been reduced to a number of simple rules for finding the dimensions, disposition, and strengths of the current sources.

In many problems it is only necessary to know whether a proposed electrode system is capable of producing a certain electron current density. By setting the current sources as described in Section II (*f*) this result is given by a single trial.

The procedure will now be considered in detail, beginning with the methods by which small concentrated current sources can be arranged to create field patterns in the model corresponding to those from distributed charges in the prototype.

(b) *The Distribution of the Injected Current*

When no space charge is present an electrolytic model may be used to determine the static distribution of potential U in an electrode system because the Laplace equation

$$\nabla^2 U = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

is true also in a uniformly conducting medium free from sources. When charge is present, the fields are given by Poisson's equation*

$$\nabla^2 U = -\rho/\varepsilon_0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

where ρ=charge density, ε_0=dielectric constant of free space.

* The rationalized M.K.S. system of units is used throughout this paper.

The corresponding expression for the model in an electrolyte of conductivity σ is

$$\nabla^2 U = -i'/\sigma, \qquad (3)$$

where i' may be termed the injected current density, although strictly speaking there is no physical means by which current may be injected in this way without disturbing the isotropy of the medium. However, the total charge δq appearing in the volume δV in the system may be represented in the model by the current δi where, from (2) and (3),

$$\delta i/\delta q = \sigma/\varepsilon_0. \qquad (4)$$

The term σ/ε_0 is a conversion constant by which the current to be injected in the model may be found when the charge distribution in the original is known. The remaining problems are, firstly, that of distributing the current flow from a number of discrete sources in the model so that the gradient and potential patterns of the charge will be simulated as closely as possible and, secondly, of finding the charge distribution consistent with the boundary conditions.

In seeking methods of distributing the current, it seemed probable that the gradient and potential patterns at a liquid surface produced by current flow from a vertical line source, completely submerged some distance below, would be generally similar to those created by a uniform cylindrical column of charge in free space.

It was found that by choosing the dimensions of the source correctly the two fields could be matched closely.

To compare these fields consider a long cylinder of charge of uniform density ρ and radius r_1. At points within the cylinder, $0 < r < r_1$,

$$\mathrm{d}U/\mathrm{d}r = -r\rho/2\varepsilon_0, \qquad (5)$$

and the potential relative to that at some external radius r_2 is

$$U = (\rho r_1^2/4\varepsilon_0)[1-(r/r_1)^2+2\ln r_2/r_1]. \qquad (6)$$

For radii outside the column

$$\mathrm{d}U/\mathrm{d}r = -r_1^2\rho/2\varepsilon_0 r, \qquad (7)$$
$$U = (\rho r_1^2/4\varepsilon_0)(2\ln r_2/r). \qquad (8)$$

These fields are to be reproduced by current from a thin source submerged as shown in Figure 1, in a layer of liquid bounded by a non-conducting floor. Let the current injected be iy_3 where y_3 is the depth of the liquid.

It may be noted immediately that at points remote from the source the current density is almost independent of depth, so that

$$\mathrm{d}U/\mathrm{d}r = -i/2\pi r\sigma, \qquad (9)$$

and the potential relative to that at radius r_2 is

$$U = (i/2\pi\sigma)\ln r_2/r. \qquad (10)$$

Thus, no matter how the current injection rod is changed in length or vertical position, the shapes of the remote gradient and potential curves match those of the charge column and also agree in magnitude when in accordance with the conversion equation (4)

$$i = \pi r_1^2 \rho\sigma/\varepsilon_0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \mathbf{(11)}$$

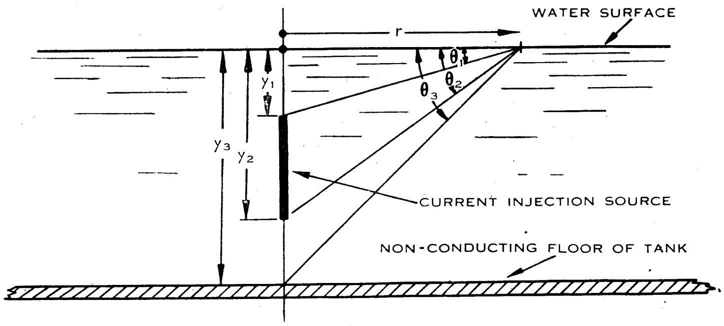

Fig. 1.—A current source used for representing space charge in a two-dimensional model.

To obtain the best match with the space charge field, the position of the gradient maximum must fall close to the radius r_1. The field about a submerged wire has been calculated approximately and measured experimentally and it has been found that the peak occurs near the radius $r = y_1$, as shown in Figure 2.

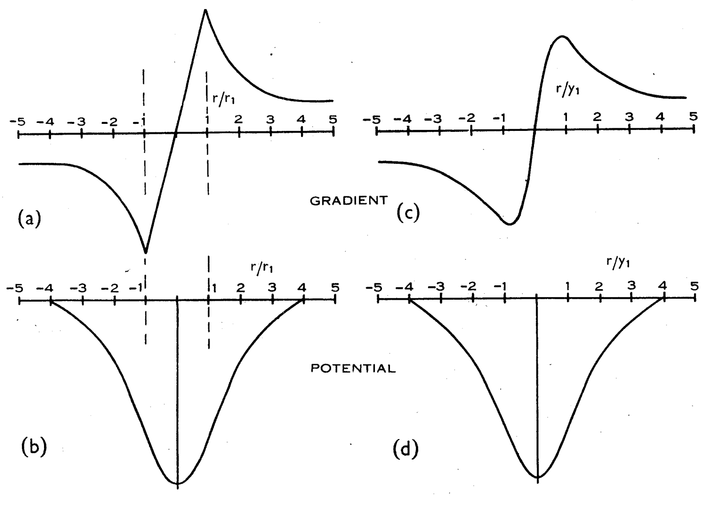

Fig. 2.—A comparison between the gradient (*a*) and potential (*b*) through a column of negative charge and the fields above a current source submerged as shown in Figure 1. The similarity between (*a*) and (*c*), (*b*) and (*d*) is used in representing space charge in two-dimensional models.

This result may be stated as a simple rule: in representing space charge columns by means of submerged line sources the depth of submersion must be equal to the radius of the column. This rule will be adhered to throughout this paper.

Although the position of the maximum is now correct, it is found that, when the source extends to the bottom of the tank ($y_2=y_3$ in Fig. 1), the gradient of the source near its maximum is considerably smaller in magnitude than that of the charge column. Since the total current and y_1 are fixed, y_2 must be varied to improve the match and this can be done without disturbing the agreement at remote points. The determination of y_2 is considered in Appendix I. It is found that, when the line source is continuous between y_1 and y_2, the depth y_2 should be equal to $0 \cdot 5 y_3$. The accuracy of match between the fields of a core of charge and current sources having $y_2/y_3=0 \cdot 5$, but with different values of y_1/y_3, is shown in Figure 3 by the curves (*b*), (*c*), and (*d*). When the ratio y_1/y_3 is less than $0 \cdot 2$ as in curve (*d*), the agreement can be improved by the method described in Appendix I, but this complication is rarely needed.

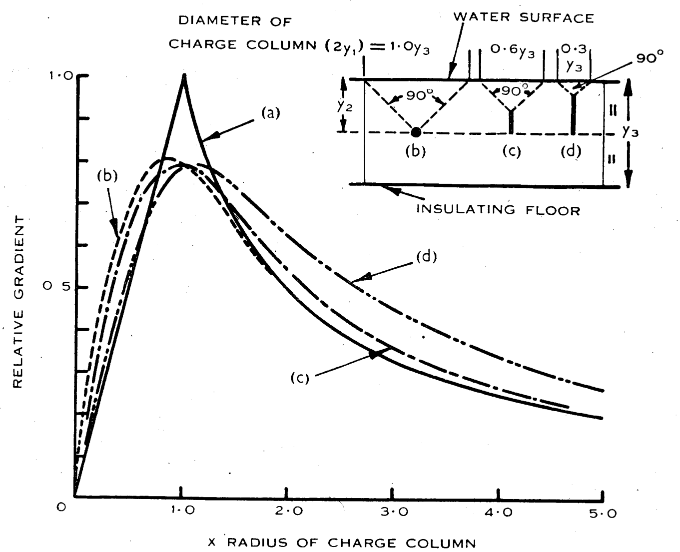

Fig. 3.—The gradient distributions about sources having different values of y_1/y_3 as shown in the key, compared with curve (*a*) for the equivalent charge column.

Thus the second rule for the representation of columns of space charge is : the depth of the bottom of the source should be made equal to half the depth of the tank.

We may now consider methods of grouping a number of simple sources to represent irregular and non-uniform regions of charge.

(*c*) *The Representation of Layers and Charge Clouds of Irregular Shape*

Most of the problems for which an electrolytic tank is useful either are two-dimensional or have axial symmetry and so may be represented in the wedge tank. The two-dimensional model will be considered first.

Suppose, for example, it is required to represent a vertical wall of charge bounded in the ZX-plane corresponding to the liquid surface as shown in

Figure 4 (a). To approximate to this layer, sources are arranged so that the corresponding columns of charge are overlapped to make the total area of the circles equal to the area of the layer. Curves 1 and 3 in Figure 4 (b) show that the gradient and potential measured along the Z-axis, with insulating sheets

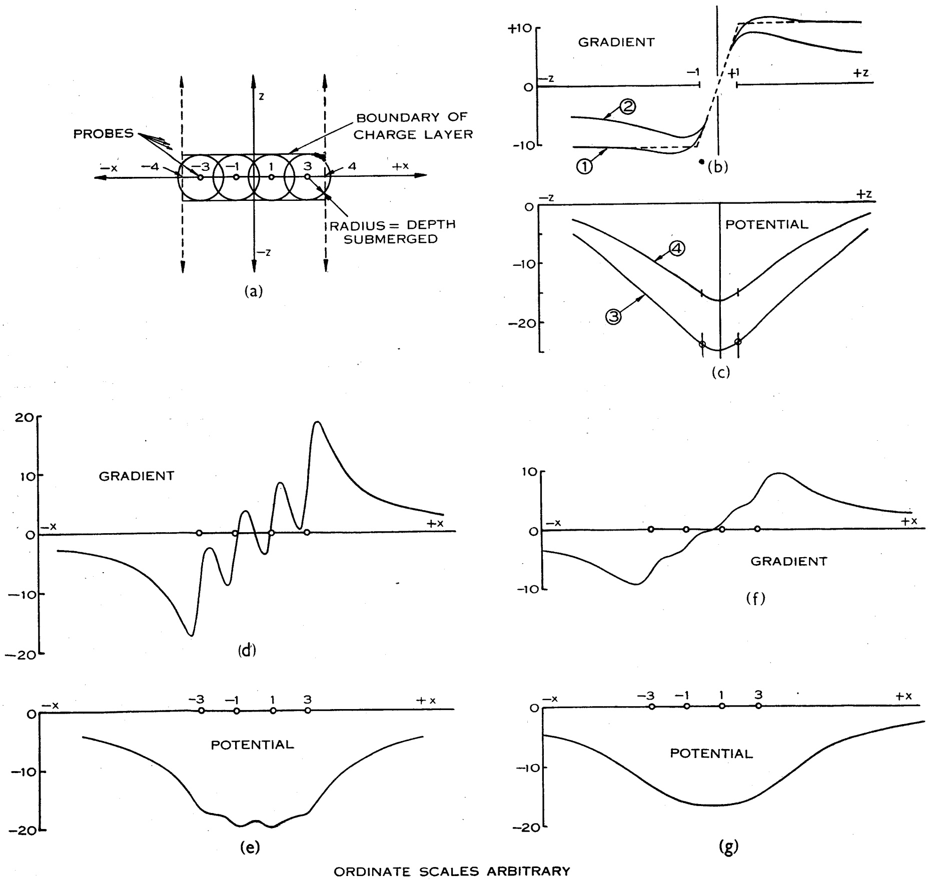

Fig. 4.—The method of combining current sources to simulate a layer of charge. (a) A view in the ZX-plane, the plane of the liquid surface, of four submerged sources and the equivalent layer of negative charge; (b) and (c) the gradient and potential along the Z-axis, curves 1 and 3 (only) were measured with insulating planes in the positions shown dotted in (a) and therefore refer to an infinite layer of charge having the gradient curve shown dotted in (b); (d) and (e) fields along the X-axis of the four sources shown in (a) submerged one-half the normal distance, $y_1=r_1/2$; (f) and (g) as above, but with the sources fully submerged, $y_1=r_1$. The local variations about each source have been greatly reduced.

in positions shown dotted in Figure 4 (a), are close to those of an infinite layer of charge. All the remaining curves in Figure 4 were measured with the insulating sheets removed and approximate to the fields of a finite layer of charge terminated at $z=\pm 4$ in Figure 4 (a). In combining sources in this way advantage is taken of the fact that the field variations localized around each probe are attenuated very rapidly as the depth of submersion is increased. This is seen when Figures 4 (*d*) and (*e*), measured along the X-axis with the array submerged one-half the correct distance, are compared with the almost smooth curves (*f*) and (*g*) from sources at the correct depth, $y_1 = r_1$.

This principle, of grouping probes to represent larger regions of charge, may be applied in the same way to charge clouds of irregular shape. The area representing the cloud in a two-dimensional tank is replaced by a pattern of overlapping circles of different sizes chosen to suit the boundary as accurately as necessary and having the same total area as the charge. The individual source currents are proportional to the charges represented.

(*d*) *The Proximity Effect*

When one or a line of sources is placed in close proximity to a conducting plane, the gradient at the liquid surface is slightly reduced by the presence of the plane. If the plane represents the surface of an emitting cathode where the magnitude of the gradient is important, it is necessary to correct for this reduction by increasing the source current. Figure 5 (*a*) shows a typical measurement

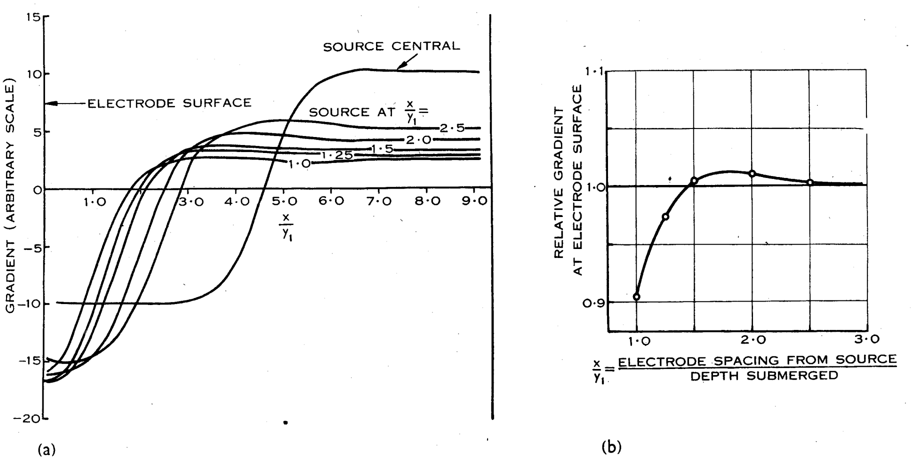

Fig. 5.—The proximity effect.

of this effect. A single constant-current source was placed in an insulated parallel-sided cell of width $\pi y_1/2$ closed by two perpendicular end electrodes and the gradient was measured for different positions of the source. As the change in gradient between the upper and lower levels of the curves in Figure 5 (*a*) should remain constant, the influence of the plane is shown by the lack of symmetry between the upper and lower crossings and in Figure 5 (*b*). At a distance $x = y_1$, the source current should be increased by 10 per cent. but at larger spacings no correction is needed. Although this method of correction raises the field on the other side of the source also, this position is usually less important than the cathode surface.

(*e*) *Charge in Axially Symmetric Systems*

Several different arrays have been used as sources in wedge tank models of axially symmetric systems. When the beam itself has axial symmetry and does not converge or diverge too rapidly, the charge may be represented by point sources placed along the axis as shown in Figure 6. Usually the spacing will be non-uniform. In order to reduce the number of points it is necessary to increase their spacing as much as possible. The spacing is limited by the need to keep local variations in the field at the edge of the beam small. Thus to find a suitable compromise it is necessary to consider the way in which these variations diminish away from the axis.

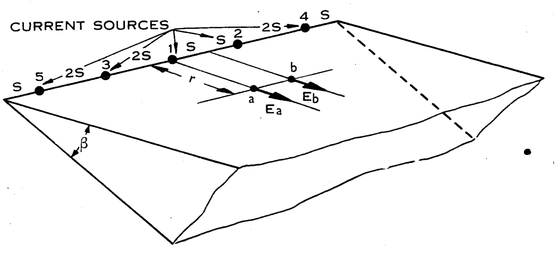

Fig. 6.—Point sources arranged along the axis of a wedge tank.

When an array is uniformly spaced, as shown in Figure 6, the maximum value of the radial gradient occurs at a and at corresponding points opposite each source. Thus, summing the contributions from an infinite line of sources each passing a current i,

$$E_a = \frac{-i\alpha}{\beta\sigma S^2}\left\{\frac{1}{2\alpha^3}+\sum_{N=1}^{N=\infty}\frac{1}{[\alpha^2+(2N)^2]^{1\cdot5}}\right\}, \quad \ldots\ldots\ldots \quad (12)$$

similarly, the minimum value E_b is given by

$$E_b = -\sum_{N=0}^{N=\infty}\frac{i\alpha}{\beta\sigma S^2}\,\frac{1}{[\alpha^2+(2N+1)^2]^{1\cdot5}}, \quad \ldots\ldots\ldots\ldots \quad (13)$$

where $\alpha = r/S$.

These expressions have been computed in the range $1<\alpha<2\cdot2$ and checked experimentally. From the results shown in Figure 7 it is seen that the field of a row of point sources is equivalent to that of a line source having a superimposed space wave that varies in the axial direction and is attenuated rapidly away from the axis. The variation allowable along the trajectory depends on the conditions of the problem ; for example α may have a smaller value where space charge is a correction rather than the controlling field. Usually it is safe to allow the trajectory to approach to $\alpha=2$, that is to within a radial distance equal to the spacing between sources, and in many cases α may be reduced to $1\cdot0$.

When the sources are placed along the axis, the trajectories being considered need not lie in the beam envelope. An inner path may be traced under space charge conditions by disregarding the shell of charge outside it and resetting the source currents to correspond to the inner beam ; but it is necessary to check whether the resulting rise in the potential of the beam as a whole introduces an appreciable error.

The more general case of a beam travelling away from the axis will now be considered. It is assumed that the beam electrode system can be represented to the desired accuracy either in a normal wedge tank or in one in which the wedge angle is π, and that the centre line of the beam always lies in the plane of the water surface. The problem then is to represent the half beam charge using sources as sparingly as possible. When the envelope is known approximately, the choice falls on submerged point sources arranged as shown in

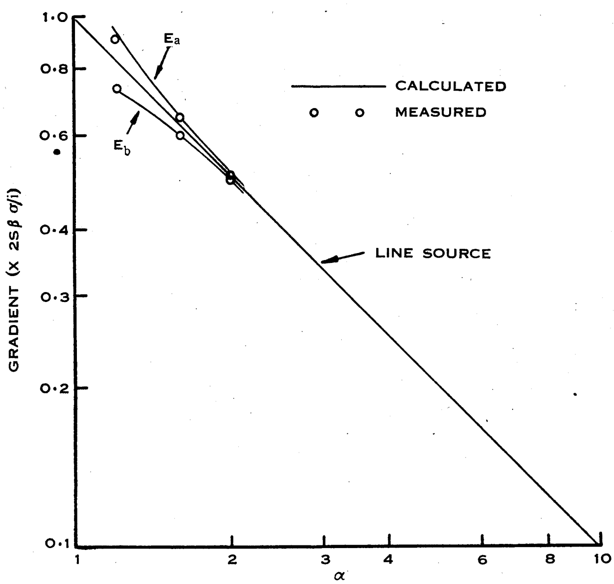

Fig. 7.—The maximum and minimum values of the radial field created by the sources shown in Figure 6.

Figure 8, the depth of submergence being made equal to the radius of the electron stream r_1 in order to bring the position of the gradient maximum to the edge of the stream and to reduce local field variations within the envelope. As the probe cannot approach closer than distance r_1 to the sources, these may be considered equivalent to a line of strength $i/\delta l$ and an image above the liquid surface. The wedge angle is assumed to be sufficiently large to allow other images to be neglected. The mean gradient at the edge ab is then

$$\mathrm{d}U/\mathrm{d}r = -i/2\pi r\sigma\delta l, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{14})$$

i.e. one-half the value obtained from sources placed at the central positions c', d', e', . . . Thus the source current calculated from the beam space charge

$$i/\delta l = \pi r^2 \rho\sigma/2\varepsilon_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{15})$$

cannot be used directly but must be increased to twice this value. The edge field is then correct and that within the beam nearly so. The remote field is too large, but in many problems this is less important. The variation in the edge field at positions *a* and *b* (Fig. 8) is less than that shown in Figure 7 for $\alpha=\sqrt{2}$ because the angle between the source and image gradients is less at the mid positions such as *b*. The variation measured experimentally was ± 4 per cent.

When the beam envelope is unknown initially it is simpler to place the line of sources near the water surface and set to the currents indicated by equation (**4**). If the sources are at the centre of the beam the gradient is correct for the edge electrons, whatever changes occur in the radius of the envelope, but paths within the beam cannot be traced using the same source currents.

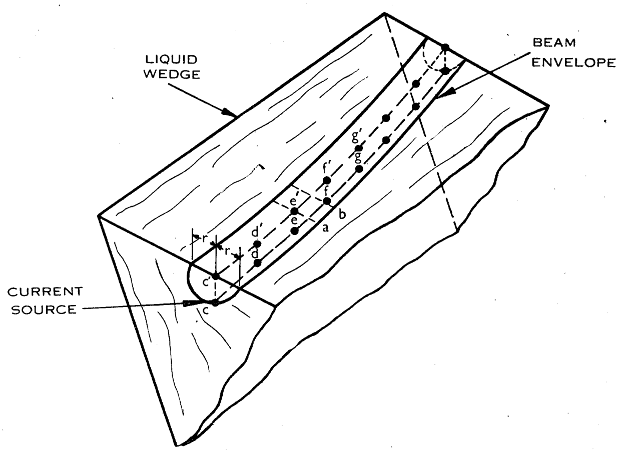

Fig. 8.—An array of sources *c*, *d*, *e*, *f*, . . . representing charge in a beam moving off the axis in a wedge model. The alternative positions *c′*, *d′*, *e′*, *f′*, . . . are used in tracing the trajectories defining the beam envelope.

When the paths converge to a focus of small radius, it is usually inconvenient to increase the number of sources sufficiently to define it exactly. In these cases it is simpler to calculate the minimum radius from the convergence and radius at some earlier position, using the space charge formulae referred to in Section III (*b*) and ignoring fields other than that of the beam charge.

(*f*) *Scaling and the Determination of Charge Density*

The electrolytic model will usually differ from the original electrode structure both in size and in the scale of the applied potentials. In representing space charge, the potential scale, which may be ignored in solving the Laplace equation, must be taken into account in determining the values of the source currents. When the electron paths are influenced by space charge, it may be shown that the paths in the model will correspond exactly with those in the original if the relationship between the current and potential in the model is such that the perveance remains the same as in the original system.

As a first step, electron paths are drawn for the space-charge-free field. These paths form boundaries of " tubes " of electron current flow. Assuming that the current in a given tube in the original is known to be I_0, the current in the corresponding tube of the model I_m is

$$I_m = I_0(U_m/U_0)^{1\cdot5}, \qquad (16)$$

where U_0, U_m are potentials in the original and model respectively.

If U_m', U_0' are the mean potentials in a small length δl of the current tube which is to be represented in the tank by one source, the total charge δq, in this length δl is

$$\delta q = \frac{\delta l I_m}{(2e/m)^{0\cdot5} U_m'^{0\cdot5}} = \frac{\delta l I_0 (U_m/U_0)^{1\cdot5}}{(2e/m)^{0\cdot5} U_m'^{0\cdot5}}, \qquad (17)$$

and the source current i is given by

$$i = \frac{\delta q \sigma}{\varepsilon_0} = I_0 \frac{\delta l U_m{}^{1\cdot5} \sigma}{U_m'^{0\cdot5} U_0 (2e/m)^{0\cdot5} \varepsilon_0}. \qquad (18)$$

(In M.K.S. units the numerical value of $1/(2e/m)^{0\cdot5}\varepsilon_0$ is $1\cdot9053 \times 10^5$.)

When sources are being set to represent axial beams in the wedge tank, the separate measurement of the conductivity σ may be avoided by measuring the gradient at one or more values of the radius r, where, if β is the wedge angle,

$$i/\delta l \beta r = \mathrm{d}U/\mathrm{d}r. \qquad (19)$$

Therefore from (**18**) and (**19**)

$$\left[\frac{\beta r}{U_m} \cdot \frac{\mathrm{d}U}{\mathrm{d}r}\right]_r = \frac{I_0/U_0{}^{3/2}}{(U_m'/U_m)^{0\cdot5}/(2e/m)^{0\cdot5}\varepsilon_0}. \qquad (20)$$

When the current I_0 is not known initially, as for example in structures which include a cathode surface, the source currents are set so that for all sources

$$i U_m'^{0\cdot5}/\delta l = K. \qquad (21)$$

The value of K is then increased until the off-cathode gradient falls to zero. The value of I_0 then follows from (**18**). For similar electrode structures the perveance is independent of size, so that the real dimensions of the original electrodes are needed only for the calculation of the cathode current density, transit time, and so on, from the model measurements.

The potential U_m' has been defined as the mean potential in the interval. When the ratio of the potentials at the beginning (U_1) and the end (U_2) of the interval differs appreciably from unity, the source current can be set more accurately by taking as the value of U_m' a potential corresponding to the mean velocity of an electron in the interval. Thus

$$U_m' = \left[(x_1 - x_2) \Big/ \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{U}}\right]^2. \qquad (22)$$

The curves in Figure 9 were calculated from (**22**) and the relation

$$U \propto x^N,$$

where $N = 1\cdot\dot{3}$ applies to a plane diode and the exponents $1\cdot2$ and $1\cdot5$ are, within the limits indicated, approximations to the results of Langmuir and Blodgett (1924) for space charge limited flow between spherical surfaces. Thus the curve in Figure 9 marked $N=1\cdot2$ refers to a comparatively rapidly expanding tube of flow and that marked $N=1\cdot5$ refers to one contracting. The small difference between these indicates that in setting currents near the cathode surface values from curve $N=1\cdot\dot{3}$ may be used in most cases and elsewhere, when $U_1/U_2>0\cdot3$, the mean potential may be taken for U_m'.

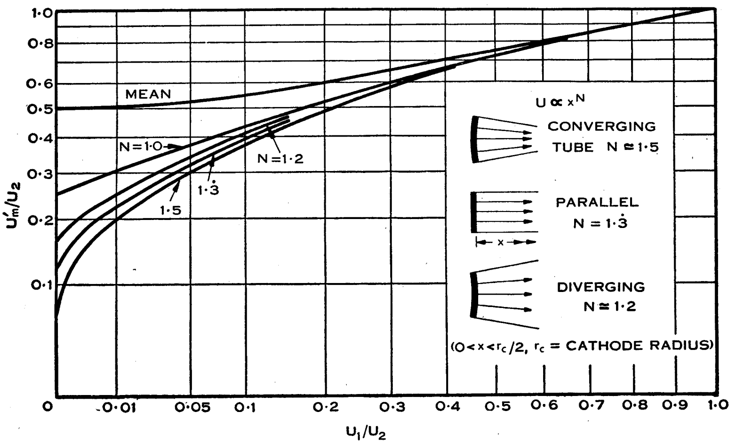

Fig. 9.—Values of U_m' in terms of U_1 and U_2 for converging, parallel and diverging space charge limited flow. In the cases $N\approx1\cdot5$ and $\approx1\cdot2$, $x=|r_c-r|$.

III. Experimental Results

(a) *The Plane Diode*

Three examples will be given to illustrate the use of these methods in space charge problems. The results of two of these may be compared with analytical solutions so that the accuracy likely to be achieved in similar problems may be estimated.

The first is a model of an infinite parallel plane diode consisting of an insulated cell having electrodes and current sources arranged as shown in Figure 10. The sources are sleeves of $0\cdot1$ mm brass supported on 3 mm glass tubes and connected through individual variable resistances to a common point. The potential U'' of this point is adjustable between zero and 110 V, 50 c/s. When, as in this model, a tube of flow terminates on an emitting cathode, the gradient at the cathode surface is much more sensitive to nearby sources than to those further away. This offers a more rapid method of finding the source currents than that based on the Laplace distribution. First the cathode gradient is reduced to zero by adjusting the first two sources alone to a common value of K (eqn. (**21**)). All the remaining sources are then set directly to this value and the potential U'' is reduced until the cathode gradient returns to zero. A sufficiently accurate solution usually follows one further relaxation of the source currents.

The diode model was solved in this way and the results are compared with the theoretical distribution in Figure 10. The perveance, calculated from the mean value of K and a separate measurement of the conductivity was within 1 per cent. and the individual values were within ± 5 per cent. of the correct result.

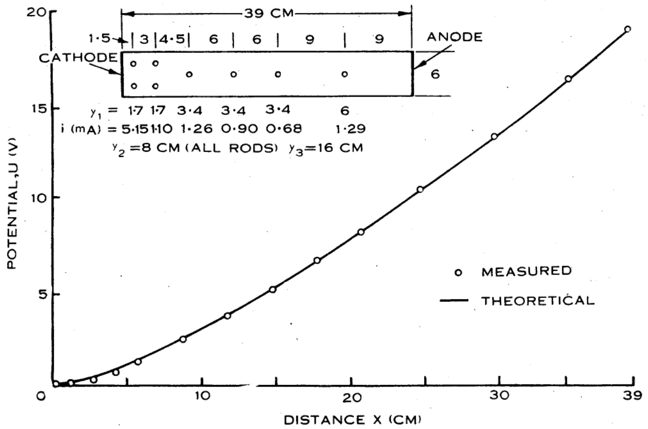

Fig. 10.—The solution for a space charge limited, infinite parallel plane diode using six sources in an insulated cell. The currents shown for the first two sources are each divided equally between two rods in order to fulfil the requirement that $y_1=r_1$.

(*b*) *Defocusing of a Convergent Beam*

The second example shows the effect of space-charge in expanding the focus of a uniformly convergent electron beam. In this test, values of the product $k\tau$, where k is a function of the perveance given by

$$k^2=33\times10^{-6}U^{1\cdot5}/I, \qquad (23)$$

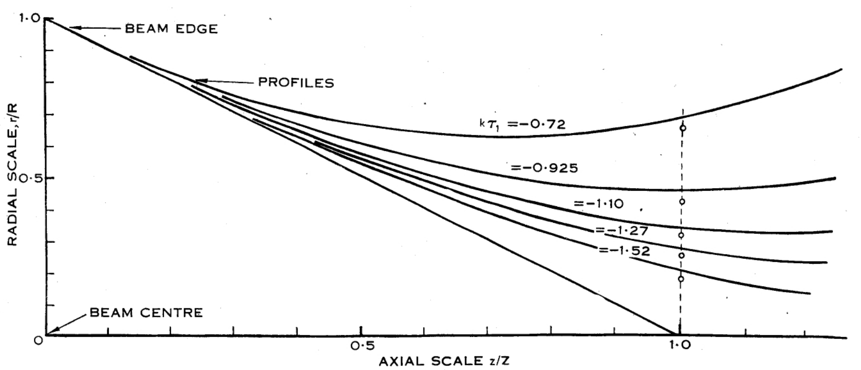

Fig. 11.—The envelope trajectories of a convergent axially symmetric beam defocused by space charge.
———— Copied from experimental tracings.
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and τ is the initial convergence R/Z (Fig. 11), were chosen to correspond to those already published (Hollway 1952). The source currents were set in accordance with equation (**20**) by the direct measurement of the gradient as described in Section II (*f*). The presence of the sources results in a variation in potential along the axis of the beam which is ignored in the calculation. By rearranging the circuit of the electron path tracer paths were drawn for both constant and varying beam potential and the difference was shown to be small. As the shapes of the calculated paths agree closely with the tracings shown in Figure 11, for clarity only a single calculated point is given for each path. While part of the difference shown is experimental error, the greater part is explained by the fact that the calculation is approximate for such rapidly converging beams because the axial electron velocity, instead of the total velocity, is assumed to be equivalent to the beam potential, whereas the experimental solution is more accurate.

(c) *A Deflected Beam*

The third is an example of the more general problem of predicting the shape of a beam moving off the axis of an axially symmetric system. In the absence of space charge the initially parallel trajectories leaving electrode A in Figure 12 (a) are brought to a focus some distance in front of the anode B by

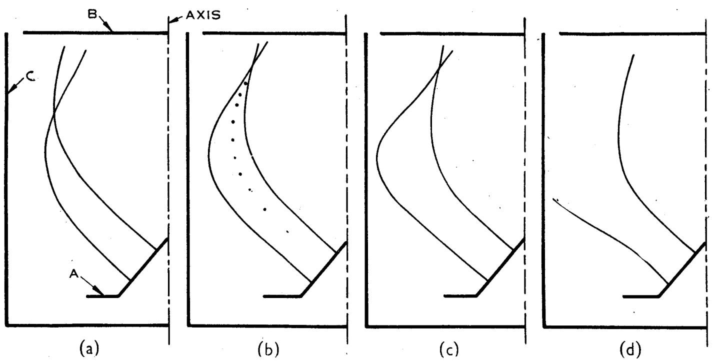

Fig. 12.—A general space charge problem showing the change in form of a deflected beam with increasing current. Electrode potentials: A 250 V, B 500 V, C zero. (a) Calculated beam current $I_0=0$, (b) $I_0=0\cdot8$ mA, (c) $I_0=1\cdot0$ mA, (d) $I_0=1\cdot8$ mA.

the field of a cylinder C at zero potential. Sources were set at unequal intervals along the centre of the beam, slightly submerged to miss the moving probe of the trajectory tracer. The spacings were made less than the local beam diameter as shown in Figure 12 (b). After measuring the potentials at the beginnings and ends of the intervals and setting the source currents, usually only one revision was necessary to obtain a sufficiently constant value of K. The paths drawn in Figures 12 (b), (c), and (d) show the expansion of the beam with increasing beam current. It may be predicted from the change in curvature of the paths approaching the electrode C that the overall focus would be broad, especially in (c). When the current is raised still further, much of the beam is dispersed.

IV. Conclusions

It is shown that by arranging current sources according to the rules given, the influence of space charge can be examined quantitatively in most of the usual electrolytic models whether or not the electron current is known initially.

Although the convergence of the source currents to their final values is quite rapid, it would be a useful extension of the technique to include in each source lead a control circuit designed to keep the current proportional to $1/\sqrt{U'_m}$. The electron current in a given tube of flow could then be set directly and the bounding trajectories found more rapidly.

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VI. References

Alma, G. A., Diemer, G., and Groendijk, H. (1953).—*Philips Tech. Rev.* **14** : 336-44.
Gabor, D. (1937).—*Nature* **139** : 373.
Hollway, D. L. (1952).—*Aust. J. Sci. Res.* A **5** : 430.
Langmuir, D. B. (1937).—*Nature* **139** : 1066.
Langmuir, D. B. (1950).—*R.C.A. Rev.* **11** : 143-54.
Langmuir, I., and Blodgett, K. B. (1924).—*Phys. Rev.* **24** : 49-59.
Marvaud, J. (1948).—*C. R. Acad. Sci., Paris* **226** : 476-8.
Marvaud, J. (1952).—*C. R. Acad. Sci., Paris* **234** : 45-7.
Musson-Genon, R. (1947).—*Ann. Télécomm.* **2** : 298-320.
Sander, K. F., Oatley, C. W., and Yates, J. G. (1949).—*Nature* **163** : 403.

Appendix I

The Optimum Length of a Vertical Line Source

As described in Section II (*b*), a cylindrical column of space charge may be represented by a vertical line source extending from the depth y_1 to y_2 in the layer of liquid shown in Figure 1. To find the optimum value of y_2/y_3 an expression must be derived for the potential gradient at the liquid surface. For the present purpose it may be assumed that at all positions along the wire the current per unit length flowing into the liquid is constant and equal to

$$iy_3/(y_2-y_1).$$

The gradient in the plane of the liquid surface is found by assuming an image of the source above the surface and, when r is not small compared with y_3, images below the floor and above the floor-image must be included also. It is not necessary to specify these exactly, however. They may be replaced by uniform line sources extending from + and $-y_3$ to infinity. The gradient then becomes

$$\frac{\mathrm{d}U}{\mathrm{d}r}=\frac{-ir}{2\pi\sigma}\left\{\frac{y_3}{y_2-y_1}\int_{y_1}^{y_2}\frac{\mathrm{d}y}{(y^2+r^2)^{1\cdot5}}+\int_{y_3}^{\infty}\frac{\mathrm{d}y}{(y^2+r^2)^{1\cdot5}}\right\} \quad \ldots\ldots (24)$$

$$=\frac{-i}{2\pi\sigma r}\left\{\frac{y_3}{y_2-y_1}\ (\sin\theta_2-\sin\theta_1)+(1-\sin\theta_3)\right\}. \quad \ldots . (25)$$

In order to match the field just outside the core of charge, (**25**) must become identical with (**9**). The bracketed term of (**25**) is therefore set equal to unity and y_2/y_3 is found as a function of r/y_1 for different values of the parameter

y_2/y_1. As y_2/y_3 is reduced, the field in the region of the maximum is increased, raising the curves (*b*), (*c*), and (*d*) in Figure 3 with respect to (*a*). As the crests of these curves are less sharp than (*a*), a compromise is needed to obtain the best general match and the value $y_2/y_3=0 \cdot 5$ has been chosen. The curves (*b*)-(*d*) show the comparatively small effect of variations in y_1 when y_2/y_3 is kept constant and equal to $0 \cdot 5$.

When $y_1/y_3<0 \cdot 2$, as in (*d*), a somewhat closer agreement can be obtained by using, instead of a line source between y_1 and y_2, one extending from y_1 to y_3 but including equally spaced gaps. Experiment shows that a suitable length for the conductors is $0 \cdot 5y_1$ and for the gaps $2 \cdot 5y_1$. Thus the sources extend from y_1 to $1 \cdot 5y_1$, $4y_1$ to $4 \cdot 5y_1$, and so on, until y_3 is reached.