CHORD CONSTRUCTION FOR CORRECTING AERIAL SMOOTHING

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[*Manuscript received January 28, 1955*]

Summary

A graphical method is given for correcting the smoothing effect which arises when a celestial distribution of brightness temperature is scanned with an aerial beam. The method is much simpler to apply than previous methods which have been used and is no less accurate.

I. Introduction

The general problem of aerial smoothing in radio astronomy has been discussed by Bracewell and Roberts (1954) with a view to clarifying the situation. In that paper current practice as regards the correction of smoothing was critically examined. In the present communication a new method of correction is proposed which has the merits of being quick to apply and accurate.

II. Graphical Method of Correction

The correction is carried out graphically as follows.

In Figure 1 let $T_a(\varphi)$ be an observed temperature distribution which it is desired to correct. Let $T(\varphi)$ be the true distribution and $A(\varphi)$ the response of the aerial to a point source. Then the integral equation connecting these functions is

$$T(\varphi)=\int_{-\infty}^{\infty}A(\varphi-u)T(u)\mathrm{d}u.$$

It has been shown (Bracewell 1955) that a solution of this equation, given $T_a(\varphi)$ and $A(\varphi)$, can be expressed in the form

$$T_a(\varphi)+\Sigma\psi_n\Delta^nT_a(\varphi),$$

where Δ^n signifies the operation of taking nth-order finite differences over an interval a. When $A(\varphi)$ is symmetrical about $\varphi=0$, n ranges over the positive even integers. If only two terms of the solution are retained we have

$$T_a(\varphi)+\psi_2\Delta^2T_a(\varphi). \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

This expression may be readily evaluated graphically, provided ψ_2 and a are stated. Thus Figure 1 shows a chord of span $2a$ drawn across an observed curve T_a. The mid point of the chord falls below the curve by an amount proportional to $\Delta^2T_a(\varphi)$. Hence, provided ψ_2 is known, the value $T_a+\psi_2\Delta^2T_a$

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may be plotted. This is a particularly simple construction, especially as, in general, it is possible to arrange that $\psi_2 = -\frac{1}{2}$. Then the amount of the correction is exactly equal to the intercept between the chord and the curve, thus eliminating any multiplication from the construction.

The necessary quantities ψ_2 and a have to be determined from $A(\varphi)$, and prove to be independent of φ. It has been shown (Bracewell 1955) that in cases arising in optics one may take $\psi_2 = -\frac{1}{2}$ and $a = \sigma$ where σ is the standard deviation of the instrumental profile. This possibility vanishes when the instrumental profile is an aerial pattern, for the following reason. The aerial aperture distribution, being necessarily finite, in general falls discontinuously to zero at the

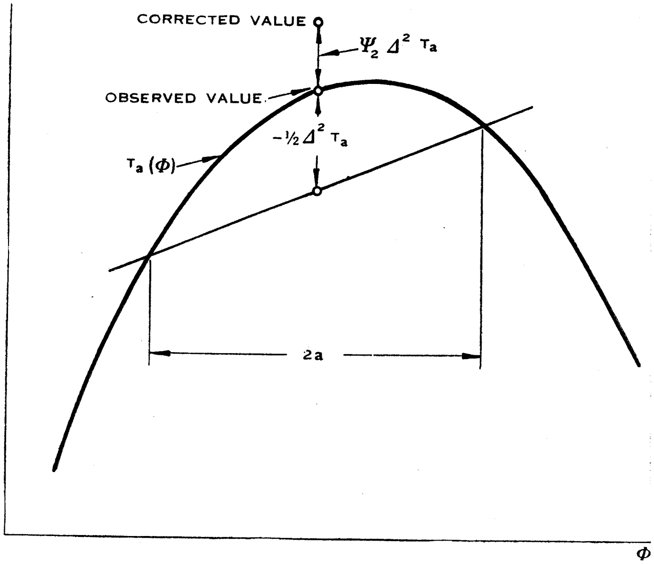

Fig. 1.—Details of the chord construction.

extremes of the aerial. The Fourier transform of such a distribution will behave asymptotically as φ^{-1} when $\varphi \rightarrow \infty$. The aerial pattern $A(\varphi)$, which is the square of this transform, will behave as φ^{-2}. Consequently the second moment of the distribution $A(\varphi)$ about $\varphi = 0$ will not, in general, exist. This circumstance necessitates a special approach to the case of aerial smoothing as distinct from other types of instrumental broadening.

By the convolution theorem,

$$\overline{T}_a(s) = \overline{A}(s)\overline{T}(s),$$

where the bars indicate Fourier transforms, and therefore

$$\overline{T}(s) = \overline{T}_a(s) + \left(\frac{1}{\overline{A}(s)} - 1\right)\overline{T}_a(s) \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

for values of s such that $A(s) \neq 0$. Now if we compare the right-hand side of (2) with the Fourier transform of (1), namely,

$$\overline{T}_a(s) - (4\psi_2 \sin^2 \pi as)\overline{T}_a(s),$$

it will be seen that the determination of ψ_2 and a reduces to arranging the best fit between $4\psi_2 \sin^2 \pi as$ and $[\bar{A}(s)]^{-1}-1$, and this may be done graphically as in the examples that follow.

(a) *Example 1: Uniform Aperture Distribution*

In this common case we have

$$A(\varphi)=b\left(\frac{\sin \pi\varphi/b}{\pi\varphi}\right)^2,$$

and therefore

$$\bar{A}(s)=1-bs, \qquad 0<s<b^{-1}. \qquad \ldots\ldots\ldots\ (3)$$

In Figure 2 we see $[\bar{A}(s)]^{-1}-1$ and $2 \sin^2 \frac{1}{2}\pi bs$ compared. The agreement is good for values of bs as large as $\frac{1}{2}$. Hence excellent correction may be expected

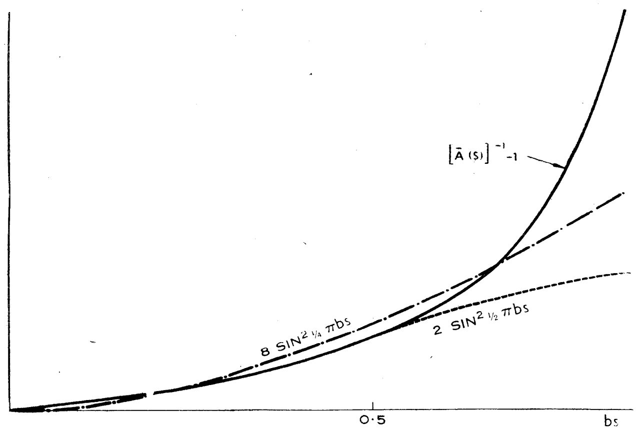

Fig. 2.—Showing the good agreement between $[\bar{A}(s)]^{-1}-1$ and $2 \sin^2\frac{1}{2}\pi bs$.

for Fourier components such that $0<s<\frac{1}{2}b^{-1}$, where b^{-1} is the cut-off value of s (see Bracewell and Roberts 1954), and partial correction may be expected for components of higher spatial frequency up to b^{-1}. As shown by Bracewell and Roberts $T_a(\varphi)$ will not contain Fourier components with more than b^{-1} waves per unit of φ.

For this case, then, we find

$$\psi_2=-\tfrac{1}{2},$$
$$a=\tfrac{1}{2}b.$$

It is useful to note that the chord span $2a$ is equal to half the beam width between zeros.

Other ways of choosing ψ_2 and a are open, and in special circumstances they may have value. For example, the curve $8 \sin^2 \frac{1}{4}\pi bs$ is shown by a dot-dash line in Figure 2. In this case the chord span is one-half the former value but the factor ψ_2 is four times larger. Probably this would not be as convenient in practice in most cases.

(*b*) *Example* 2 : *Tapered Distribution*

The distribution in the aperture plane of a paraboloidal reflector normally falls off towards the rim and we may take for an example the aperture distribution of Figure 3, namely,

$$E(x)=2-x^2, \qquad |x|<1.$$

The function $\bar{A}(s)$, which is proportional to the autocorrelation function of $E(x)$, is readily calculated. Although in the present case the integral can be performed without difficulty, it will be found much quicker to calculate $\bar{A}(s)$ numerically (Fig. 3 (*b*)). The resulting curve of $[\bar{A}(s)]^{-1}-1$ is compared with the uniform case in Figure 3 (*c*). The agreement is seen to be rather close, and further examination shows that $2\sin^2 \frac{1}{2}\pi bs$ is as good a match for the tapered as for the uniform case.

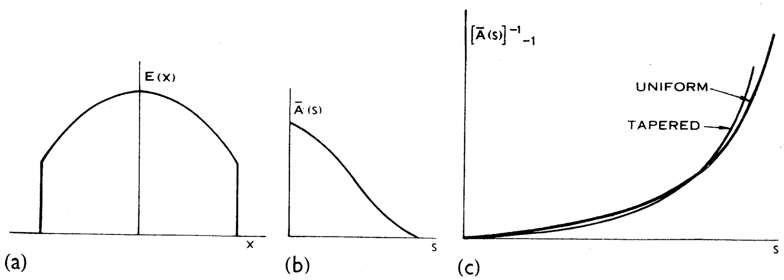

Fig. 3 (*a*).—Tapered aperture distribution.
Fig. 3 (*b*).—Corresponding $\bar{A}(s)$.
Fig. 3 (*c*). $[\bar{A}(s)]^{-1}-1$ for the uniform and tapered aperture distributions.

It therefore appears that moderate tapering of the aperture distribution does not seriously affect ψ_2 and a. However, in a particular case this matter could always be studied very easily by the method indicated, with a view to modifying ψ_2 and a to obtain a best fit over the range of s considered most appropriate to the nature of $T_a(\varphi)$.

III. Comparison with the Method of Successive Substitutions

The method of correcting aerial smoothing which has been most widely adopted in radio astronomy (Bolton and Westfold 1950 ; Bracewell and Roberts 1954) is referred to as the method of successive substitutions. In this method one tests an approximate solution by subjecting it to aerial smoothing and compares the result with $T_a(\varphi)$. The discrepancy is then applied as a correction to the approximate solution, and the process repeated. This method engenders confidence, because it is self-checking, and is quite popular in fields other than radio astronomy (van Cittert 1931). One or two stages are usually found to bring the discrepancy within the observational error.

In the uniform aperture case, the spectrum of T_a bears a ratio to that of T which is the function of s shown in curve A of Figure 4 (see equation (**3**)). After one stage of correction this spectral factor is brought closer to unity (curve B) and after two and three stages the effect is as represented by the dotted curves.

Curve D represents the effect of a very large number of stages, i.e. full restoration. The equation of the n-stage curve may be shown to be

$$y = 1 - (bs)^{n+1}.$$

With the chord construction here described, the degree of restoration jumps straight to the high values of curve C whose equation is

$$y = (1 - bs)(1 + 2 \sin^2 \tfrac{1}{2}\pi bs).$$

This impressive result shows that the simple chord construction is as good as several stages of successive substitutions. Evidently the series of finite differences converges more rapidly than the series on which the method of successive substitutions is based.

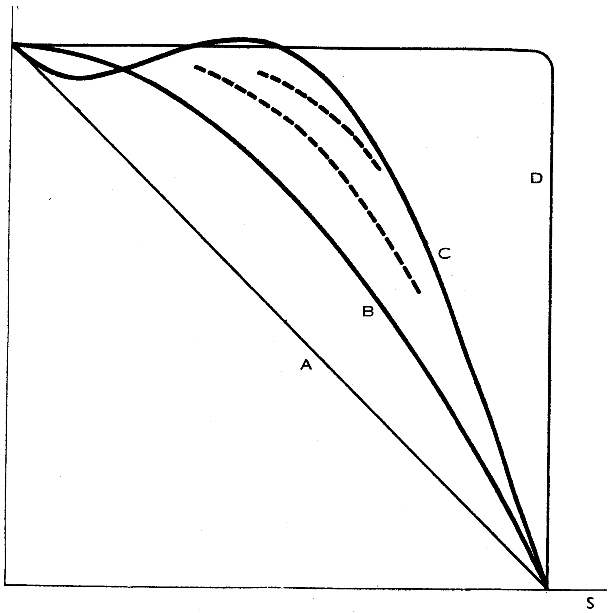

Fig. 4.—Spectral content of various distributions relative to that of $T(\varphi)$.

It should not be forgotten, however, that the precise nature of $T_a(\varphi)$ also enters into a comparison between methods. Thus for small values of s it will be noticed that, although the corrections are small, nevertheless successive substitutions give them better than the chord method. Therefore, to take an extreme case, if $T_a(\varphi)$ contains strong components for $s < 0 \cdot 2$, and weak components for $s > 0 \cdot 2$, then better results might be obtained by successive substitutions.

As a comparative illustration we may take the stripwise distribution across the quiet Sun at 21 cm observed by Christiansen and Warburton (1953). Figure 5 shows $T_a(\varphi)$, $A(\varphi)$, and the chord used for correction. Curve (a), shown displaced, is the result of correcting by the chord construction, and curve (b) shows for comparison the known result of full restoration calculated by Bracewell and

Roberts. The present result is, for most purposes, as good as the full restoration, and better than the result of one stage of successive substitutions. In this example the low values of s are present in considerable strength since the distribution is wide compared with the aerial beam width. The effect discussed in the previous paragraph therefore appears and its magnitude will be looked into. The effect is to give zero correction over the flat top of $T_a(\varphi)$, and this leads to values about 2 per cent. too low in this range of φ. Thus the effect is

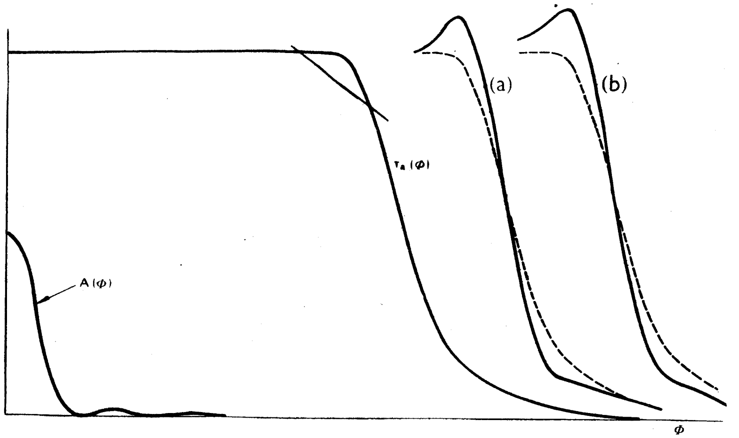

Fig. 5.—Stripwise distribution of brightness across the quiet Sun at 21 cm ; (*a*) corrected by chord construction, (*b*) accurate solution.

unimportant, being within the limits of error. Furthermore, it has no effect on the *shape* of the distribution. It therefore seems likely that experience with the chord construction will show it to be very generally satisfactory in accuracy as well as in speed.

IV. References

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