THE TRANSPORT OF HEAT THROUGH NARROW CHANNELS BY LIQUID HELIUM II

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Summary

In narrow channels ($\sim 10^{-4}$ cm) the observed heat transport is considerably larger than calculated by the internal convection theory. It is suggested that, because of the anisotropy of the distribution of phonons in the channel walls resulting from the temperature gradient, the normal fluid in the immediate vicinity of the walls is not at rest but flows towards the colder region. The magnitude of the resulting heat transport is in reasonable agreement with the observed discrepancy.

I. INTRODUCTION

Heat transport in liquid helium II is considered to be an internal convection process (London and Zilsel 1948; Gorter and Mellink 1949). A temperature gradient gives rise to a fountain pressure gradient

$$\text{grad } p = \rho S \text{ grad } T, \qquad \qquad (1)$$

where ρ is the total density of the fluid and S its entropy per unit mass. The fountain pressure drives the normal fluid towards the cold region, and there is an equal return flow of superfluid. In narrow channels and for small temperature gradients this flow is limited by the viscous friction of the normal fluid. We shall not consider here cases of wide channels and large temperature gradients, where the mutual friction between the normal fluid and the superfluid becomes important, leading to a non-linear dependence of flow on grad T (Gorter and Mellink 1949). The flow (volume per unit time) through a slit of unit breadth and width d due to the fountain pressure is

$$V = -\frac{d^3}{12\eta} \text{ grad } p, \qquad \qquad (2)$$

where η is the coefficient of viscosity of the normal fluid. Since the returning superfluid has no entropy, this circulation mechanism gives rise to a heat current

$$Q = \rho S T V = -\frac{\rho^2 S^2 T d^3}{12\eta} \text{ grad } T \qquad \qquad (3)$$

leading to an apparent thermal conductivity $\varkappa = -Q/(d \text{ grad } T)$, which varies with temperature and is proportional to d^2.

When comparing the heat flow measured by Keesom and Duyckaerts (1947) and by Meyer and Mellink (1947) against (**3**), as was done by London

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and Zilsel (1948), it is found that the observed heat flow exceeds the values calculated from (**3**) for slit widths of order 10^{-4} cm or less, the relative discrepancy becoming larger for small slit widths and low temperatures (see Table 1, column 5).

Gorter and Mellink (1949) have suggested that this discrepancy arises because the slit width d is less than the mean free path l_0 of those excitations in the fluid which determine the bulk viscosity, and the heat flow is increased by a factor of order l_0/d. However, the observed thermal conductivities do not seem to vary as d, as required by this explanation.

An alternative explanation of this discrepancy is suggested here. In the derivation of (**2**) the assumption has been made that the normal fluid in the immediate vicinity of the channel walls is at rest. This will only be so if the walls are isothermal, for, if the normal fluid exchanges momentum with the phonon gas at the walls, an anisotropic phonon distribution will tend to induce in the normal fluid a bulk flow tangential to the walls in the opposite direction to the temperature gradient which causes the phonon anisotropy. This additional flow is again compensated by a return flow of superfluid, and the additional heat transport can account for the discrepancies between the observed flow and equation (**3**).

II. Quasi-Equilibrium between the Fluid and the Phonons of the Wall

It can be shown (Klemens 1951) that the distribution of phonons in a temperature gradient is of the form

$$N_{\mathbf{k}}=[e^{\alpha\omega-\lambda.\mathbf{k}}-1]^{-1}\simeq[e^{\alpha\omega}-1]^{-1}+\lambda.\mathbf{k}e^{\alpha\omega}[e^{\alpha\omega}-1]^{-2}, \quad \dots . \quad (4)$$

where N is the average occupation of the normal mode of frequency ω and wave-vector $\mathbf{k}$, and $\alpha=h/2\pi KT$. The vector λ characterizes the anisotropy of the distribution, and is given by

$$\lambda=-\text{grad } T\,\frac{l}{T}\,\frac{hc}{2\pi KT}, \quad \dots\dots\dots\dots\dots\dots \quad (5)$$

where l is the phonon mean free path and c the velocity of sound.

The distribution function of the bosons which describe the fluid is of the form

$$N_{\mathbf{k}}=[e^{(E-E_0)/KT-\lambda'.\mathbf{k}}-1]^{-1}, \quad \dots\dots\dots\dots\dots \quad (6)$$

where E is the energy and $\mathbf{k}$ the wave-vector of the boson state considered, and E_0 is a parameter. If the fluid is at rest relative to the observer, $\lambda'=0$, leading to the normal Planck distribution. In general, (**6**) is the distribution of a fluid moving relative to the frame of reference with a bulk velocity $\mathbf{v}$ given by

$$\lambda'=h\mathbf{v}/2\pi KT \quad \dots\dots\dots\dots\dots\dots \quad (7)$$

(e.g. Dingle 1952, p. 137).

If in a system of interacting bosons energy is conserved for each interaction process, it is easily seen that equation (**6**), with $\lambda'=0$, describes a stationary state; if two subsystems interact so as to conserve energy, they will tend to a mutual equilibrium with a common value of T in their distribution functions. Similarly, if both energy and momentum (that is, $\mathbf{k}$) are conserved in individual interactions, (**6**) will be a stationary distribution for any set of parameters T and λ', and two subsystems, conserving energy and momentum in each interaction process, will tend to a quasi-equilibrium with common values of T and λ' in their distribution functions.

If the interactions of the bosons of the fluid with the phonons of the wall conserve the tangential component of $\mathbf{k}$, then these will be in mutual equilibrium if $\lambda=\lambda'$ in their distribution functions (**6**) and (**4**) and if the component of λ normal to the wall vanishes. Under isothermal conditions $\lambda=0$, hence $\lambda'=0$ for the normal fluid in the immediate vicinity of the walls, and it is therefore at rest. But in the presence of a temperature gradient $\lambda\neq0$, and, to satisfy $\lambda=\lambda'$, the bulk velocity of the normal fluid near the wall will, from (**5**) and (**7**), be given by

$$\mathbf{v}=-\frac{cl}{T}\operatorname{grad} T. \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{8})$$

With (**8**) as boundary condition instead of $\mathbf{v}=0$, there will be an additional flow through the slit given by

$$V'=-d\frac{cl}{T}\operatorname{grad} T, \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{9})$$

and thus an additional heat flow given by

$$Q'=\rho STV'=-cld\rho S\operatorname{grad} T, \qquad \ldots\ldots\ldots\ldots \quad (\mathbf{10})$$

which corresponds to an additional thermal conductivity

$$\varkappa'=S\rho cl, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{11})$$

which is independent of the cross section of the channel.

III. Comparison with Observations

In Figure 1 are plotted differences between the thermal conductivities observed for various slit widths ($\varkappa_0$) and the conductivities calculated from (**3**) ($\varkappa_{(3)}$). According to the above theory this difference should be independent of slit width, and a function of temperature only. Considering the uncertainties of the observations of Q and d, the observed points could well fall on a single curve, though the scatter for high temperatures is considerable. In any case the plotted points do not reveal any systematic dependence of $\varkappa'$ on slit width, so that an equation of the form (**11**) could account for the discrepancies.

To calculate the additional heat flow from (**11**), the velocity and the mean free path of the phonons in the walls must be known. In the experiments of Keesom and Duyckaerts (1947) and of Meyer and Mellink (1947) the slits were formed between two optically flat surfaces of glass. One can take for glass $c\sim2\times10^5$ cm/sec. The mean free path for phonons in quartz glass is known from measurements of the thermal conductivity at low temperatures (Berman 1951) and the theory of Klemens (1951). Thus, for longitudinal phonons of wave number k, $l \sim 3 \cdot 6 \times 10^{9} k^{-2}$ cm. The mean free path of transverse phonons is probably smaller by a factor of about 50.

Since λ is not a constant, λ' must be equated to some average value of λ. There arises a difficulty because the appropriate averaging procedure depends upon the nature of the interactions between the phonons of the wall and the

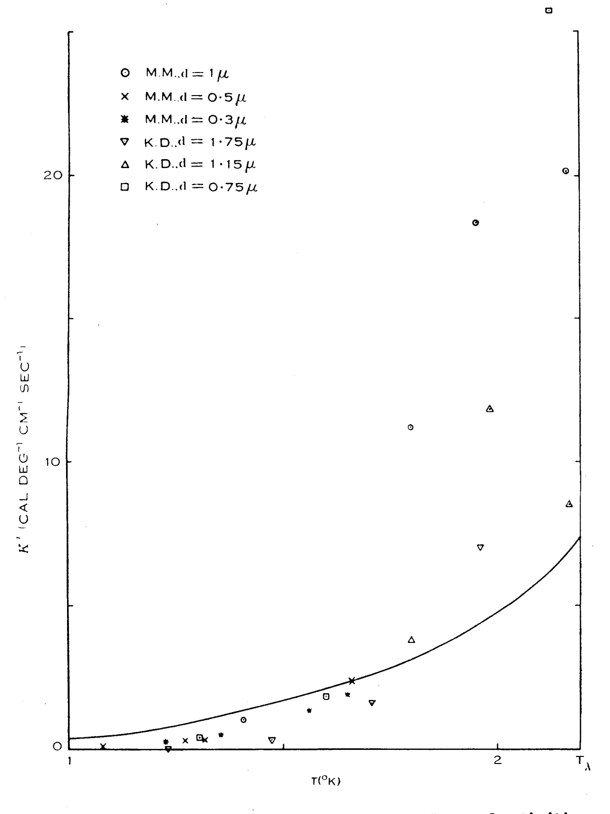

Fig. 1.—Difference between the observed conductivities and conductivities calculated from equation (**3**). Full line shows $\varkappa'$ calculated from equation (**11**).

bosons of the normal fluid. If one takes for this average, admittedly somewhat arbitrarily, the mean free path for the frequency $2\pi KT/h$, averaged over the three polarizations, one obtains $l = 2 \cdot 8 \times 10^{-3} T^{-2}$ cm.

Values of $\varkappa'$ calculated from (**11**), using the above values of l and c, and values of S taken from Kramers, Wasscher, and Gorter (1952), are plotted as a function of temperature in Figure 1. Although the ratio $\varkappa_0/(\varkappa_{(3)}+\varkappa')$, given in Table 1, clearly does not depart from unity to the extent that $\varkappa_0/\varkappa_{(3)}$ does, there is considerable lack of agreement between $\varkappa'$ and the experimental points. This is of such a form as to indicate that l is not as strongly temperature dependent as one would expect. This may be due to a difference between the phonon mean free paths near a polished surface and in the interior, or it may be due to the role of the phonons of different polarizations changing with temperature. It should also be remembered that the experimental results are uncertain, particularly just below the λ-point, and that the mean free paths of phonons differ significantly from glass to glass. It is significant that $\varkappa'$ appears to be a

TABLE 1

APPARENT HEAT CONDUCTIVITY OF HELIUM II THROUGH SLITS OF WIDTH d

$\varkappa_0$ values observed by Keesom and Duyckaerts (1947)—K.D., and by Meyer and Mellink (1947)—M.M.; $\varkappa_{(3)}$ theoretical values calculated from equation (3) (London and Zilsel 1948); $\varkappa'$ correction calculated from equation (11)

d (10^{-4} cm)	T (°K)	$\varkappa_0$ (cal deg^{-1} cm^{-1} sec^{-1})	$\varkappa_{(3)}$	$\frac{\varkappa_0}{\varkappa_{(3)}}$	$\frac{\varkappa_0}{\varkappa_{(3)}+\varkappa'}$
1·75 (K.D.)	1·960	17·2	10·2	1·69	1·18
	1·705	3·5	1·86	1·88	0·80
	1·476	0·61	0·32	1·92	0·32
	1·223	0·066	0·032	2·05	0·09
1·15 (K.D.)	2·170	24	15·5	1·55	1·18
	1·989	17·1	5·28	3·25	1·74
	1·799	5·35	1·55	3·45	1·14
1·0 (M.M.)	2·159	31	10·8	2·9	1·75
	1·948	21·4	3·06	7·0	2·9
	1·802	12·4	1·19	10·4	2·8
	1·411	1·05	0·060	17·5	0·75
0·75 (K.D.)	2·097	30·2	4·3	7·0	2·9
	1·600	1·85	0·156	11·8	0·82
	1·403	0·39	0·031	12·5	0·27
0·5 (M.M.)	1·659	2·48	0·108	23	0·99
	1·315	0·354	0·0064	55	0·35
	1·274	0·277	0·0043	64	0·31
	1·086	0·124	0·0006	202	0·28
0·3 (M.M.)	1·652	1·92	0·0365	54	0·82
	1·558	1·35	0·018	75	0·69
	1·358	0·48	0·0034	140	0·42
	1·226	0·25	0·00097	260	0·33

function of temperature independent of slit width, and that the present theory gives the observed order of magnitude of the heat flow in all cases, while equation (3) fails to do so.

It would be interesting to observe the heat flow through slits formed of materials with a different phonon mean free path, such as crystalline quartz, diamond, or artificial sapphire; the heat transport through these slits should be considerably enhanced.

IV. References

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