THE EFFECTS OF COLLIMATION AND OBLIQUE INCIDENCE IN LENGTH INTERFEROMETERS. I

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Summary

An investigation has been made of the effects of position and size of diaphragm apertures on the interference fringes in length interferometry. The results indicate that, while the well-known correction formula for the effect of the oblique rays from a point source off the optic axis is always applicable, the effect of finite area of aperture is a more complex one and is not in general proportional to the length measured. This effect depends essentially on the phase differences arising from different points of the aperture and is oscillatory in character. For a narrow slit the maximum fringe displacement arising from this effect is not greater than about 0·15 fringe. For a circular aperture the maximum effect is about 0·5 fringe and for a square aperture about 0·35 fringe.

I. Introduction

In precision length interferometry, measurements are made most frequently with Fizeau-type and Michelson-type interferometers (Kösters 1926; Pérard 1930; Barrell 1948). In all cases collimation of light is obtained by placing the entrance aperture at the principal focus of the collimating lens. Since the aperture has a finite area the emergent beam from the collimating lens is not ideally collimated. In the case of Fizeau systems the aperture is also very frequently placed off the optic axis so that the light as a whole falls obliquely on to the interferometer. The effects of oblique incidence on the disposition of the fringes is well known and their significance is becoming of greater importance as isotopic light sources extend the range of direct measurement by interferometry.

Figures 1 and 2 show the optical schemes of the National Physical Laboratory Gauge Interferometer and the Kösters Gauge Interferometer which are Fizeau and Michelson systems respectively. In the Fizeau system the illuminating (entrance) and viewing (exit) apertures are off the optic axis so that the effects of both oblique incidence and area of aperture are involved. In the Kösters interferometer both entrance and exit apertures lie on the optic axis so that any effect from oblique rays can only come from the resultant effect of all rays from the area of the aperture. For convenience the effect of aperture area alone can be termed a " collimation effect " and the effect of non-normal incidence an " incidence effect ". In both cases of course the effects arise from oblique rays.

The correction factors for the " incidence effect " are derived from the classical relation $n\lambda=2t\cos\theta$ where θ is the angle of incidence and t corresponds to the length measured. The correction factor per unit length is $(1-\cos\theta)$

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or $\frac{1}{2}\theta^2$ for small values of θ. If in the case of the Fizeau system the distance between entrance and exit apertures is $2s$, and f is the focal length of the collimating lens, the correction factor for the incidence effect is

$$C_1 = s^2/2f^2. \qquad (1)$$

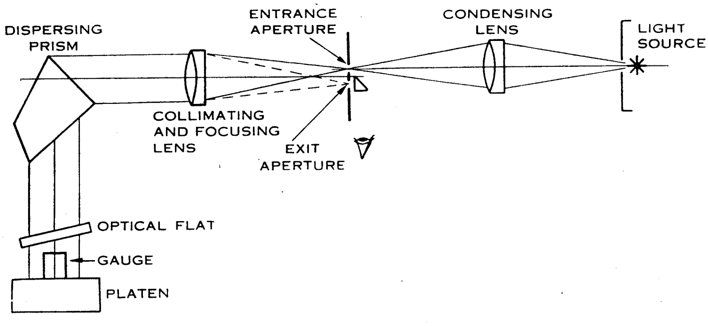

Fig. 1.—Optical scheme of N.P.L. gauge interferometer.

The effect of area of aperture has in the past been considered to be the integrated sum of obliquity effects as determined by equation (**1**) for all elementary point sources within the area of the aperture. Thus for a circular aperture of radius r (Fig. 3) the correction would be

$$C_2 = \frac{\int_0^{2\pi}\int_0^r \frac{x^2}{2f^2} x\,\mathrm{d}\varphi\,\mathrm{d}x}{\int_0^{2\pi}\int_0^r x\,\mathrm{d}\varphi\,\mathrm{d}x} = \frac{r^2}{4f^2}.$$

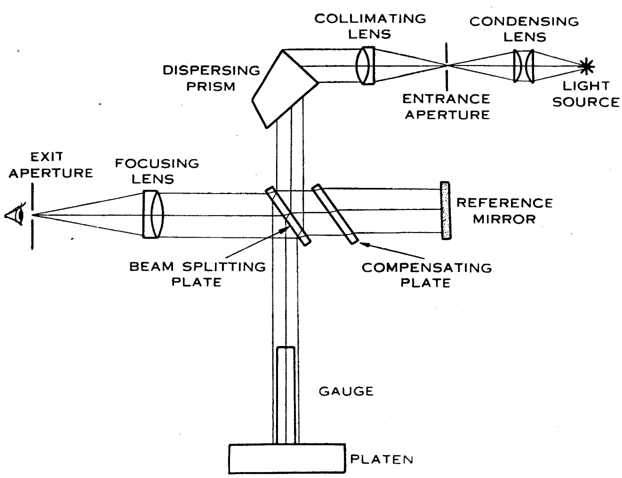

Fig. 2.—Optical scheme of Kösters gauge interferometer.

For a rectangular aperture of length l and breadth b, $C_2 = (l^2 + b^2)/24f^2$ Hamon (1951). The total correction for circular apertures in the Fizeau system would be

$$C = C_1 + C_2 = s^2/2f^2 + r^2/4f^2. \qquad (2)$$

For the Kösters interferometer with rectangular apertures, the total correction would be

$$C_2=(l^2+b^2)/24f^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

The above integration does not take into account the actual intensity distribution of the fringes and it is considered that the actual observed shape and position of a fringe depends on the integrated intensity from all elementary fringes formed from all elementary sources within the entrance aperture. It should be stated, of course, that the dimensions of the entrance aperture will control the effect from its area provided its image, focused in the plane of the exit aperture, is smaller than the size of the exit aperture. If the size of the image of the entrance aperture is greater than the size of the exit aperture, then clearly the dimensions of the latter and the focal length of the focusing lens associated with it will determine the range of oblique rays collected by the observer's eye.

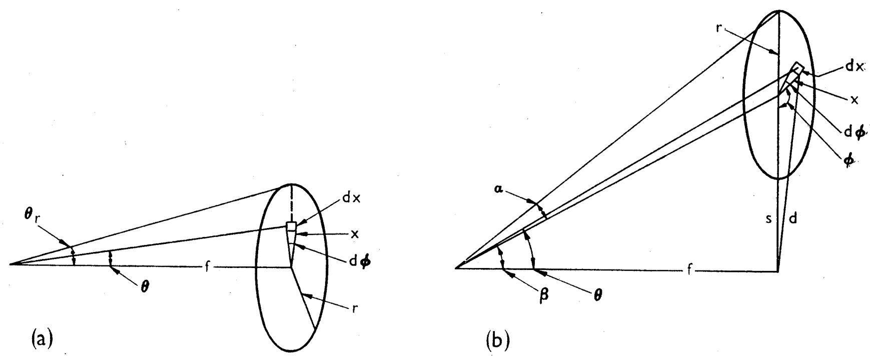

Fig. 3.—Theory of fringe intensity. (*a*) Circular aperture on centre, (*b*) circular aperture off centre.

It is the purpose of this paper (paper I) to describe an investigation of the effects of oblique incidence and area of aperture and the accuracy of the above relationships. Paper II (Thornton 1955), which follows this paper, describes the theory of the effects in greater detail for the various possible cases in both two-beam and multiple-beam systems. It is assumed that fringe displacements are observed from the centre of fringes since there was no real evidence that settings are made on different parts of different sets of fringes as a result of fringe asymmetry.

II. Fringe Intensity Distributions

The intensity for two-beam fringes is given by $I=\cos^2 K$ where $K=\frac{1}{2}$ (phase difference) and where for simplicity the amplitude factor is treated as unity.

(*a*) *Circular Aperture*

For a circular aperture centred on the optic axis (Fig. 3 (*a*)) the intensity for an elementary point source making an angle θ with the axis is

$$\delta I=\cos^2 (K \cos \theta) x \mathrm{d}\varphi \mathrm{d}x.$$

The resultant fringe intensity due to the whole aperture is

$$I=\int_0^{2\pi}\int_0^r \cos^2 (K \cos \theta)x d\varphi dx.$$

As θ is of the order of $0 \cdot 001$ radian a legitimate approximation for $\cos \theta$ is $1-x^2/2f^2$ where f is the focal length of the collimating lens as indicated in Figure 3. The integral then becomes

$$I=2\pi\int_0^r \tfrac{1}{2}[\cos 2K(1-x^2/2f^2)+1]x dx.$$

By substituting $u=2K(1-x^2/2f^2)$ and integrating

$$I=\frac{\pi r^2}{2}-\frac{\pi f^2}{2K}(\sin 2K \cos Kr^2/f^2-\cos 2K \sin Kr^2/f^2-\sin 2K). \quad .. \ (4)$$

A convenient form of equation (4) is

$$I=\tfrac{1}{2}A\left[1+\frac{\sin \Delta}{\Delta}\cos (2K-\Delta)\right], \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (5)$$

where $A=\pi r^2$, which is the area of the aperture, and $\Delta=\frac{1}{2}K\theta^2$, $\theta=r/f$, $K=2\pi t/\lambda$, and t is the length measured. The phase factor Δ is the phase difference between a point source at the centre and a point source at the edge of the aperture. The relation indicates that the fringe would be symmetrical and the phase factor Δ determines where fringe minima (or maxima) will occur. If Δ is small $\sin \Delta/\Delta$ is approximately unity and equation (5) becomes

$$I=A \cos^2 (K-\tfrac{1}{2}\Delta), \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (6)$$

which shows that fringe minima will be displaced from the position of those for a point source on the axis by $\frac{1}{2}\Delta$. It can be seen, however, from equation (5) that for $0 \leqslant \Delta \leqslant \pi$, the factor $\sin \Delta/\Delta$ is positive throughout and the fringe displacement will still be given by $\frac{1}{2}\Delta$.

For values of Δ greater than π, the effects are different due to the change in sign of $\sin \Delta/\Delta$. At $\Delta=n\pi$ ($n=1, 2, 3 \ldots$) the intensity is the same for all values of the phase factor $\cos (2K-\Delta)$ and the fringes should therefore vanish for these values of Δ. For the range $\pi \leqslant \Delta \leqslant 2\pi$ the intensity is

$$I=\tfrac{1}{2}A\left[1+\frac{\sin (\pi+\delta)}{\pi+\delta}\cos (2K-\pi-\delta)\right]$$

$$=\tfrac{1}{2}A\left[1+\frac{\sin \delta}{\pi+\delta}\cos (2K-\delta)\right],$$

where $\Delta=\pi+\delta$ and $0 \leqslant \delta \leqslant \pi$. The fringe displacement is therefore $\frac{1}{2}\delta$ or $\frac{1}{2}(\Delta-\pi)$.

In general for the range $n\pi \leqslant \Delta \leqslant (n+1)\pi$ the fringe displacement will be $\frac{1}{2}\delta$ or $\frac{1}{2}(\Delta-n\pi)$ where $\Delta=n\pi+\delta$ and the maximum fringe displacement will not exceed $\frac{1}{2}\pi$ or 0·5 fringe. The form of the obliquity-fringe displacement curve will therefore be as shown in curve A of Figure 4.

As the fringes vanish for $\Delta=n\pi$ their visibility is of some interest. The "visibility" is usually defined as

$$V=\frac{I_{\text{max.}}-I_{\text{min.}}}{I_{\text{max.}}+I_{\text{min.}}}.$$

The maximum and minimum values of intensity will be at $2K-\Delta=2n\pi$ and $(2n+1)\pi$ respectively, for $0\leqslant\Delta\leqslant\pi$, at $2K-\Delta=(2n+1)\pi$ and $2n\pi$ respectively, for $\pi\leqslant\Delta\leqslant2\pi$, and so on. The visibility V will be given by $\sin\Delta/\Delta$ and the maximum value for V is given by $\mathrm{d}V/\mathrm{d}\Delta=0$ which gives the condition $\Delta=\tan\Delta$

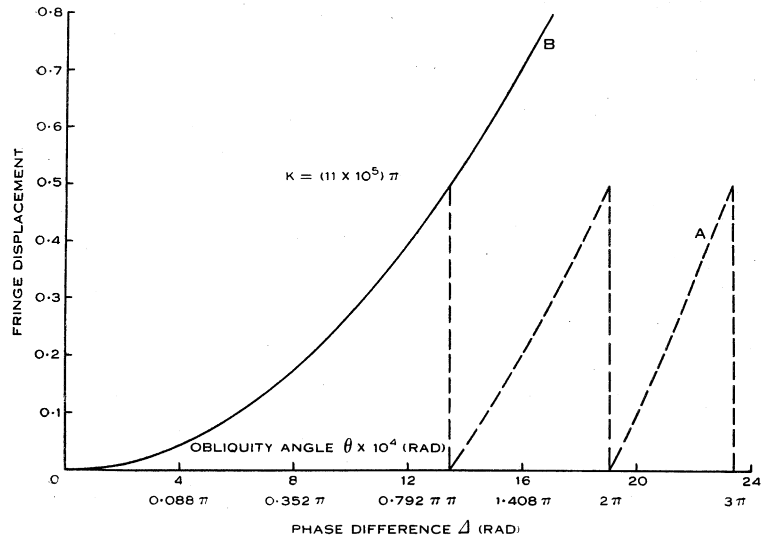

Fig. 4.—Fringe displacement-obliquity curve for a circular aperture. Curve A from $I=\frac{1}{2}\Delta[1+(\sin\Delta/\Delta)\cos(2K-\Delta)]$, curve B from $C=d^2/16f^2$ $(d=2r)$.

for best visibility. Thus for very small values of Δ the maximum value for visibility will approach 1. For $\pi\leqslant\Delta\leqslant2\pi$ the visibility maximum will be about 0·2 with Δ near $3\pi/2$ and for $2\pi\leqslant\Delta\leqslant3\pi$ it will be about 0·1 with Δ near $5\pi/2$. Fringes are very faint but visible for values of Δ up to 5π at least in the experimental work.

For a circular aperture off centre (Fig. 3 (*b*)) the intensity will be

$$I=\int_0^{2\pi}\int_0^r\cos^2(K\cos\theta)x\mathrm{d}x\mathrm{d}\varphi$$

$$=\int_0^{2\pi}\int_0^r\tfrac{1}{2}[1+\cos(2K-Kd^2/f^2)]x\mathrm{d}x\mathrm{d}\varphi,$$

where $\theta=d/f$ and $d^2=x^2+s^2+2xs\cos\varphi$.

Substituting and expanding,

$$I=\tfrac{1}{2}\pi r^2+\pi\cos 2K\int_0^r J_0(z)\cos\frac{K(x^2+s^2)}{f^2}x\,dx$$

$$+\pi\sin 2K\int_0^r J_0(z)\sin\frac{K(x^2+s^2)}{f^2}x\,dx,$$

where $J_0(z)$ is the zero order Bessel function in which

$$z=2Kxs/f^2.$$

The intensity can therefore be represented by

$$I=\tfrac{1}{2}\pi r^2+\pi\int_0^r\cos\left[2K-\frac{K(x^2+s^2)}{f^2}\right]J_0(2Kxs/f^2)x\,dx. \quad \ldots\ldots \quad (7)$$

As xs/f^2 is of the order of 10^{-6}, the function $J_0(2Kxs/f^2)$ would approach unity only if K is less than, for example, 10^5. With this limitation equation (7) becomes

$$I=\tfrac{1}{2}\pi r^2+\pi\int_0^r\cos\left[2K-\frac{K(x^2+s^2)}{f^2}\right]x\,dx,$$

which reduces to

$$I=\frac{\pi r^2}{2}+\frac{\pi f^2}{K}[\sin\tfrac{1}{2}K\alpha^2\cos(2K-\tfrac{1}{2}K\alpha^2-K\beta^2)],$$

where $\alpha=r/f$ and $\beta=s/f$.

This equation can be put in the following form

$$I=\tfrac{1}{2}A\left[1+\frac{\sin\Delta}{\Delta}\cos(2K-\Delta-K\beta^2)\right], \quad \ldots\ldots\ldots\ldots \quad (8)$$

where $\Delta=\tfrac{1}{2}K\alpha^2$.

This equation is similar to equation (5) and for $\Delta \gg \pi$ reduces to

$$I=A\cos^2(K-\tfrac{1}{2}\Delta-\tfrac{1}{2}K\beta^2). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

Thus the fringe displacement would be $\tfrac{1}{2}\Delta+\tfrac{1}{2}K\beta^2$, which is equivalent to a normal obliquity effect from the centre of the aperture ($\tfrac{1}{2}K\beta^2$) plus the effect of area of aperture as found for the centred case. This result is not exact in view of the approximation $J_0(z)=1$ which is not legitimate for higher values of K, but it is some indication of the conditions for smaller values of K.

(b) Narrow Slit

The intensity distribution can be examined by a summation of intensities from elementary point sources making up the aperture and can be done graphically or analytically. This method can be used conveniently for the case of a narrow slit which is generally used in the Kösters interferometer.

The intensities for a number of point sources P_0, P_1, P_2 . . . (Fig. 5) can be expressed as

$$I_0=\cos 2K,$$
$$I_1=\cos^2 (K \cos \theta),$$
$$I_2=\cos^2 (K \cos 2\theta),$$
$$I_n=\cos^2 (K \cos n\theta).$$

The condition for fringe minima can be found by differentiating these expressions with respect to K and equating the result to zero. As $n\theta$ is always small, terms involving θ^2 as factor may be neglected and the result becomes

$$\sin 2K(1+\cos 2\varphi+\cos 8\varphi+. . .+\cos 2n^2\varphi)$$
$$=\cos 2K (\sin 2\varphi+\sin 8\varphi+. . .+\sin 2n^2\varphi),$$

where $\varphi=\frac{1}{2}K\theta^2$.

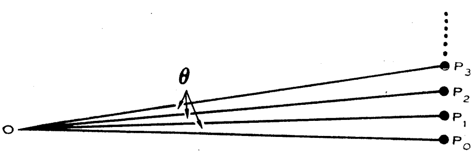

Fig. 5.—Theory of fringe intensity for a rectangular aperture.

The values of K at which there are fringe minima or maxima is therefore given by

$$\tan 2K=\frac{\sum_1^n \sin 2n^2\varphi}{1+\sum_1^n \cos 2n^2\varphi}+n\pi. \qquad \text{(10)}$$

Values of K derived from expressions of this type for a range of values of n agree closely with those obtained from direct graphical summation of intensities. The general character of the obliquity-fringe displacement curves are shown in Figures 6 (a) and (b) for values of K of about $(4\times10^5)\pi$ and $(11\times10^5)\pi$ respectively. For $\lambda=5461$ Å these values would correspond to lengths of about 110 mm and 300 mm respectively. Curve B is obtained by using equation (**3**) with $b=0$.

The curves indicate that the maximum fringe displacement possible is about 0·15 fringe and up to this value the displacements are approximately of the same order as that derivable from equation (**3**) if b is taken to be zero in comparison with l. This maximum occurs when Δ is about $\frac{1}{2}\pi$ and thereafter there is no agreement with equation (**3**) as the fringe displacement is oscillatory in character. It is of interest to note that the phase value of $\Delta=\frac{1}{2}\pi$ corresponds to the approximate limiting condition (Rayleigh criterion) for no deterioration in the visibility of the fringes. The fringes of course are clearly observable at greater values of this phase difference. For long lengths, phase differences greater than $\frac{1}{2}\pi$ may very well occur if the focal length of the lens is not sufficiently great or the dimensions of the aperture are not sufficiently small. For example, with a 500 mm length and $\lambda=5000$ Å, K is about $(2\times10^6)\pi$, and if the maximum obliquity angle θ was no greater than 0·001 radian the phase factor $\Delta=\frac{1}{2}K\theta^2$ would be π radians.

An exact treatment of the centred narrow slit case is given in paper II in terms of Fresnel integrals and shows that the curves given here indicate adequately the oscillatory character of the fringe displacement and its order of magnitude. Using Fresnel integrals, the conditions for a narrow slit off the axis can be examined to see if the fringe displacement effect can be treated as a normal obliquity effect from the centre of the slit plus the effect of the area of the slit treated as if it were centred on the axis.

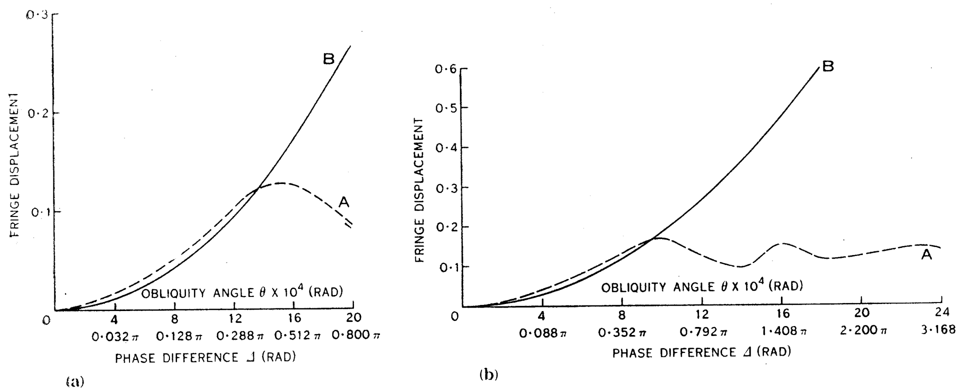

Fig. 6.—Fringe displacement-obliquity curves for a narrow slit. Curve A from summation of elements, curve B from $C=l^2/24f^2$. (*a*) $K=(4\times10^5)\pi$; (*b*) $K=(11\times10^5)\pi$.

For the narrow slit off centre the relevant part of the intensity relation is

$$I=\int_{\theta_1}^{\theta_2}\cos^2(K\cos\theta)\mathrm{d}\theta=\int_{\theta_1}^{\theta_2}\cos^2 K(1-\tfrac{1}{2}\theta^2)\mathrm{d}\theta,$$

where θ_1 and θ_2 are the obliquity angles at the extreme edges of the slit relative to the optic axis. Differentiating under the integral sign, as done in paper II, with respect to K and ignoring terms with θ^2 as factor, since θ is small

$$\begin{aligned}\frac{\partial I}{\partial K}&=\int_{\theta_1}^{\theta_2}\sin(2K-K\theta^2)\mathrm{d}\theta\\&=\int_0^{\theta_2}\sin(2K-K\theta^2)\mathrm{d}\theta-\int_0^{\theta_1}\sin(2K-K\theta^2)\mathrm{d}\theta\\&=\sin 2K\left[\int_0^{\theta_2}\cos K\theta^2\mathrm{d}\theta-\int_0^{\theta_1}\cos K\theta^2\mathrm{d}\theta\right]\\&\quad-\cos 2K\left[\int_0^{\theta_2}\sin K\theta^2\mathrm{d}\theta-\int_0^{\theta_1}\sin K\theta^2\mathrm{d}\theta\right].\end{aligned}$$

The condition for fringe minima is therefore

$$\tan 2K=\frac{S(v_2)-S(v_1)}{C(v_2)-C(v_1)}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{11})$$

where $S(v)$ and $C(v)$ are the Fresnel integrals

$$\int_0^v \sin \tfrac{1}{2}\pi v^2 \mathrm{d}v \text{ and } \int_0^v \cos \tfrac{1}{2}\pi v^2 \mathrm{d}v$$

respectively, in which $v=\sqrt{(2K/\pi)}\theta$. The fringe displacement for typical values of θ_1 and θ_2 can be found by evaluating the integrals for the appropriate values of v_1 and v_2.

It was also shown in the case of the centred narrow slit that there was reasonable agreement between the fringe displacement calculated from equation (**3**) with $b=0$ and that obtained from the summation of elements analysis described, provided the phase difference between the centre and edge of the slit did not exceed about $\frac{1}{2}\pi$. Accepting this result, the fringe displacement from the effect of area of aperture in the off-centre case will be $\frac{1}{6}K(\theta_2-\theta)^2$ provided $\frac{1}{2}K(\theta_2^2-\theta^2)$ is not greater than about $\frac{1}{2}\pi$ where $\theta=\theta_1+\frac{1}{2}\theta_2$ is the obliquity angle corresponding to the centre of the slit. The fringe displacement due to obliquity of the centre of the slit will be $\frac{1}{2}K\theta^2$. The combined effect is therefore

$$D=\tfrac{1}{2}K\theta^2+\tfrac{1}{6}K(\theta_2-\theta)^2. \quad \ldots\ldots\ldots\ldots\ldots \quad (\mathbf{12})$$

Fringe displacements calculated from equations (**11**) and (**12**) for typical values of θ_1, θ_2, and K agree to well within $0\cdot05$ fringe. This indicates again that an off-centre aperture can be treated without great error as a centred aperture for the area effect together with a direct obliquity effect from the centre of the aperture. Both effects are, of course, included in equation (**11**) which can be used to obtain directly the fringe displacement arising from both oblique incidence and collimation effects.

(*c*) *Rectangular Aperture*

For a rectangular aperture centred on the axis whose width may be comparable with its length the intensity is given by

$$I=\int_{-l/2}^{l/2}\int_{-b/2}^{b/2} \cos^2 (K \cos \theta)\mathrm{d}y\mathrm{d}x$$

$$=\int_{-l/2}^{l/2}\int_{-b/2}^{b/2} \tfrac{1}{2}(1+\cos 2K \cos \theta)\mathrm{d}y\mathrm{d}x.$$

Differentiating the relevant portion with respect to K and ignoring terms with θ^2 as factor

$$\frac{\partial I}{\partial K}=\int_{-l/2}^{l/2}\int_{-b/2}^{b/2} \sin (2K-K\theta^2)\mathrm{d}y\mathrm{d}x.$$

$$=2 \sin 2K\int_0^{l/2}\int_0^{b/2} \cos K\theta^2\mathrm{d}y\mathrm{d}x-2 \cos 2K\int_0^{l/2}\int_0^{b/2} \sin K\theta^2\mathrm{d}y\mathrm{d}x.$$

Fringe minima (or maxima) will occur when $\partial I/\partial K=0$ which gives

$$\tan 2K=\frac{\int_0^{l/2}\int_0^{b/2}\sin K\theta^2 \mathrm{d}y\mathrm{d}x}{\int_0^{l/2}\int_0^{b/2}\cos K\theta^2 \mathrm{d}y\mathrm{d}x}.$$

Making the substitution $\theta^2=x^2+y^2/f^2$ where f is the focal length of the lens concerned

$$\begin{aligned}\int_0^{l/2}\int_0^{b/2}\sin K\theta^2\mathrm{d}y\mathrm{d}x&=\int_0^{l/2}\int_0^{b/2}\sin\frac{Kx^2}{f^2}\cos\frac{Ky^2}{f^2}\mathrm{d}y\mathrm{d}x\\&\quad+\int_0^{l/2}\int_0^{b/2}\cos\frac{Kx^2}{f^2}\sin\frac{Ky^2}{f^2}\mathrm{d}y\mathrm{d}x\\&=f^2\frac{\pi}{2K}[S(u)C(v)+C(u)S(v)],\end{aligned}$$

where $S(u)$, $C(u)$, $S(v)$, and $C(v)$ are the Fresnel integrals

$$\int_0^u \sin\tfrac{1}{2}\pi u^2\mathrm{d}u,\quad \int_0^u \cos\tfrac{1}{2}\pi u^2\mathrm{d}u,\quad \int_0^v \sin\tfrac{1}{2}\pi v^2\mathrm{d}v,\text{ and } \int_0^v \cos\tfrac{1}{2}\pi v^2\mathrm{d}v$$

respectively, in which

$$u=\sqrt{\left(\frac{2K}{\pi}\right)}\frac{x}{f},$$
$$v=\sqrt{\left(\frac{2K}{\pi}\right)}\frac{y}{f}.$$

In a similar manner it can be shown that

$$\int_0^{l/2}\int_0^{b/2}\cos K\theta^2\mathrm{d}y\mathrm{d}x=f^2\frac{\pi}{2K}[C(u)C(v)-S(u)S(v)].$$

The condition for fringe minima is therefore given by

$$\tan 2K=\frac{S(u)C(v)+C(u)S(v)}{C(u)C(v)-S(u)S(v)}. \qquad \text{(13)}$$

If the width of the aperture is very small compared with its length v will be very small and $S(v)\to 0$ and the equation becomes

$$\tan 2K=\frac{S(u)}{C(u)},$$

which is the case of the narrow slit as derived in paper II.

The fringe displacements can be found by direct evaluation of the Fresnel integral products in equation (**13**) for different length/width ratios. As an example Figure 7 shows the fringe displacement curve for a square aperture ($u/v=1$) and for a narrow slit. Values of $u=\sqrt{\left(\frac{2K}{\pi}\right)}\theta$ and $\Delta=\frac{1}{2}K\theta^2$ are shown as abscissae. The obliquity angle θ refers to that at the edge of the aperture and Δ is the phase difference from the centre to the edge. For the square aperture $\theta = l/2f = b/2f$, and for the equivalent circular aperture $\theta = \sqrt{(l^2+b^2)}/2f$.

Displacement curves for values of u/v greater than 1 can be found in a similar manner. Examination of these cases indicates that the fringe displacement curves are of the same general character as those for the narrow slit and rectangular aperture and give values for the fringe displacement that are intermediate between these two cases but which gradually increase in the oscillatory region. For a length/width ratio equal to or greater than 4 ($u/v \geqslant 4$) the aperture can be treated as a narrow slit and this treatment gives correct fringe displacements to within 0·05 fringe for values of Δ of up to at least 2π.

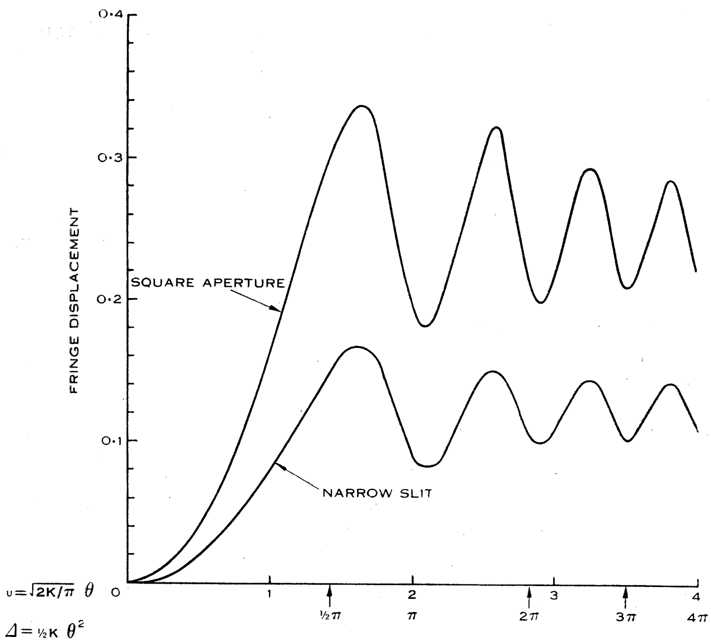

Fig. 7.—Fringe displacement curves for square aperture and narrow slit.

(d) *Multiple-Beam Effects*

In Fizeau systems it is possible to make use of multiply reflected beams in some degree by coating the optical flat with a highly reflecting layer. Since large air gaps are a necessary condition in practical length measurement full use cannot be made of multiple-beam interference.

The intensity relation for multiple-beam fringes in the centred circular aperture case would be

$$I = \int_0^{2\pi}\int_0^r \frac{4R\sin^2(K\cos\theta)}{(1-R)^2+4R\sin^2(K\cos\theta)}x\mathrm{d}x\mathrm{d}\varphi$$

$$= 2\pi f^2\int_0^{\theta_r}\frac{A\sin^2 K(1-\frac{1}{2}\theta^2)}{1+A\sin^2 K(1-\frac{1}{2}\theta^2)}\theta\mathrm{d}\theta,$$

where $A=4R/(1-R)^2$. Substituting $u=1-\frac{1}{2}\theta^2$,

$$I=2\pi f^2\int_{u_0}^{u_r}\frac{A\sin^2 Ku}{-(A+1)+A\cos^2 Ku}\mathrm{d}u.$$

Integrating,

$$I=2\pi f^2\left[\frac{1}{K\sqrt{F}}\{\tan^{-1}(\sqrt{F}\tan K(1-\tfrac{1}{2}\theta^2))-\tan^{-1}(\sqrt{F}\tan K)\}+\tfrac{1}{2}\theta^2\right]. \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

The fringes should therefore be symmetrical, if this was the only effect. The actual effect, however, of a large aperture is to produce a marked asymmetry in the fringes in the direction of increasing order or increasing air gap, for low order, high definition multiple-beam fringes. This is due to the changed phase conditions introduced by the linear displacement of multiply reflected beams at higher angles of incidence (Brossell 1947).

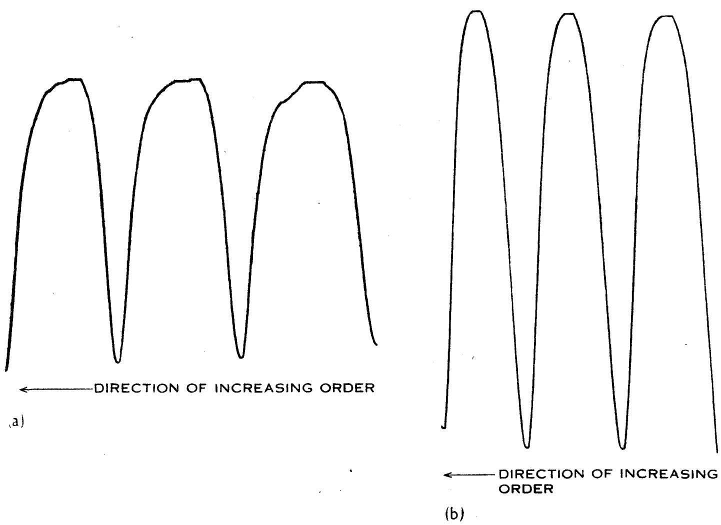

Fig. 8.—Microphotometer record of fringe intensities. (*a*) Fizeau interferometer, (*b*) Kösters interferometer.

Microphotometer records were made of the fringe intensity distribution with the interferometers described here at various aperture sizes and at a range of path differences. The aperture size had no observable influence on the fringe shape of the two-beam fringes of the Kösters interferometer or the fringes of the Fizeau interferometer which may be two-beam or partially multiple-beam in character. Typical results are shown in Figures 8 (*a*) and (*b*).

It is of interest to note that the Kösters interferometer gave symmetrical fringes and the Fizeau interferometer fringes of smaller half width, but with a distinct asymmetry in the direction of decreasing order, which is the opposite direction to that to be expected from pure collimation effects. This indicates that phase effects at the silvered optical flat are the predominating influence on the fringe shape and that any effects due to area of aperture are negligible in comparison. The effects of using fringes of the asymmetrical multiple-beam type from the surface of a length standard together with broad two-beam fringes from the platen to which the standard is wrung has been investigated (Terrien 1954).

A treatment of the multiple-beam case by summation of elements for values of the reflection coefficient R in the range from $0 \cdot 6$ to $0 \cdot 9$ gives obliquity-fringe displacement curves of the same form as those for two-beam cases. It should be noted, however, that under true multiple-beam conditions the value of the length t must be very small which in turn makes K much smaller. The fringe displacement factor which is a function of K and θ^2 will therefore be extremely small, and of little practical significance.

III. Experimental Observations

The Kösters interferometer made in this Laboratory permitted the apertures to be varied in size over a small range during observations. For example a particular setting allowed the entrance aperture to be varied from $0 \cdot 3 \times 0 \cdot 08$ mm to $0 \cdot 6 \times 0 \cdot 3$ mm. Observations were made initially with an exit (or viewing) aperture $1 \cdot 0 \times 0 \cdot 5$ mm. As the focal lengths of the lenses associated with the entrance and exit apertures were 208 mm and 390 mm respectively, the image of the entrance aperture was smaller than the size of the exit aperture, with which it is coincident. The dimensions of the entrance aperture therefore controlled the limits on obliquity effects. For a 300 mm length and a focal length of 208 mm a change of $0 \cdot 4$ fringe should occur in the fringe displacement as the entrance aperture changes from $0 \cdot 3 \times 0 \cdot 08$ mm to $0 \cdot 6 \times 0 \cdot 3$ mm, if equation (**3**) is valid. The displacement is that between the fringes from the length gauge surface and the fringes from the surface to which this gauge is wrung. The observed displacement change was certainly no greater than $0 \cdot 1$ fringe.

Further tests were made with a set of circular and rectangular apertures. These were made and used so that their centres were precisely located on the optic axis in order that no off-centre effects would occur. This was further verified by autocollimation tests. A 305 mm (12 in.) length was used and the wavelength 5461 Å from a mercury 198 lamp gave clear fringes at this path difference which corresponds to a value of about $(11 \times 10^5)\ \pi$ for the phase factor K. The set of apertures was used first at the entrance end of the interferometer and then at the exit end. In each case the fixed aperture, whether it was at the exit end or entrance end, was sufficiently large to ensure that the set of apertures of variable sizes controlled the range of oblique rays received by the observer's eye.

Tables 1 and 2 give a series of results using the exit aperture of the interferometer to control the obliquity due to area of aperture. As the focal length of the associated lens was 390 mm the aperture did not need to be as small as when the entrance aperture was used ($f=208$ mm) to give a range of values for Δ below $\frac{1}{2}\pi$.

It is to be noted that for rectangular apertures there is approximate agreement between the observed fringe displacement difference and those calculated from equation (3) for values up to about $\frac{1}{2}\pi$ only. Thereafter there is disagreement and the observed effects conform quite closely with that indicated by the oscillatory curve of Figures 5 and 6 for a narrow slit.

TABLE 1

RECTANGULAR APERTURES

No.	Dimensions		Obliquity Angle $\theta=\sqrt{(l^2+b^2)}/2f$ (rad)	Phase Difference $\Delta=\frac{1}{2}K\theta^2$ (rad)	Fringe Displacement Difference	
	l (mm)	b (mm)			From $l^2+b^2/24f^2$	Observed Mean
1	0·310	0·108	$4\cdot20\times10^{-4}$	$0\cdot10\pi$	0	0
2	0·409	0·221	5·96	0·20	0·04	0·02
3	0·644	0·105	8·36	0·39	0·10	0·08
4	0·622	0·332	9·04	0·46	0·12	0·11
5	0·818	0·411	11·73	0·77	0·23	0·15
6	1·030	0·099	13·27	0·98	0·30	0·17
7	1·026	0·611	15·31	1·31	0·41	0·19
8	2·025	0·120	26·01	3·78	1·23	0·13

TABLE 2

CIRCULAR APERTURES

No.	Diameter d (mm)	Obliquity Angle $\theta=d/2f$ (rad)	Phase Difference $\Delta=\frac{1}{2}K\theta^2$ (rad)	Fringe Displacement Difference	
				From $d^2/16f^2$	Observed Mean
1	0·198	$2\cdot54\times10^{-4}$	$0\cdot03\pi$	0	0
2	0·408	5·23	0·15	0·06	0·04
3	0·612	7·85	0·34	0·15	0·13
4	0·784	10·05	0·56	0·26	0·25
5	0·990	12·69	0·90	0·43	0·30
6	1·180	15·13	1·28	0·62	0·45
7	1·415	18·14	1·84	0·90	0·53
8	1·512	19·38	2·10	1·02	0·48
9	2·010	25·77	3·71	1·81	0·48
10	2·504	32·10	5·75	2·86	0·48

For the circular apertures the observed displacement generally agrees with equation (2) with $S=0$ for values of Δ up to π. The theoretical analysis given in this paper also predicts this effect exactly, but for values of Δ greater than π an oscillatory condition again occurs. As Δ approaches π the visibility of the fringes decreases markedly and they are difficult to observe which may

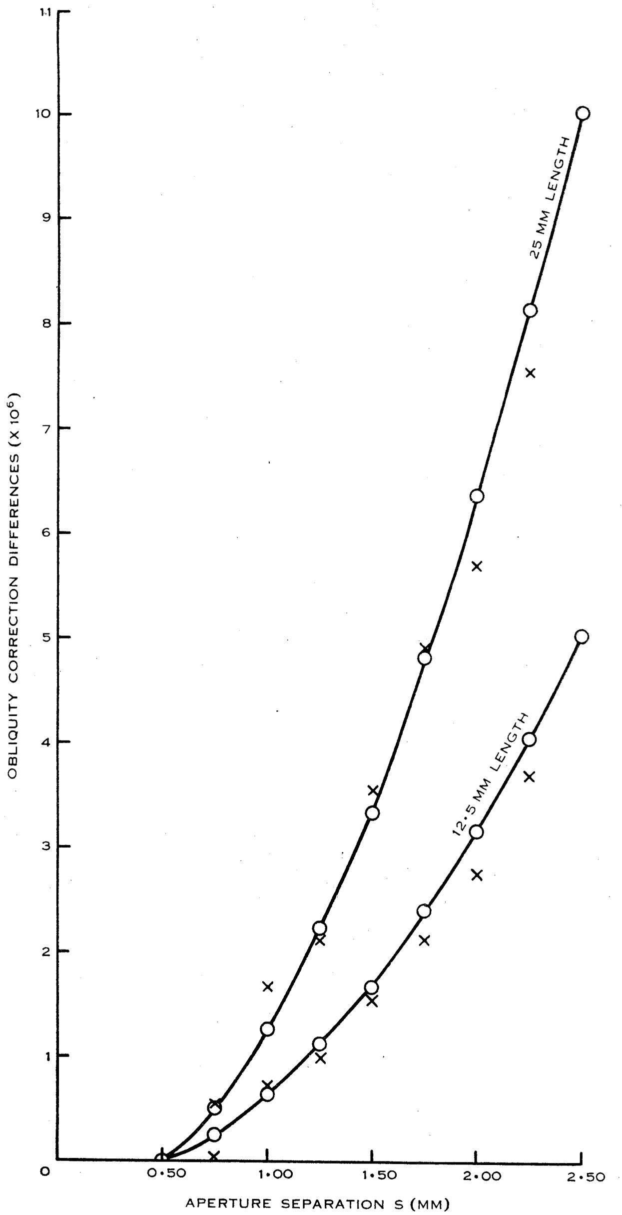

Fig. 9.—Obliquity correction differences for different aperture separations.

account for the rather large disagreement between observed and predicted results for aperture No. 5. For values of Δ greater than π visibility is too low to determine with confidence if the fringe displacement agrees closely with the predicted oscillatory curve in Figure 4, but there was no evidence of an increase beyond 0·5 fringe.

Experimental tests with the Fizeau interferometer are more difficult since large path differences cannot be used. In addition only a limited separation or enlargement of the apertures can be tolerated if the light beams are to pass through the optical system on their return from the interferometer unit. With increasing aperture separation, the effects of varying obliquity, due to size of aperture as the eye is moved, become more serious.

A series of observations with different apertures were made with lengths from 12·5 to 25 mm corresponding to path differences of 25 and 50 mm. The fringes were sharpened by using a highly reflecting layer on the optical flat. The fringe displacement changed as the aperture separation increased in a manner closely agreeing with equation (**1**). Figure 9 shows for two lengths obliquity correction differences corresponding to changes in separation of apertures. The full line curve has been drawn from obliquity corrections calculated from equation (**1**). The observed obliquity correction differences from that for an aperture separation of 0·50 mm are marked $\times$.

An attempt was made to test the effect of aperture size, but the predicted effects were too small to allow really reliable observation, because of the small path differences and to the limitations mentioned above in the interferometer used. It did not seem profitable to continue such tests when fairly definitive results are possible for centred apertures with the Kösters interferometer.

IV. Conclusions

The study of the intensity distribution of the fringes together with microphotometer records indicate that the effects of finite area of aperture on symmetry are insignificant. Any asymmetry which arises in the Kösters system is not appreciable and not observable in the case of partially sharpened Fizeau fringes because of the predominating influence of phase effects at the silvered glass surface.

The effect of finite area of aperture on the position of the fringe maximum is a more complex one. For a centred aperture a fringe displacement does occur which progressively increases provided the phase difference Δ between the centre of the aperture and its edge does not exceed a certain value. For narrow slits and rectangular apertures this value is about $\frac{1}{2}\pi$ and for circular holes about π. Beyond these values the fringe displacement does not increase as equations (**2**) and (**3**) would indicate but oscillates. While there happens to be approximate agreement theoretically and experimentally with these equations for the limited values of Δ mentioned, there is no evidence that they can be applied generally as corrections per unit length. For example it has been quite common to use a narrow slit in the Kösters interferometer. The analysis described here shows that the maximum fringe displacement possible, due to the size of the slit, will be about 0·15 fringe. Much larger values for the displacement would be possible if a formula of the type exemplified by equation (**3**) was applied without consideration to the value of the phase factor Δ involved.

A rectangular aperture may be treated as a narrow slit without great error if its length is equal to or greater than four times its width.

In the case of off-centred apertures, fringe displacement effects appear to be adequately covered by treating the effect as one of direct obliquity from the centre of the aperture plus the effect from the area of the aperture treated as a centred aperture. In the case of a narrow slit off centre, the fringe displacement effect owing to both the position and size of aperture can be calculated from an exact relation.

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