THE EFFECTS OF COLLIMATION AND OBLIQUE INCIDENCE IN LENGTH INTERFEROMETERS. II

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Summary

It is important in length determinations by interferometry to know the effect on the fringe pattern of an entrance aperture of finite size through which the light enters the interferometer. The correction generally applied has been found to be in error (Bruce 1955). This paper is a further discussion of this correction. Both two-beam and multiple-beam systems are discussed and show, in accordance with experiment, that the correction at present applied for the finite size of the aperture appears to be invalid. Formulae are developed and curves are given from which the correction to be applied for any particular length and aperture size can be very easily determined.

I. INTRODUCTION

The usual obliquity correction formula involves a term due to the displacement of the aperture off the optic axis and another due to the finite size of the aperture itself. If the centre of the aperture is displaced a distance x from the optical axis and f is the focal length of the collimator then the angle subtended at the collimator between the aperture and the optic axis is

$$\theta = \tan^{-1} x/f.$$

Now

$$\begin{aligned} p\lambda &= 2t\cos\theta \\ &\simeq 2t(1-x^2/2f^2). \end{aligned}$$

In this equation p is the fringe order and t is the path difference.

The obliquity is seen to be $x^2/2f^2$ per unit length. There remains the correction for the finite aperture size which is derived by taking the average correction for the whole area of the aperture calculated in the manner above. This correction is therefore given by

$$\frac{\displaystyle\iint\left(\frac{x^2}{2f^2}+\frac{y^2}{2f^2}\right)\mathrm{d}x\mathrm{d}y}{\displaystyle\iint \mathrm{d}x\mathrm{d}y}$$

per unit length, and for a circular aperture of diameter d is $d^2/16f^2$ whilst for a rectangular aperture of length a and width b is $(a^2+b^2)/24f^2$. Since $x=0$ in the Kösters interferometer the only correction applied is the latter. However, in tests made at large path differences with a variable entrance slit, the fringe

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displacements predicted by the usual formula were not observed and further investigation of the effect seemed desirable. The method of investigation presented in this paper involves a different approach to that used in paper I (Bruce 1955) in that differentiation under the integral sign is used and the results expressed in terms of Fresnel integrals.

II. The Fringe Displacement for a Finite Aperture

(a) *Two-Beam Systems*

The intensity distribution of two-beam interference fringes when the finite size of the entrance aperture is considered is easily shown to be

$$I_1=\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\cos^2(K\cos\theta)\mathrm{d}x\mathrm{d}y\simeq bf\int_0^{\theta_r}2\cos^2(K\cos\theta)\mathrm{d}\theta, \quad \ldots \quad (1)$$

for a rectangular aperture of dimensions b in the x direction and a in the y direction $(a\gg b)$, $\theta_r=a/2f$, and

$$I_2=\int_0^{2\pi}\int_0^{r}\cos^2(K\cos\theta)x\mathrm{d}x\mathrm{d}\varphi=2\pi f^2\int_0^{\theta_r}\cos^2(K\cos\theta)\theta\mathrm{d}\theta, \quad \ldots \quad (2)$$

for a circular aperture of radius r. Also $K=2\pi t/\lambda$, t is the path difference and x, θ and φ, and r are as in Figure 1.

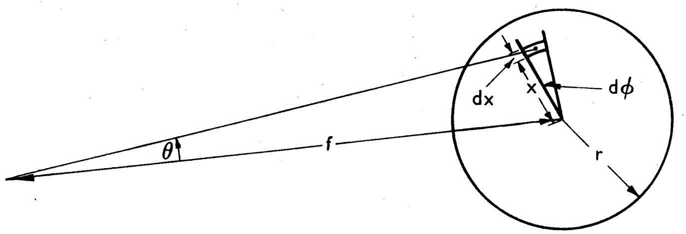

Fig. 1

The displacement of the fringe maxima and minima from their positions corresponding to an ideal point source will be determined in each case by finding the values of K for which $\partial I/\partial K=0$. For the narrow slit, using differentiation under the integral sign,

$$\frac{\partial I_1}{\partial K}\propto\int_0^{\theta_r}\sin(2K\cos\theta)\cos\theta\mathrm{d}\theta.$$

Putting $\cos\theta=1-\frac{1}{2}\theta^2$ we have

$$\frac{\partial I_1}{\partial K}\propto\sqrt{\left(\frac{\pi}{2K}\right)}\{\sin 2KC(u)-\cos 2KS(u)\},$$

where

$$S(u)=\int_0^u \sin\tfrac{1}{2}\pi z^2\mathrm{d}z,$$

$$C(u)=\int_0^u \cos\tfrac{1}{2}\pi z^2\mathrm{d}z,$$

are the Fresnel integrals and

$$u=\sqrt{\left(\frac{2K}{\pi}\right)}\theta_r.$$

Thus we have $\partial I_1/\partial K=0$ when

$$K=\tfrac{1}{2}\tan^{-1}\frac{S(u)}{C(u)}+\tfrac{1}{2}n\pi \quad (n=0,\ 1,\ 2,\ 3,\ .\ .\ .), \qquad \ldots\ldots \quad (3)$$

whereas for two-beam fringes from an ideal point source $\partial I/\partial K=0$ when $K=\frac{1}{2}n\pi$. With an accuracy of a few parts in a million, K may be taken as $2\pi l/\lambda$ in the arguments of the Fresnel integrals, l being the length of the end bar and the correction δ_1 to be applied is thus

$$\delta_1=\frac{\lambda}{4\pi}\tan^{-1}\frac{S(v)}{C(v)}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

with

$$v=\sqrt{\left(\frac{4l}{\lambda}\right)}\theta_r.$$

The maximum value of the correction is approximately 0·17 fringe which occurs when the ratio of the Fresnel sine and cosine integrals is a maximum. This position corresponds to a phase difference $\frac{1}{2}K\theta_r^2$ between central and extreme rays through the aperture of the order of half a wavelength which corresponds to the Rayleigh criterion for two-beam fringes of good definition. For an end bar of length 300 mm, the fringe displacement for a narrow slit of length 0·3 mm used with a collimator of focal length 208 mm (which is a typical case occurring in practice) is obtained from equation (4) as 0·11 fringe. The displacement for a slit of length 0·6 mm is almost the same.

Observations showed that the displacement (if any) on increasing the slit length from 0·3 to 0·6 mm was verified by several observers to be definitely less than 0·1 fringe. Application of the correction formula $(a^2+b^2)/24f^2$ per unit length indicates that the fringe displacement would be approximately 0·3 fringe. The correction $2\delta_1/\lambda$ (=fringe displacement) as a function of $\sqrt{(4l/\lambda)}\theta_r(=2\sqrt{\Delta/\pi})$ is shown in Figure 2. Points on the axis where $\Delta=\pi/2,\ \pi,\ 2\pi,\ .\ .\ .$ are also shown.

A similar result to that given in equation (4) can be obtained for the circular aperture

$$\frac{\partial I_2}{\partial K}\propto\int_0^{\theta_r}\sin\,(2K\cos\theta)\theta\cos\theta d\theta$$

$$=\theta_r I(\theta_r)-\int_0^{\theta_r}I(\theta)d\theta$$

integrating by parts, where

$$I(\theta)=\int_0^{\theta}\sin\,(2K\cos\theta)\cos\theta d\theta,$$

which was evaluated for the case of the slit and

$$I(\theta_r)=\int_0^{\theta_r} \sin\,(2K\cos\theta)\cos\theta d\theta.$$

Thus $\partial I_2/\partial K=0$ when

$$\theta_r I(\theta_r)=\int_0^{\theta_r} I(\theta)d\theta,$$

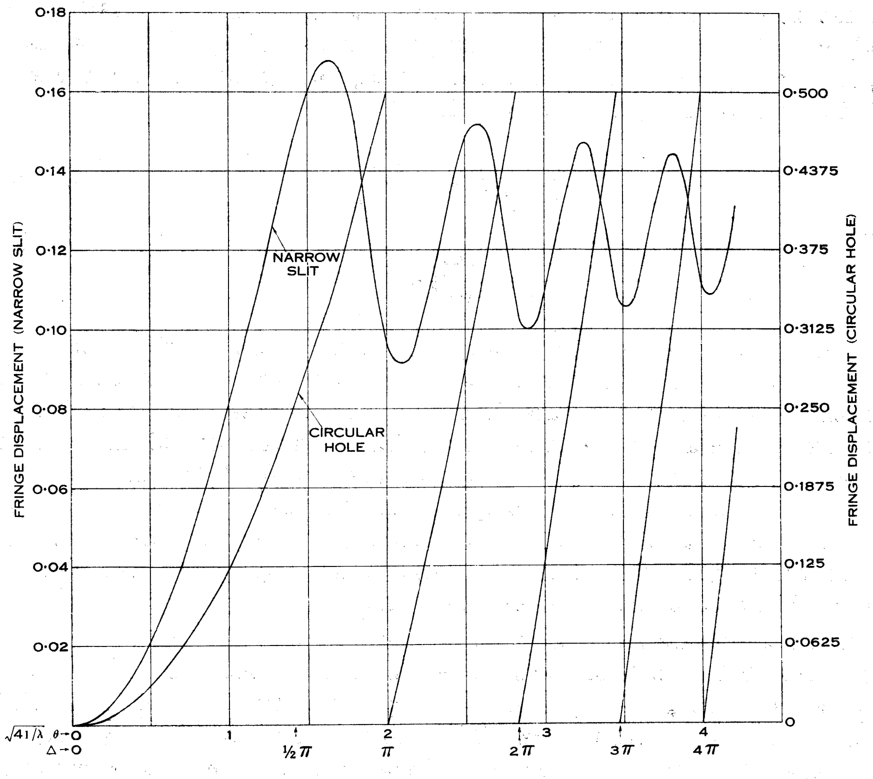

Fig. 2

which leads to a correction δ_2 of amount

$$\delta_2=\frac{\lambda}{4\pi}\tan^{-1}\left\{\frac{S(v)-\frac{1}{v}\int_0^v S(v)dv}{C(v)-\frac{1}{v}\int_0^v C(v)dv}\right\}. \qquad (5)$$

A plot of δ_2 as a function of $\sqrt{(4l/\lambda)}\theta_r$ is given in Figure 2 also. Note that in this case the phase difference between central and extreme rays, $\frac{1}{2}K\theta_r^2$, is a multiple of π at positions of maximum fringe displacement. The curve obtained agrees precisely with that obtained in paper I by a different approach.

The correction δ_1 and δ_2 of course both reduce to zero for $\theta_r=0$.

(b) *Multiple-Beam Systems*

In multiple-beam systems the effect of obliquity is not as important as for the two-beam case since the path differences involved are necessarily very small. The case of a rectangular aperture used in a multiple-beam interferometer will be discussed as an example. For the transmission case the intensity distribution is given by

$$T_4=\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\frac{\mathrm{d}x\mathrm{d}y}{1+R^2\sin^2(K\cos\theta)}$$

$$=\frac{4bf}{2+R^2}\int_0^{\theta_r}\frac{\mathrm{d}\theta}{1-C\cos(2K\cos\theta)}, \qquad \ldots\ldots\ldots\ldots\ (6)$$

where $R^2=4r^2/(1-r^2)^2=$ Fabry's "Coefficient of Finesse" and $C=R^2/(2+R^2)$. Expanding the integrand in equation (6) as a Fourier series with $2K\cos\theta$ as the independent variable we have

$$T_4=\frac{2bf}{2+R^2}\int_0^{\theta_r}[1+\sum_{n=1}^{\infty}A_n\cos(2nK\cos\theta)]\mathrm{d}\theta, \qquad \ldots\ldots\ldots\ (7)$$

where

$$A_n=\frac{2}{\pi}\int_0^{\pi}\frac{\cos nz}{1-C\cos z}\mathrm{d}z=\frac{2}{\sqrt{(1-C^2)}}\left[\frac{1-\sqrt{(1-C^2)}}{C}\right]^n.$$

Therefore

$$\frac{\partial T_4}{\partial K}\propto\int_0^{\theta_r}\sum_{n=1}^{\infty}A_n n\cos\theta\sin(2nK\cos\theta)\mathrm{d}\theta$$

$$=\sum_{n=1}^{\infty}A_n n\sqrt{\frac{\pi}{2nK}}[\sin 2nKC(\sqrt{n}u)-\cos 2nKS(\sqrt{n}u)].$$

The same form can be obtained for the reflection case.

Now multiple-beam interference conditions require K to be small ($K\sim10^3$), so the argument of the Fresnel integrals, for θ_r equal to $0\cdot001$ and n as large as 40, is approximately $0\cdot2$ and up to such values the Fresnel sine integral is very small and may be neglected in comparison with the Fresnel cosine integral and the latter is very closely given by

$$\int_0^{\beta}\cos\tfrac{1}{2}\pi z^2\mathrm{d}z=\beta.$$

Therefore

$$\frac{\partial T_4}{\partial K}\propto\sum_{n=1}^{\infty}A_n n\sin 2nK,$$

$$\frac{\partial R_4}{\partial K}\propto\sum_{n=1}^{\infty}D_n n\sin 2nK,$$

and

$$\frac{\partial T_4}{\partial K}=\frac{\partial R_4}{\partial K}=0,$$

when $K=0$, $\tfrac{1}{2}\pi$, π, . . . as for an ideal point source.

Thus it may be concluded that the correction due to the above effect is negligible for multiple-beam systems. The implied symmetry of the fringes in the above analysis is not correct, for consideration of the effect of phase changes introduced by multiply reflected beams shows that the fringes will be asymmetrical (Brossell 1947).

III. The Intensity Distributions

The fringe intensity distributions are themselves not of great importance but can be found if required in several ways in the different cases.

The formulae for the two-beam intensity distributions and those for the multiple-beam cases, can in each instance be shown to reduce to the corresponding formula for an ideal point source provided K is not too large. As an example the case of a rectangular aperture in a two-beam system will be examined.

Substituting $y=f \tan \theta$ in equation (**1**), we obtain

$$I_1=K^2bf\int_0^{\theta_r}\frac{\cos^2(K\cos\theta)}{(K\cos\theta)^2}\mathrm{d}\theta, \qquad (\mathbf{8})$$

which can be compared with the expression $\int \sin^2(K\sin\alpha)\mathrm{d}\alpha/(K\sin\alpha)^2$ occurring in the formula for the diffracted energy from a rectangular aperture. The integral in equation (**8**) can be evaluated as follows using a Maclaurin expansion :

$$\cos^2(K\cos\theta)=\cos^2 K+K\theta^2\sin 2K-\frac{\theta^4}{4!}\{6K^2\cos 2K+K\sin 2K\}+\ldots .$$

Therefore

$$I_1=bf\left[\cos^2 KJ_1+K\sin 2KJ_2-\frac{(6K^2\cos 2K+K\sin 2K)}{4!}J_3+\ldots\right],$$

where

$$J_1=\tan\theta_r,$$

$$J_2=\theta_r^2\tan\theta_r+2\sum_{n=1}^{\infty}\frac{(-1)^n2^{2n}(2^{2n}-1)}{(2+2n-1)(2n)!}B_{2n}\theta_r^{2+2n-1},$$

$$J_3=\theta_r^4\tan\theta_r+4\sum_{n=1}^{\infty}\frac{(-1)^n2^{2n}(2^{2n}-1)}{(4+2n-1)(2n)!}B_{2n}\theta_r^{4+2n-1},$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

and B_{2n} are Bernoulli's numbers,

$$B_2=\frac{1}{30},\; B_4=\frac{1}{30},\; B_6=0\cdot253114,\; B_8=7\cdot092157,\; \ldots .$$

Taking $\tan\theta_r=\theta_r=a/f$, I_1 is given by

$$I_1=ab[\cos^2 K+0\cdot978\theta_r^2 K\sin 2K-0\cdot001710\theta_r^4\{6K^2\cos 2K+K\sin 2K\}+\ldots$$
$$=ab\cos^2 K \quad \text{for } \theta_r\sim 0\cdot001 \text{ and } K\gg 10^5 \text{ or so,}$$

IV. Conclusions

The effect of the finite size of the aperture in practice does not exceed 0·17 fringe approximately for a narrow slit and 0·5 fringe for a circular aperture and can be conveniently found using the formulae and curves given in this paper.

The usual correction due to the finite size of the aperture appears to be invalid for long lengths.

V. Acknowledgment

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VI. References

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