THE DISTRIBUTION OF THE RATIO OF TWO QUADRATIC FORMS*

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Summary

The exact probability distribution of the ratio of two quadratic forms is given in the case where the quadratic forms and the multivariate normal distribution of the variables is such that the ratio is a ratio of linear functions of gamma variables of integral order.

I. Introduction

The distribution of the ratio of two quadratic forms in normal variables is required for a variety of problems, especially in the analysis of time series. In general it is required for non-null distributions of serial correlation coefficients. Dixon (1944) proposed the problem of finding the distribution of

$$r=\frac{\sum_{i=1}^{N}(x_i-2x_{i+1}+x_{i+2})^2}{\sum_{i=1}^{N}(x_i-x_{i+1})^2}, \quad \ldots \ldots \ldots \ldots \quad (1)$$

with $x_{N+1}=x_1$, $x_{N+2}=x_2$, when $x_1, \ldots, x_N$ are N.I.D. (0,1). Dixon found the mean and variance of the smoothed distribution of this ratio.

Gurland (1953) has shown how to express the distribution of

$$r=\frac{\mathbf{x}'\mathbf{A}\mathbf{x}}{\mathbf{x}'\mathbf{B}\mathbf{x}}, \quad \ldots \ldots \ldots \ldots \quad (2)$$

where $\mathbf{x}$ is a vector of N.I.D. (0,1) variables and $\mathbf{A}$ and $\mathbf{B}$ are real symmetric matrices, $\mathbf{B}$ positive definite, by a Laguerrian expansion. Pitman and Robbins (1949) give a different series solution for the same problem. In many cases of interest, however, these series converge very slowly and are therefore of no practical value. The Pitman and Robbins method was applied by the present writer to the distribution of von Neumann's ratio and the series found by using the EDZAC, Cambridge Mathematical Laboratory; the terms of the series did not begin to decrease until the 37th term! Less elegant methods (which give no estimate of error) are thus still necessary in many problems. It is therefore worth while to have an exact distribution, which is available in certain cases, to provide a check.

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II. An Exact Distribution

We suppose that the matrices **A** and **B** of (**2**) are commutative so that they may be diagonalized by the same orthogonal transformation and, furthermore, that their latent roots are equal in pairs. Then we may write

$$r=\frac{\sum_{1}^{n}\lambda_j w_j}{\sum_{1}^{n}\mu_j w_j}=\frac{l}{m}, \text{ say}, \quad \ldots\ldots\ldots\ldots \quad (\mathbf{3})$$

where w_j $(j=1, \ldots, n)$ are independent γ (1) variables, and where λ_j, μ_j $(j=1, \ldots, n)$ satisfy the conditions

$$\left.\begin{aligned} J_{jj'}=\begin{vmatrix} \lambda_j & \mu_j \\ \lambda_{j'} & \mu_{j'} \end{vmatrix} &>0 \qquad (\text{all } j<j'), \\ \mu_j &>0 \qquad (\text{all } j), \\ \begin{vmatrix} 1 & \lambda_k & \mu_k \\ 1 & \lambda_j & \mu_j \\ 1 & \lambda_{j'} & \mu_{j'} \end{vmatrix} &\neq 0 \qquad (\text{all } k\neq j\neq j'). \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (\mathbf{4})$$

It may be verified that the ratio (**1**) may be put into a form (**3**) satisfying all these conditions if N is odd. The same is true of the circular lag–1 serial correlation with a mean correction when the population is a circular second order autoregressive process.

Before we can proceed we need the following

Lemma

If, for any real numbers λ_k, μ_k $(k=1, \ldots, n)$,

$$\begin{vmatrix} 1 & \lambda_k & \mu_k \\ 1 & \lambda_j & \mu_j \\ 1 & \lambda_{j'} & \mu_{j'} \end{vmatrix} \neq 0 \qquad (\text{all unequal } k,\ j,\ j'),$$

then

$$\frac{1}{\prod_{k=1}^{n}(1-\lambda_k u-\mu_k v)}=\sum_{j<j'}\sum\frac{\begin{vmatrix} \lambda_j & \mu_j \\ \lambda_{j'} & \mu_{j'} \end{vmatrix}^{n-2}}{\prod_{k\neq j, j'}\begin{vmatrix} 1 & \lambda_k & \mu_k \\ 1 & \lambda_j & \mu_j \\ 1 & \lambda_{j'} & \mu_{j'} \end{vmatrix}}\,\frac{1}{(1-\lambda_j u-\mu_j v)(1-\lambda_{j'}u-\mu_{j'}v)}.$$

Proof

For $n=3$, the lemma may be simply proved by considering the identity $1=\sum_{k=1}^{3}A_k(1-\lambda_k u-\mu_k v)$. Successive applications of the result for $n=3$ then give the general result.

The joint characteristic function of l and m, $\varphi(u, v)$, say, is given by

$$\left.\begin{aligned}\varphi(u, v) &= \prod_{j=1}^{n} (1-\mathrm{i}u\lambda_j-\mathrm{i}v\mu_j)^{-1}\\ &= \sum_{j<j'}\sum \frac{\begin{vmatrix} \lambda_j & \mu_j \\ \lambda_{j'} & \mu_{j'} \end{vmatrix}^{n-2}}{\prod\limits_{k\neq j, j'} \begin{vmatrix} 1 & \lambda_k & \mu_k \\ 1 & \lambda_j & \mu_j \\ 1 & \lambda_{j'} & \mu_{j'} \end{vmatrix}} \frac{1}{(1-\mathrm{i}\lambda_j-\mathrm{i}\mu_j v)(1-\mathrm{i}\lambda_{j'}u-\mathrm{i}\mu_{j'}v)},\end{aligned}\right\} \quad \ldots\ldots (5)$$

by the lemma. Thus the joint probability density of l and m is given by

$$f(l, m) = \frac{1}{(2\pi\mathrm{i})^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(u, v)\mathrm{e}^{-\mathrm{i}ul-\mathrm{i}vm}\mathrm{d}u\mathrm{d}v, \quad \ldots\ldots (6)$$

where the integration is to be performed separately on every term of (5). But

$$\frac{1}{(2\pi\mathrm{i})^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp(-\mathrm{i}ul-\mathrm{i}vm)}{(1-\mathrm{i}\mu\lambda_j-\mathrm{i}v\mu_j)(1-\mathrm{i}u\lambda_{j'}-\mathrm{i}v\mu_{j'})}\mathrm{d}u\mathrm{d}v,$$

is the joint probability density of

$$l_{jj'} = \lambda_j w_j + \lambda_{j'} w_{j'}, \qquad m_{jj'} = \mu_j w_j + \mu_{j'} w_{j'},$$

if

$$\frac{\lambda_j}{\mu_j} \geqslant \frac{l}{m} \geqslant \frac{\lambda_{j'}}{\mu_{j'}},$$

or zero otherwise. This density may be found *ab initio* and is given by

$$\frac{\mathrm{e}^{-1}}{J_{jj'}} \exp\left\{\frac{1}{J_{jj'}}\begin{vmatrix} 1 & l_{jj'} & m_{jj'} \\ 1 & \lambda_j & \mu_j \\ 1 & \lambda_{j'} & \mu_{j'} \end{vmatrix}\right\}, \quad \ldots\ldots\ldots\ldots (7)$$

since $J_{jj'} > 0$ by (4). Thus,

$$\frac{\lambda_i}{\mu_i} \geqslant \frac{l}{m} \geqslant \frac{\lambda_{i+1}}{\mu_{i+1}}.$$

$$f(l, m) = \sum_{j=1}^{i}\sum_{j'=i+1}^{n} \frac{J_{jj'}^{n-3}\mathrm{e}^{-1}}{\prod\limits_{k\neq j, j'}\begin{vmatrix} 1 & \lambda_k & \mu_k \\ 1 & \lambda_j & \mu_j \\ 1 & \lambda_{j'} & \mu_{j'} \end{vmatrix}} \exp\left\{\frac{1}{J_{jj'}}\begin{vmatrix} 1 & l & m \\ 1 & \lambda_j & \mu_j \\ 1 & \lambda_{j'} & \mu_{j'} \end{vmatrix}\right\}. \quad \ldots (8)$$

Now

$$\frac{1}{J_{jj'}}\begin{vmatrix} 1 & l & m \\ 1 & \lambda_j & \mu_j \\ 1 & \lambda_{j'} & \mu_{j'} \end{vmatrix} = 1 - \frac{1}{J_{jj'}}\{(\mu_{j'}-\mu_j)l - (\lambda_{j'}-\lambda_j)m\},$$

so the joint density of the variables $r = l/m$ and m is

$$f(r, m) = \sum_{j=1}^{i} \sum_{j'=i+1}^{n} \frac{J_{jj'}^{n-3} m}{\prod_{k \neq j, j'} \begin{vmatrix} 1 & \lambda_k & \mu_k \\ 1 & \lambda_j & \mu_j \\ 1 & \lambda_{j'} & \mu_{j'} \end{vmatrix}} \times \exp\left\{ -\frac{m}{J_{jj'}} [(\mu_{j'} - \mu_j) r - (\lambda_{j'} - \lambda_j)] \right\}, \quad \ldots\ldots (9)$$

for $\lambda_i/\mu_i \geqslant r \geqslant \lambda_{i+1}/\mu_{i+1}$. For fixed r, m may run between zero and infinity. Integrating out m, the density of r is found to be

$$f(r) = \sum_{j=1}^{i} \sum_{j'=i+1}^{n} \frac{J_{jj'}^{n-1}}{\prod_{k \neq j, j'} \begin{vmatrix} 1 & \lambda_k & \mu_k \\ 1 & \lambda_j & \mu_j \\ 1 & \lambda_{j'} & \mu_{j'} \end{vmatrix}} \frac{1}{[(\mu_{j'} - \mu_j) r - (\lambda_{j'} - \lambda_j)]^2}, \quad \ldots (10)$$

for $\lambda_i/\mu_i \geqslant r \geqslant \lambda_{i+1}/\mu_{i+1}$. Integrating with respect to r, the distribution function of r is given by

$$P(r \geqslant r_0) = \sum_{i=1}^{i_0} \sum_{j=1}^{i} \sum_{j'=i+1}^{n} \frac{J_{jj'}^{n-1}}{\prod_{k \neq j, j'} \begin{vmatrix} 1 & \lambda_k & \mu_k \\ 1 & \lambda_j & \mu_j \\ 1 & \lambda_{j'} & \mu_{j'} \end{vmatrix}} \frac{J_{i,\, i-1}}{\begin{vmatrix} 0 & \lambda_{i-1} & \mu_{i-1} \\ 1 & \lambda_j & \mu_j \\ 1 & \lambda_{j'} & \mu_{j'} \end{vmatrix} \begin{vmatrix} 0 & \lambda_i & \mu_i \\ 1 & \lambda_j & \mu_j \\ 1 & \lambda_{j'} & \mu_{j'} \end{vmatrix}}$$

$$+ \sum_{j+1}^{i} \sum_{j'=i+1}^{n} \frac{J_{jj'}^{n-1}}{\prod_{k \neq j, j'} \begin{vmatrix} 1 & \lambda_k & \mu_k \\ 1 & \lambda_j & \mu_j \\ 1 & \lambda_{j'} & \mu_{j'} \end{vmatrix}} \frac{(\lambda_{i_0} - r_0 \mu_{i_0})}{\begin{vmatrix} 0 & \lambda_i & \mu_i \\ 1 & \lambda_j & \mu_j \\ 1 & \lambda_{j'} & \mu_{j'} \end{vmatrix} \begin{vmatrix} 0 & r & 1 \\ 1 & \lambda_j & \mu_j \\ 1 & \lambda_{j'} & \mu_{j'} \end{vmatrix}}, \quad \ldots\ldots (11)$$

for

$$\frac{\lambda_{i_0}}{\mu_{i_0}} \geqslant r_0 \geqslant \frac{\lambda_{i_0+1}}{\mu_{i_0+1}}.$$

III. An Alternative Form

The expression (11) would be difficult to calculate, even for small n. However, the use of the ratio device leads to a much simpler result. Box (1954) has given a general theorem on the distribution of ratios of quadratic forms of the type we are dealing with. Instead of stating his theorem, it is preferable to derive the distribution directly by his methods.

Writing $\nu_j = R\mu_j - \lambda_j$, we have

$$\begin{aligned} P(r \leqslant R) &= P(\Sigma(\lambda_j - R\mu_j) w_j \leqslant 0) \\ &= P(\Sigma \nu_j w_j \geqslant 0). \quad \ldots\ldots (12) \end{aligned}$$

To evaluate (**12**), we notice that the characteristic function of $\Sigma\nu_j w_j$, $\varphi(t)$, admits the partial fraction expansion

$$\varphi(t)=\sum_{j=1}^{n}\frac{\nu_j^{n-1}}{\prod\limits_{\substack{j'=1\\(j'\neq j)}}^{n}(\nu_j-\nu_{j'})}\,\frac{1}{(1-\mathrm{i}\nu_j t)}\quad(\nu_j\text{'s distinct}).\quad\ldots.\ (\mathbf{13})$$

Thus the density of $\Sigma\nu_j w_j$, $f(x)$, say, is given by

$$f(x)=\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}xt}\varphi(t)\mathrm{d}t$$

$$=\sum_{j=k}^{n}\frac{\nu_j^{n-2}}{\Pi(\nu_j-\nu_{j'})}\exp\left(-\frac{x}{\nu_j}\right),\quad\ldots\ldots\ldots.\ (\mathbf{14})$$

where k is defined by $\lambda_k/\mu_k\leqslant R\leqslant\lambda_{k-1}/\mu_{k-1}$. Thus

$$P(r\leqslant R)=\int_0^{\infty}f(x)\mathrm{d}x$$

$$=\sum_{j=k}^{n}\frac{\nu_j^{n-1}}{\Pi(\nu_j-\nu_{j'})}$$

$$=\sum_{j=k}^{n}\frac{(R\mu_j-\lambda_j)^{n-1}}{\prod\limits_{j'}[(\mu_j-\mu_{j'})R-(\lambda_j-\lambda_{j'})]}.\quad\ldots\ldots\ldots.\ (\mathbf{15})$$

This is the required distribution. It is important to note that, had we sought by these methods an expression for $P(r\geqslant R)$ we would have found

$$P(r\geqslant R)=\sum_{j=1}^{k}\frac{(\lambda_j-R\mu_j)^{n-1}}{\prod\limits_{j'}[(\lambda_j-\lambda_{j'})-R(\mu_j-\mu_{j'})]},\quad\ldots\ldots\ (\mathbf{16})$$

where k would be defined by $\lambda_k/\mu_k\geqslant R\geqslant\lambda_{k+1}/\mu_{k+1}$.

Since $P(r\leqslant R)+P(r\geqslant R)=1$, we can use for calculation the formula with the shorter summation. In applications of Box's theorem, two formulae will always be available and the shorter will naturally be used.

It will be noted that, if all the μ_j equal unity, the formula (**16**) reduces to that given by Anderson (1942) for the distribution of the first circular serial correlation coefficient, as it should.

The expressions (**15**) and (**16**) are not as convenient as (**10**) for the calculation of the exact moments of r. In practice, approximate moments could be found by using the easily determined moments of l and m.

IV. Acknowledgment

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V. References

Anderson, R. L. (1942).—Distribution of the serial correlation coefficient. *Ann. Math. Statist.* **13**: 1–13.

Box, G. E. P. (1954).—Some theorems on quadratic forms . . . *Ann. Math. Statist.* **25**: 290–302.

Dixon, W. J. (1944).—Further contributions to the problem of serial correlation. *Ann. Math. Statist.* **15**: 119–44.

Gurland, J. (1953).—Distribution of quadratic forms and ratios of quadratic forms. *Ann. Math. Statist.* **24**: 416–27.

Pitman, E. J. G., and Robbins, H. (1949).—Application of the method of mixtures to quadratic forms in normal variates. *Ann. Math. Statist.* **20**: 552–60.