EIGEN OSCILLATIONS OF COMPRESSIBLE, IONIZED FLUIDS*

By R. E. Loughhead†

The frequencies of eigen oscillations of regions of uniform magnetic field may be derived very simply from the hydromagnetic equations for a compressible fluid of infinite electrical conductivity in the cases where the regions are bounded by cylindrical and plane-parallel surfaces.

For small oscillations these equations may be written in the forms :

$$\frac{\partial \mathbf{h}}{\partial t}=\operatorname{curl}\,(\mathbf{v}\times\mathbf{H}), \qquad (1)$$

$$4\pi\mu\frac{\partial \mathbf{v}}{\partial t}=(\operatorname{curl}\,\mathbf{H})\times\mathbf{h}+(\operatorname{curl}\,\mathbf{h})\times\mathbf{H}-4\pi\,\operatorname{grad}\,p, \qquad (2)$$

$$\frac{\partial \rho}{\partial t}=-\operatorname{div}\,(\mu\mathbf{v}), \qquad (3)$$

$$\frac{\partial p}{\partial t}=q^2\frac{\partial \rho}{\partial t}, \qquad (4)$$

where **h** is the perturbation in the steady magnetic field **H**, **v** the corresponding change in the fluid velocity, ρ the variation in the steady mass density μ, p the variation in the gas pressure, and q the velocity of sound in the fluid.

* Manuscript received June 1, 1955.

† Division of Physics, C.S.I.R.O., University Grounds, Sydney.

Suppose a uniform magnetic field H_z is confined within a circular cylinder of radius R whose central line coincides with the z-axis. In cylindrical coordinates the equations for radial oscillations may be written in the forms :

$$\mathrm{i}\omega h_z = -\frac{1}{r} \cdot \frac{\partial}{\partial r}(r v_r H_z), \quad \ldots\ldots\ldots\ldots\ldots \quad (5)$$

$$4\pi\mu \mathrm{i}\omega v_r = -\frac{\partial}{\partial r}(H_z h_z + 4\pi q^2 \rho), \quad \ldots\ldots\ldots\ldots \quad (6)$$

$$\mathrm{i}\omega\rho = -\frac{1}{r}\frac{\partial}{\partial r}(r\mu v_r), \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

in which the variables are taken to be proportional to $e^{\mathrm{i}\omega t}$. The boundary conditions across the surface $r=R$ follow from equations (5) . . . (7), namely,

$$\left.\begin{aligned} \Delta(v_r H_z) &= 0, \\ \Delta(H_z h_z + 4\pi q^2 \rho) &= 0, \\ \Delta(\mu v_r) &= 0. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

Inside the cylinder, where the magnetic field has the constant value

$$H_z = H_1, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

the variation of v_r is found to be governed by the equation

$$\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} + \left(\frac{\omega^2}{V^2} - \frac{1}{r^2}\right) v_r = 0, \quad \ldots\ldots\ldots\ldots \quad (10)$$

where

$$V = \left(\frac{H_1^2}{4\pi\mu_1} + q^2\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

is the velocity of hydromagnetic disturbances propagated at right angles to the field H_z, μ_1 being the fluid density inside the cylinder. The solution of (10), finite on the axis $r=0$, is

$$v_r = A\mathrm{J}_1\left(\frac{\omega r}{V}\right), \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

where $\mathrm{J}_1(x)$ is Bessel's function of the first kind of order unity and A is an arbitrary constant. Outside the magnetic region v_r obeys the equation

$$\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} + \left(\frac{\omega^2}{q^2} - \frac{1}{r^2}\right) v_r = 0, \quad \ldots\ldots\ldots\ldots \quad (13)$$

whose solution, finite at infinity, takes the form

$$v_r = B\mathrm{J}_1\left(\frac{\omega r}{q}\right) + C\mathrm{Y}_1\left(\frac{\omega r}{q}\right). \quad \ldots\ldots\ldots\ldots\ldots \quad (14)$$

In this equation $\mathrm{Y}_1(y)$ is Bessel's function of the second kind of order unity and B and C are arbitrary constants.

The boundary conditions (**8**) impose three linear relations on the constants A, B, and C and the condition for the existence of non-trivial solutions reduces to the equation

$$J_1\left(\frac{\omega R}{V}\right) \cdot \frac{\omega R}{q}\left\{J_1\left(\frac{\omega R}{q}\right)Y_0\left(\frac{\omega R}{q}\right)-J_0\left(\frac{\omega R}{q}\right)Y_1\left(\frac{\omega R}{q}\right)\right\}=0. \qquad \textbf{(15)}$$

However, by a standard result in the theory of Bessel functions (cf. Watson 1944)*

$$y\{J_1(y)Y_0(y)-J_0(y)Y_1(y)\}=2/\pi,$$

and hence equation (**15**) is satisfied if

$$J_1\left(\frac{\omega R}{V}\right)=0. \qquad \textbf{(16)}$$

Equation (**16**) has an infinity of real roots corresponding to the zeros of the Bessel function of the first kind of order unity. Let j_n ($n=1, 2, 3, \ldots$) denote the infinity of real roots of the equation

$$J_1(x)=0,$$

then the eigen-frequencies of the various modes of radial oscillation of the magnetic cylinder are given by

$$\omega_n=\frac{j_n}{R}\left\{\frac{H_1^2}{4\pi\mu_1}+q^2\right\}^{\frac{1}{2}}, \qquad \textbf{(17)}$$

where $n=1, 2, 3, \ldots$

In the second case consider a uniform magnetic field H_z confined within a pair of parallel planes $x=\pm a$, referred to Cartesian axes Ox, y, z. Then, by a similar analysis, the eigen-frequencies of unidimensional oscillations in the x-direction at right angles to the magnetic field are found to be given by

$$\omega_n=\frac{n\pi}{2a}\left\{\frac{H_1^2}{4\pi\mu_1}+q^2\right\}^{\frac{1}{2}}, \qquad \textbf{(18)}$$

where $n=1, 2, 3, \ldots$

The analysis may be generalized to cover the situation in which the magnetic field takes the values $H_z=H_1$ for $r<R$ or $-a<x<a$, and $H_z=H_2$ for $r>R$ or $|x|>a$, respectively. It is found that the eigen-frequencies given by (**17**) and (**18**) remain unchanged provided, of course, that $H_1\neq H_2$.

* Watson, G. N. (1944).—"Theory of Bessel Functions." p. 77. (Cambridge Univ. Press.)