RECIPROCAL STATIC SOLUTIONS OF THE EQUATIONS OF THE GRAVITATIONAL FIELD

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Summary

This paper deals with reciprocal static line-elements, previously defined by the author, the condition that their Ricci tensors vanish being no longer imposed. If the indices i, k run from 1 to n with the exception of the fixed index a (the line-elements being static with respect to x^a) a certain quantity Q_μ^ν appears with the remarkable property that in a reciprocal transformation Q_i^k is invariant, whilst Q_a^a merely changes sign. Q_μ^ν is closely related to the Hamiltonian derivative of the Gaussian curvature, so that the general results obtained may be applied to the field equations of General Relativity Theory, with $n=a=4$. Q_4^4 is then the total energy density; and formally every static distribution of matter has a " reciprocal distribution " associated with it. In particular, the equation of state of a distribution of fluid reciprocal to a distribution of fluid possessing a given equation of state may be obtained directly from the latter, i.e. without the solution of the field equations being known.

I. INTRODUCTION

In a previous paper (Buchdahl 1954), hereafter referred to as RI, the line-element *reciprocal* to the static line-element

$$ds^2=g_{ik}(x^j)dx^idx^k+g_{aa}(x^j)(dx^a)^2 \quad \ldots\ldots\ldots\ldots \quad (1.1)$$

was defined to be

$$ds^2=(g_{aa})^{2/(n-3)}g_{ik}dx^idx^k+(g_{aa})^{-1}(dx^a)^2. \quad \ldots\ldots \quad (1.2)$$

Here roman indices run over the range $1, \ldots, n$ ($\geqslant 4$) with the omission of the *fixed* index a. It was shown that, if (**1.1**) satisfies the equations

$$G_{\mu\nu}=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1.3)$$

then so does (**1.2**), $G_{\mu\nu}$ being the Ricci tensor of the V_n the metric of which is (**1.1**). (Greek indices run over the entire range $1, \ldots, n$.) In particular, when $n=4$ and a is chosen to be 4, (**1.3**) are the equations of the gravitational field in empty space.

I now consider reciprocal static line-elements without assuming them to satisfy (**1.3**). In particular it will be shown (Section II) that, if

$$P_\mu^\nu=\tfrac{1}{2}\delta_\mu^\nu G-G_\mu^\nu, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1.4)$$

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and P_{μ}^{ν} is the corresponding tensor-density, then the identities

$$\left.\begin{aligned} \bar{\mathrm{P}}_{i}^{k}&=\mathrm{P}_{i}^{k},\\ \left(\bar{\mathrm{P}}_{a}^{a}-\frac{1}{n-3}\bar{\mathrm{P}}_{i}^{i}\right)&=-\left(\mathrm{P}_{a}^{a}-\frac{1}{n-3}\mathrm{P}_{i}^{i}\right)\end{aligned}\right\} \quad \ldots\ldots\ldots (1.5)$$

hold, unbarred and barred quantities being associated with (**1.1**) and (**1.2**) respectively. When $n=a=4$ it then follows from (**1.5**) that if one has any formal* static solution of the gravitational field equations in the presence of matter, then one can immediately write down another solution which is "reciprocal" to it (Section III (*a*)), and the total energies of two such distributions, supposed finite and regular, are equal and opposite in sign (Section III (*b*)). In particular, if one contemplates "reciprocal distributions"† of fluids in thermodynamic equilibrium (Section IV) whose equations of state are $\rho=f(p)$ and $\bar{\rho}=\bar{f}(\bar{p})$ respectively, then the second of these equations may be deduced from the first without either of the corresponding solutions of the field equations being known (Section V), and two such equations of state will be said to be reciprocal to one another.

II. The Ricci Tensors of Reciprocal Metrics

It was shown in RI, Section 2, that the components of the Ricci tensor belonging to the static line-element

$$ds^2=e^{2\sigma}\underset{\circ}{g}_{ik}dx^i dx^k+e^{2\gamma}(dx^a)^2 \quad \ldots\ldots\ldots\ldots\ldots (2.1)$$

may be written

$$\left.\begin{aligned} G_{ik}&=\underset{\circ}{G}_{ik}+(n-3)(\sigma_{;ik}-\sigma_{;i}\sigma_{;k})+\gamma_{;ik}+\gamma_{;i}\gamma_{;k}-(\sigma_{;i}\gamma_{;k}+\sigma_{;k}\gamma_{;i})\\ &\qquad+\underset{\circ}{g}_{ik}[\Delta\sigma+(n-3)\widehat{\sigma\sigma}+\widehat{\sigma\gamma}],\\ G_{ia}&=0,\\ G_{aa}&=e^{2\gamma-2\sigma}[\Delta\gamma+\widehat{\gamma\gamma}+(n-3)\widehat{\sigma\gamma}].\end{aligned}\right\} \ldots (2.2)$$

Here a subscript following a semicolon denotes covariant differentiation in the $\underset{\circ}{V}_{n-1}$ whose metric is $ds^2=\underset{\circ}{g}_{ik}dx^i dx^k$, and $\underset{\circ}{G}_{ik}$ is the Ricci tensor of this V_{n-1}, whilst if ξ, η are any two scalars, $\widehat{\xi\eta}$ stands for $\underset{\circ}{g}^{ik}\xi_{;i}\eta_{;k}$ and $\Delta\xi$ for $\underset{\circ}{g}^{ik}\xi_{;ik}$. From (**1.4**) and (**2.2**) one then has

$$\left.\begin{aligned} P_{ik}&=\underset{\circ}{P}_{ik}-(n-3)(\sigma_{;ik}-\sigma_{;i}\sigma_{;k})-(\gamma_{;ik}+\gamma_{;i}\gamma_{;k})+(\sigma_{;i}\gamma_{;k}+\sigma_{;k}\gamma_{;i})\\ &\qquad+\underset{\circ}{g}_{ik}[(n-3)\Delta\sigma+\Delta\gamma+\tfrac{1}{2}(n-3)(n-4)\widehat{\sigma\sigma}+(n-4)\widehat{\sigma\gamma}+\widehat{\gamma\gamma}],\\ P_{ia}&=0,\\ P_{aa}&=e^{2\gamma-2\sigma}\left(\frac{1}{n-3}\underset{\circ}{P}+(n-2)\Delta\sigma+\tfrac{1}{2}(n-2)(n-3)\widehat{\sigma\sigma}\right).\end{aligned}\right\} \ldots (2.3)$$

* See the remarks at the beginning of Section III (*b*).

† I.e. distributions such that the respective solutions $g_{\mu\nu}$, $\bar{g}_{\mu\nu}$ of the field equations are reciprocal to one another.

The tensor $P_{\mu\nu}$ associated with the V_n whose metric is (**1.1**) is obtained from (**2.3**) by setting $\sigma=0$, $\gamma=\tau=\frac{1}{2}\log g_{aa}$ and omitting the subscript 0 :

$$\left.\begin{aligned} P_{ik}&=P'_{ik}-(\tau_{;ik}+\tau_{;i}\tau_{;k})+g_{ik}(\Delta\tau+\widehat{\tau\tau}),\\ P_{ia}&=0,\\ P_{aa}&=\frac{1}{n-3}e^{2\tau}P'. \end{aligned}\right\} \quad \text{.. (2.4)}$$

Here P'_{ik} is associated with the V_{n-1} whose metric is $ds^2=g_{ik}dx^idx^k$. Similarly, setting $\sigma=2\tau/(n-3)$ and $\gamma=-\tau$ in (**2.3**) one has the tensor $\bar{P}_{\mu\nu}$ associated with $\bar{V}_n$, that is, the V_n whose line-element is reciprocal to (**1.1**). Thus

$$\left.\begin{aligned} \bar{P}_{ik}&=P'_{ik}-(\tau_{;ik}+\tau_{;i}\tau_{;k})+g_{ik}(\Delta\tau+\widehat{\tau\tau}),\\ \bar{P}_{ia}&=0,\\ \bar{P}_{aa}&=\frac{1}{n-3}\exp\left(-\frac{2(n-1)}{n-3}\tau\right)[P'+2(n-2)(\Delta\tau+\widehat{\tau\tau})]. \end{aligned}\right\} \quad \text{.. (2.5)}$$

Raising one of the indices in each of the equations (**2.4**) and (**2.5**) it then follows at once that

$$\left.\begin{aligned} \bar{P}^k_i&=(g_{aa})^{-2/(n-3)}P^k_i,\\ \bar{P}^a_a&=-(g_{aa})^{-2/(n-3)}\left(P^a_a-\frac{2}{n-3}P^i_i\right). \end{aligned}\right\} \quad \text{.. (2.6)}$$

The components P^ν_μ, $\bar{P}^\nu_\mu$ in which just one index equals a are of course zero, and they will not be explicitly referred to hereafter. Keeping in mind that

$$\sqrt{(-\bar{g})}=(g_{aa})^{2/(n-3)}\sqrt{(-g)}, \quad \text{............ (2.7)}$$

the equations (**1.5**) follow at once. If one defines

$$Q^\nu_\mu=P^\nu_\mu-\frac{1}{n-3}\delta^\nu_\mu P^i_i, \quad \text{................ (2.8)}$$

then (**2.6**) may be written in the curious form

$$\bar{Q}^k_i=Q^k_i,\quad \bar{Q}^a_a=-Q^a_a. \quad \text{................ (2.9)}$$

In transformations of the x^i, Q^k_i behaves as a tensor and Q^a_a as a scalar.

III. Reciprocal Distributions of Matter

(*a*) If one chooses $n=4$, the preceding results may be applied to the field equations of General Relativity Theory which are (in suitable units)

$$P^\nu_\mu=T^\nu_\mu, \quad \text{.................... (3.1)}$$

T^ν_μ being the components of the stress-energy tensor of the medium filling the region of interest. x^a will be taken to be the coordinate time x^4. The T^ν_μ

and $\bar{T}^{\nu}_{\mu}$ associated with any metric and its reciprocal are then related as follows, according to (**2.6**):

$$\bar{T}^{k}_{i}=(g_{44})^{-2}T^{k}_{i}, \quad \bar{T}^{4}_{4}=-(g_{44})^{-2}(T^{4}_{4}-2T^{i}_{i}). \quad \dots\dots\dots\dots \quad (\mathbf{3.2})$$

The components of the stress-energy tensor $\underset{\circ}{T}_{\mu\nu}$ as measured by an observer at rest with respect to the coordinates employed above, who uses proper coordinates $\underset{\circ}{x}^{\mu}$ at any point of interest, are (Tolman 1934, p. 215)

$$\underset{\circ}{T}_{ik}=p_{ik}, \quad T_{i4}=0, \quad \underset{\circ}{T}_{44}=\rho, \quad \dots\dots\dots\dots\dots \quad (\mathbf{3.3})$$

ρ, p_{ik} being the density and mechanical stresses of the medium under consideration. The results of measurements* carried out by such an observer on distributions represented by a metric tensor $g_{\mu\nu}$ on the one hand, and by its reciprocal $\bar{g}_{\mu\nu}$ on the other are, in view of (**3.2**), therefore related by

$$\bar{p}_{ik}=(g_{44})^{-2}p_{ik}, \quad \bar{\rho}=-(g_{44})^{-2}(\rho-2\sum_{i=1}^{3}p_{ii}). \quad \dots\dots \quad (\mathbf{3.4})$$

(*b*) From (**3.4**), or more directly from (**2.9**), it follows that of two finite, regular reciprocal distributions T^{ν}_{μ} and $\bar{T}^{\nu}_{\mu}$ only one can be physically realizable. A result due to Tolman (1934, p. 234) states that the total energy U of such a distribution of matter, including the energy of the field produced by it, is given by

$$U=\int Q^{4}_{4}dx_{1}dx_{2}dx_{3}. \quad \dots\dots\dots\dots\dots\dots\dots \quad (\mathbf{3.5})$$

If $\bar{U}$ is the total energy of the " reciprocal distribution ", it follows at once from the second of equations (**2.9**) that

$$U=-\bar{U}. \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad (\mathbf{3.6})$$

The stated result is implied whenever it is granted that any physically realizable distribution must have positive total energy.

IV. Fluid Distributions

An interesting special case is that of a distribution of fluid in thermodynamic equilibrium. For this

$$T^{k}_{i}=-p\delta^{k}_{i}, \quad T^{4}_{i}=T^{i}_{4}=0, \quad T^{4}_{4}=\rho, \quad \dots\dots\dots\dots \quad (\mathbf{4.1})$$

where ρ and p are the proper density and pressure of the fluid. Equations (**3.2**) then read

$$\bar{p}=(g_{44})^{-2}p, \quad \bar{\rho}=-(g_{44})^{-2}(\rho+6p). \quad \dots\dots\dots \quad (\mathbf{4.2})$$

* See the remarks at the beginning of Section III (*b*).

Now the equation

$$T^{\nu}_{\mu;\nu}=0, \quad \ldots\ldots \quad (4.3)$$

which is identically satisfied, can here be reduced to the form (Tolman 1934, p. 317)

$$\text{grad } p+\tfrac{1}{2}(\rho+p) \text{ grad log } g_{44}=0. \quad \ldots\ldots \quad (4.4)$$

Let the equation of state of the fluid be given in the form $\rho=f(p)$. Assuming $\rho+p\neq 0$, (**4.4**) then gives at once

$$\log g_{44}=-2\int \mathrm{d}p/(\rho+p). \quad \ldots\ldots \quad (4.5)$$

Equations (**4.2**) may therefore be written

$$\bar{p}=p \exp \int\frac{4\mathrm{d}p}{\rho+p}, \quad \bar{\rho}=-(\rho+6p) \exp \int\frac{4\mathrm{d}p}{\rho+p}. \quad \ldots\ldots \quad (4.6)$$

These equations are a parametric representation of the equation of state* of the fluid constituting the reciprocal distribution.

V. Reciprocal Equations of State

One may look upon (**4.6**) as directly defining an equation of state $\bar{\rho}=\bar{f}(\bar{p})$ which is " reciprocal to " the given equation of state $\rho=f(p)$. Note that the explicit form of the distribution, *qua* function of the coordinates, is not required. A particularly simple example is furnished by the equation of state

$$\rho=sp, \quad \ldots\ldots \quad (5.1)$$

where $s(\neq -1)$ is a constant. Then

$$\bar{p}=\beta p^{c}, \quad \bar{\rho}=-(s+6)\beta p^{c}, \quad \ldots\ldots \quad (5.2)$$

where $c=(s+5)/(s+1)$, and β is a positive constant. Hence

$$\bar{\rho}=-(s+6)\bar{p}. \quad \ldots\ldots \quad (5.3)$$

The equation $\bar{\rho}=-3\bar{p}$ is evidently reciprocal to itself.

VI. Electrostatic Fields

If T^{ν}_{μ} is the stress-energy tensor of an electrostatic field, then the same cannot be true of $\bar{T}^{\nu}_{\mu}$. For otherwise the vanishing of the spur of the electromagnetic stress-energy tensor would require

$$T^{i}_{i}=-T^{4}_{4} \text{ and } \bar{T}^{i}_{i}=-\bar{T}^{4}_{4} \quad \ldots\ldots \quad (6.1)$$

to hold simultaneously. By (**3.2**) the first of these implies

$$\bar{T}^{4}_{4}=3\bar{T}^{i}_{i}, \quad \ldots\ldots \quad (6.2)$$

which is inconsistent with (**6.1**) since T^{4}_{4} cannot vanish.

* Actually they define a one-parameter set of equations of state since a generally non-trivial constant of integration occurs in (**4.6**). A change of this constant is, however, simply equivalent to a change of units in which $\bar{\rho}$ and $\bar{p}$ are measured.

VII. Inclusion of the Cosmical Constant

The possible presence of a cosmological term in the field equations has hitherto been ignored. If this term be restored, (**3.1**) must be replaced by

$$P_{\mu}^{\nu}-\delta_{\mu}^{\nu}\lambda=T_{\mu}^{\nu}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{7.1})$$

where λ is the cosmical constant. Equations (**3.2**) are therefore appropriately modified by replacing T_{μ}^{ν} by $T_{\mu}^{\nu}+\delta_{\mu}^{\nu}\lambda$ throughout. In the same way, equations (**4.6**) become

$$\bar{p}=(p-\lambda)J+\lambda, \quad \bar{\rho}=-(\rho+6p-5\lambda)J-\lambda, \quad \ldots\ldots\ldots\ldots \quad (\mathbf{7.2})$$

where

$$J=\exp 4\int \mathrm{d}p/(\rho+p). \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{7.3})$$

VIII. References

Buchdahl, H. A. (1954).—*Quart. J. Math.* **5**: 116.
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