THE ACCELERATED MOTION OF DROPLETS AND BUBBLES

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Summary

The drag force acting upon small particles such as droplets and bubbles, moving through a viscous medium, depends upon the rate of change of the state of motion of the medium and upon the diffusion of vorticity from the surface of the particle. For accelerated particles the drag force changes with time and depends upon all previous accelerations. The equation of motion of a particle takes the form of an integro-differential equation, which has been solved numerically in the case of deceleration to rest from uniform motion with no impressed body forces. In any experiments designed according to the principles of dynamical similarity, the ratios of viscosities and densities of the medium and the particle must be maintained constant in the scale transformations involved.

I. INTRODUCTION

Recent studies of cloud and droplet physics have raised the question of what may be the detailed behaviour of very small droplets subject to forces which change with time; in particular whether or not it is possible to simulate the motion of such droplets in the laboratory by suitable changes in the scale of the variables (Sartor 1954). In theoretical studies of droplet dynamics it has so far been usual to assume that the only drag force experienced by a droplet, moving through a viscous medium at any instant with a Reynolds number R, to be identical with the drag it would experience if it were moving uniformly at that same Reynolds number, and the acceleration has been assumed to depend only upon the difference between this drag force and any time-variable body forces acting upon the droplet only.

The actual accelerations suffered by a body, immersed in a viscous medium and subject to time-variable forces, are affected by two factors. As is well known, the momentum imparted by such forces is distributed between the body and the surrounding medium, giving rise to an effect which, in a non-viscous medium, is equivalent to an increase in the mass of the body. In fact, for a spherical droplet the effective increase in mass is equal to half the mass of the medium displaced by the droplet (Lamb 1932, p. 124). A further effect occurs owing to the finite rate at which vorticity diffuses from the surface of the body. The distribution of vorticity throughout the medium depends upon the past velocity of the body and thus upon its history. The actual drag experienced at any particular time is more affected by the recent past history than by the distant past.

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The case of the motion of a lamina under given forces and immersed in a viscous medium has been discussed previously by others (Lamb 1932, p. 591). The variation with time of the effect of past motions upon the current drag force for a sphere is here found to be similar to the case of a lamina.

In the case of droplets and bubbles the effect of past history on the equations of rectilinear motion is determined to a degree of approximation corresponding to that given by Oseen's approximation to the problem of viscous flow around a sphere. The resulting integro-differential equation shows that any experiment, which endeavours to simulate the case of droplets and bubbles by changing the scales of the physical constants involved, must maintain constant both the Reynolds number and the ratios of viscosities and densities of the droplet and medium.

The equation of motion has been solved numerically in the case of a droplet, initially moving with uniform velocity, which is allowed to come to rest as a result of the drag forces arising from the surrounding medium.

The solutions show that the effect of past history on the motion of a small droplet or bubble may in certain circumstances affect, by a relatively large amount, its velocity and the distance it travels so that neglect of this effect in theoretical studies would cause serious error.

II. The Drag Force for Impulsive Motion of a Droplet

We shall restrict consideration at first to the case of a droplet of radius a supposed to be at rest for $t<0$ and to move with uniform velocity U for $t>0$. The drag force experienced by the droplet will be the same as that which would apply if the droplet were held at rest and the medium allowed to flow past it with a main stream velocity $-U$.

The flow may be expressed in terms of a Stokes type of stream function, Ψ, and Oseen's approximation to the inertia terms in the hydrodynamical equations for small U may be adopted. Thus Ψ satisfies the partial differential equation

$$\left.\begin{aligned}&\left(\nu D^2-\frac{\partial}{\partial t}-U\frac{\partial}{\partial x}\right)D^2\Psi=0,\\ \text{where}\quad &D^2\equiv\frac{\partial^2}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right),\end{aligned}\right\}\quad\ldots\quad(1)$$

ν is the kinematic viscosity of the medium, and r,θ,φ are spherical polar coordinates with the centre of the droplet as origin. The velocity U is directed along the positive axis of x corresponding to $\theta=0$. For $t>0$ we may assume the component of the stream function corresponding to the uniform motion of the medium, namely $-Ur^2\sin^2\theta/2$, to be omitted, and restrict attention to the disturbance caused by the droplet.

The components of fluid velocity caused by the presence of the droplet are

$$u_r=\frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta},\quad u_\theta=-\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r},$$

and the boundary conditions, assuming no slipping to occur at the surface for $t > 0$, become

$$u_r = U\cos\theta,\ u_\theta = -U\sin\theta, \qquad r = a,$$
$$u_r = u_\theta = 0, \qquad r = \infty.$$

The remaining components of the stream function may be expressed as the sum of an irrotational function $\Psi'^{(1)}$ and a solenoidal function $\Psi'^{(2)}$, each of which is obtainable in the form of an infinite series of functions which satisfy the boundary conditions at infinity and whose coefficients are determined by the conditions at the surface of the droplet.

The irrotational component is of impulsive type, whilst the solenoidal component corresponds to the diffusion of vorticity from the surface of the droplet required to maintain the boundary conditions. Both components provide contributions to the drag experienced by the droplet which behave differently with time.

Taking Laplace transforms in the variable p, $\overline{\Psi'^{(1)}}$ and $\overline{\Psi'^{(2)}}$, where $\overline{\Psi'} = \overline{\Psi'^{(1)}} + \overline{\Psi'^{(2)}}$, then

$$\left.\begin{aligned}\overline{\Psi'^{(1)}} &= \sum_{n=1}^{\infty} A_n(p)/r^n.(1-\mu^2)P'_n(\mu),\\ \overline{\Psi'^{(2)}} &= e^{\mu kr}\sum_{n=1}^{\infty} B_n(p)\chi_n(qr)(1-\mu^2)P_n(\mu),\end{aligned}\right\} \quad \ldots\ldots (2)$$

where

$$\chi_n(z) = z^{\frac{1}{2}}K_{n+\frac{1}{2}}(z),$$
$$q^2 = k^2 + p/\nu,$$
$$k = U/2\nu = R/4a.$$

R is the Reynolds number $2Ua/\nu$, K_m is the modified Bessel's function of second kind of order m, and $P_n(\mu)$ is the Legendre function of order n.

The boundary conditions require that

$$A_n + a^n \sum_{m=1}^{\infty} B_m\chi_m(qa)X_{nm}(ka) = Ua^{n+2}\delta_{1n}/2p,$$

$$\sum_{m=1}^{\infty} B_m[n\chi_m(qa)X_{nm}(ka) + ka\chi_m(qa)X_{nm}'(ka) + qa\chi_m'(qa) + X_{nm}(ka)]$$
$$= 3Ua^2\delta_{1n}/2p, \quad \ldots\ldots (3)$$

where the δ_{1n} is the Kronecker function and X_{nm} is the function introduced by Goldstein (1929), and is defined by the identity

$$e^{\mu z}P_m'(\mu) = \sum_{n=1}^{\infty} X_{nm}(z)P_n'(\mu). \quad \ldots\ldots (4)$$

Solution of the equations for the A_n and B_n, followed by inversion of the Laplace transform, would provide the solution for the flow pattern of the stream function. So far as the drag on the sphere is concerned we require only to integrate the appropriate stress components over the surface of the droplet.

Thus we integrate the axial component of the tangential or viscous stress $p_{r\theta}$ and the normal or pressure stress p_{rr}, where

$$\left.\begin{aligned} p_{rr} &= -p+2\eta\frac{\partial u_r}{\partial r}, \\ p_{r\theta} &= \eta\left\{r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial \theta}\right\}, \end{aligned}\right\} \quad r=a,$$

p is the hydrostatic pressure, and η is the coefficient of viscosity of the medium. The total drag force D is then given by

$$D=2\pi a^2\int_0^\pi(-p_{rr}\cos\theta+p_{r\theta}\sin\theta)\sin\theta\, d\theta.$$

Expressing this in terms of the stream function at $r=a$, and using the relation between the hydrostatic pressure p and the stream function,

$$\frac{\partial p}{\partial\theta}=\frac{\rho}{\sin\theta}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\frac{\partial\Psi'^{(1)}}{\partial r},$$

where ρ is the density of the medium and, taking Laplace transforms, then

$$\bar{D}=\frac{4\pi\rho a}{3}\left[Ua^2-\frac{3p}{a}A_1\right]+\frac{8\pi\rho U}{5a^2}A_2. \quad \ldots\ldots\ldots\ldots (5)$$

Actually a complete determination of D requires knowledge of all the A's and B's in order that A_1 and A_2 may be obtained. In the case of small droplets some approximations may be made. Thus A_1 will be of order U, and A_2 of order U^2. For sufficiently small U, A_2 may be neglected and in the determination of A_1 all terms for $n>1$ may be omitted. Thus we find

$$A_1=(Ua^3/2p)[1-3/\{1+qa\chi'(qa)/\chi(qa)+kaX_{11}'(ka)/X_{11}(ka)\}],$$

where

$$\chi_1(z)=\sqrt{(\tfrac{1}{2}\pi)}e^{-z}(1+z^{-1}),$$
$$X_{11}(z)=z^{-\frac{1}{2}}I_{3/2}(z),$$

and $I_{3/2}(z)$ is the modified Bessel function of first kind and order 3/2. For sufficiently small ka, therefore,

$$kaX_{11}'/X_{11}=1+k^2a^2/5+O(k^4a^4).$$

If, however, k^2a^2 be neglected throughout compared with $(p/\nu)a$, then upon carrying out the inversion on the Laplace transformation, the drag force reduces to

$$D=6\pi\eta aU[1+a/\sqrt{(\pi\nu t)}]+Um\delta(t)/2, \qquad t\geqslant 0, \quad \ldots\ldots (6)$$

where m is the mass of the medium displaced by the droplet and $\delta(t)$ is the Dirac δ-function. Thus D, the force required instantaneously to raise a droplet from rest to uniform motion, consists of three terms: (1) the viscous force which would be experienced in the case of uniform translation, in this case the drag according to Stokes' theory of a uniformly translated droplet; (2) a time-variable component which is infinite at $t=0$ and decreases as $t^{-\frac{1}{2}}$; and (3) a transient force which occurs only at $t=0$ and is due to the momentum transferred to the medium by the sudden change in velocity.

It will be shown later, in Section IV, that the equation of motion of a droplet, depending upon (**6**) for the drag, implies physically inadmissible conditions. Although a droplet, subject to no body forces after time $t=0$, may come to rest, (**6**) implies that it must travel an infinite distance before doing so, owing to the complete neglect of terms of the order k^2a^2. If k^2a^2 be retained, the behaviour differs for large elapsed times and the distances travelled become finite. Thus we find

$$D=6\pi\eta a U\left[1+\frac{1}{\sqrt{(\pi T)}}-\frac{1}{\sqrt{\pi}}\,\frac{(1+\gamma_1)\gamma_1^2}{(\gamma_1-\gamma_2)}\exp\left(-\gamma_1^2 T\right)\int_0^T \exp\left(\gamma_1^2 T\right)\mathrm{d}T/\sqrt{T}\right.$$

$$\left.+\frac{1}{\sqrt{\pi}}\,\frac{(1+\gamma_2)\gamma_2^2}{(\gamma_1-\gamma_2)}\exp\left(-\gamma_2^2 T\right)\int_0^T \exp\gamma_2^2 T\,\mathrm{d}T/\sqrt{T}\right]+\tfrac{1}{2}Um\delta(T), \quad \ldots (\mathbf{7})$$

where $T=\nu t/a^2$ and γ_1 and γ_2 are the roots of the quadratic

$$80z^2-R^2z+4R^2=0.$$

For small T, D approximates to

$$D=6\pi\eta a U\left[1+\frac{1}{\sqrt{(\pi T)}}\left\{1+\frac{3R^2}{40}T+\mathrm{O}(R^4T^2)\right\}\right]+\tfrac{1}{2}Um\delta(T), \qquad 0\leqslant T<16/R^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{8.1})$$

which corresponds to (**6**), and for large T becomes

$$D=6\pi\eta a U[1+(\pi T)^{-\frac{1}{2}}\{25/2R^2T+\mathrm{O}(R^{-4}T^{-2})\}], \qquad T>16/R^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{8.2})$$

thus, for $T>16/R^2$, D decreases more rapidly than predicted by (**6**).

III. The Impulsive Motion of Bubbles

The main assumption made in Section II was that of rigidity of the droplet. This applies only if the droplet is highly viscous compared with the surrounding medium, and does not apply to bubbles. When internal motions are considered, the stream function in the interior is represented by Ψ_1, and its Laplace transform is

$$\overline{\Psi}_1=\sum_1^\infty[C_nr^{n+1}+\mathrm{e}^{-k\mu r}E_n\psi_n(q_1r)](1-\mu^2)P_n'(\mu), \qquad 0\leqslant r\leqslant a,$$

in which the unit suffix indicates reference to the variables and physical constants of the medium of the bubble, and ψ_n involves the modified Bessel function of first kind

$$\psi_n(z)=z^{\frac{1}{2}}I_{n+\frac{1}{2}}(z).$$

The boundary conditions at the surface require the viscous stresses and the velocities of the two media to be continuous, that is,

$$u_r(a)=0,$$

$$\operatorname*{Lim}_{\varepsilon=0}[u_\theta(a+\varepsilon)-u_\theta(a+\varepsilon)]=0,$$

$$\operatorname*{Lim}_{\varepsilon=0}\left[\eta\frac{\partial}{\partial a}u_r(a+\varepsilon)-\eta_1\frac{\partial}{\partial a}u_r(a-\varepsilon)\right]=0,$$

and

$$\operatorname*{Lim}_{\varepsilon=0}\left[\eta\left\{\frac{\partial}{\partial a}-\frac{1}{a}\right\}u_\theta(a+\varepsilon)-\eta_1\left\{\frac{\partial}{\partial a}-\frac{1}{a}\right\}u_\theta(a-\varepsilon)\right]=0.$$

These conditions are sufficient to determine the coefficients A_n, B_n, C_n, and E_n, although complete evaluation would be impracticable.

If the drag force be evaluated to the same degree of approximation as was adopted in equation (**6**) we find

$$D=6\pi\eta aU\xi[1+a/\sqrt{(\pi\nu t)}]+Um\delta(t)/2, \quad t\geqslant 0, \quad \ldots\ldots\ldots \quad (\mathbf{9})$$

where ξ is a parameter depending only upon the coefficients of viscosity of the two media, that is,

$$\xi=(3\eta_1+2\eta)/(3\eta_1+3\eta).$$

The form of the expression (**9**) is the same as that for a rigid droplet, although in this case the component due to viscosity is modified by the factor ξ which varies between the limit 1, which it takes in the case of highly viscous droplets ($\eta_1\gg\eta$), and 2/3 in the case of bubbles ($\eta\gg\eta_1$). This factor arises from the internal motions which allow flow to occur at the surface, thus reducing the viscous stresses. The drag force in the case of uniform translation (when $t=\infty$) agrees with that quoted by Lamb (1932, pp. 600–1). The force arising from the transfer of momentum to the surrounding fluid remains unchanged to the degree of approximation adopted. This is as might be expected since the bubble is presumed to retain its shape during the impulsive change of velocity. This would be so in practice if the internal pressure due to surface tension were large enough.

IV. The Equations of Motion of Droplets and Bubbles

To the approximation adopted, the drag force is a linear function of the velocity U, and consequently, for continuously variable velocities, the method of superposition of solutions may be adopted in which U is replaced by the increment in velocity over an infinitesimally small interval of time between T and $T+\mathrm{d}T$, i.e. $\dot{U}(t-T)\mathrm{d}T$, the summation being taken over all previous elapsed time for which U is non-zero. Making the substitution for the drag force we find the equation of rectilinear motion to be

$$(M+\tfrac{1}{2}m)\dot{U}+6\pi\eta a\xi U+\frac{6\pi\eta a^2\xi}{\sqrt{(\pi\nu)}}\int_0^\infty\frac{\dot{U}(t-t')}{\sqrt{t'}}\mathrm{d}t'=F,$$

where F represents the applied or body force acting upon the mass, M, of the droplet or bubble. The term $\frac{1}{2}m$ constitutes the effective increase in mass arising from the transfer of momentum to the medium as the acceleration occurs and is not due to viscosity. The equation of motion may be reduced to non-dimensional form by the transformations

$$\left.\begin{aligned} \tau&=\nu\pi\lambda^2 t/\xi^2 a^2, \quad \lambda^2=9\rho\xi/\pi(2\rho_1+\rho),\\ \text{and}\quad V&=u/U_0, \quad f=F/6\pi\eta a\xi U_0, \end{aligned}\right\} \qquad (\mathbf{10.1})$$

where U_0 is some standard velocity, whence

$$\frac{dV}{d\tau}+V+\lambda\int_0^\infty \frac{dV(\tau-\tau')}{d(\tau-\tau')} \cdot \frac{d\tau'}{\sqrt{\tau'}}=f. \qquad (\mathbf{10.2})$$

It is clear that the motion of a small droplet or bubble at any instant will depend markedly upon its previous motions, since the effect upon the drag force of past accelerations decreases slowly with elapsed time t, as $t^{-\frac{1}{2}}$, only so long as the maximum Reynolds number is sufficiently small (cf. (**8.1**) and (**8.2**)). The actual magnitude of the effect depends upon the relative densities of the two media and their coefficients of viscosity, and any experiment conducted and designed according to principles of dynamical similarity must be arranged so that the value of the parameter λ is maintained constant during the transformation of scale.

The value of λ is affected only little by variations of the viscosity of the medium since $\xi^{\frac{1}{2}}$ may vary only between the limits $(\frac{2}{3})^{\frac{1}{2}}$ and 1. In this respect the need to maintain a constant ratio of viscosity may well be relaxed. However, the variation with the density is important as λ may be about $(9\rho/2\pi\rho_1)^{\frac{1}{2}}$ for droplets ($\rho_1 \gg \rho$), and $(6/\pi)^{\frac{1}{2}}$ for bubbles. In particular,

$$\begin{aligned} \lambda&=0\cdot 0428, && \text{for water droplets in air,}\\ &=1\cdot 38, && \text{for air bubbles in water.} \end{aligned}$$

V. Solution of the Equations of Motion

To illustrate the effect of past history upon the velocity and position of droplets and bubbles, a simple situation is considered in which, up to the time $t=0$, motion is uniform and rectilinear and determined by a constant impressed body force such as that due to gravity. At time $t=0$ and for all subsequent time this force is supposed removed, so that the particle decelerates to rest under the action of viscous forces.

Suppose therefore that U_0 be taken so that f is equal to unity for $\tau<0$ and zero thereafter. In the absence of any effects of past acceleration, i.e. $\lambda=0$, the velocity for $\tau>0$ would be $e^{-\tau}$. The difference in the velocity due to the effect of past history, ω, satisfies the integro-differential equation

$$\frac{d\omega}{d\tau}+\omega+\lambda\int_0^\tau \frac{d\omega(\tau-\tau')}{d(\tau-\tau')}\,\frac{d\tau'}{\sqrt{\tau'}}=\lambda e^{-\tau}\int_0^\tau e^{\tau'}\frac{d\tau'}{\sqrt{\tau'}}, \quad \tau>0, \quad (\mathbf{11.1})$$

and

$$\omega=0, \text{ at } \tau=0.$$

The difference in the distance travelled will be linearly related to s, where

$$s=\int_0^\tau \omega \mathrm{d}\tau. \qquad \qquad (\mathbf{11.2})$$

Solution of the equation (**11**) for ω as a power series in τ gives an expression which is useful when τ is sufficiently small, thus

$$\omega=\sum_{n=0}^{\infty} a_n \tau^{2n+3/2},$$

where

$$a_0=\frac{4\lambda}{3}, \qquad a_1=-X,$$

$$a_2=\frac{8\lambda}{15}(1-X), \qquad a_3=X\left(1-\frac{2X}{3}\right),$$

$$a_4=\frac{16\lambda}{35}\left(1-\frac{8X}{3}+\frac{4X^2}{3}\right), \qquad a_5=-\frac{X}{2}\left(1-\frac{5X}{3}+\frac{7X^2}{6}\right),$$

and $X=\frac{1}{2}\pi\lambda^2$. Expansion into power series of λ gives

$$\omega=\sum_{r=1}^{\infty} \omega_r \lambda^r,$$

where

$$\omega_r=-\mathrm{e}^{-\tau}\int_0^\tau \mathrm{e}^{\mu}\int_0^\mu \omega'_{r-1}(\mu-\tau')\frac{\mathrm{d}\tau'}{\sqrt{\tau'}} \cdot \mathrm{d}\mu, \qquad r>1,$$

and

$$\omega_1=\mathrm{e}^{-\tau}\int_0^\tau\int_0^\mu \mathrm{e}^{\tau'}\frac{\mathrm{d}\tau'}{\sqrt{\tau'}}\mathrm{d}\mu.$$

For large elapsed times, $\tau\gg1$, ω_1 is at most, of the order $\tau^{-\frac{1}{2}}$, and from the differential equation it is seen that the ω_r is then of order $\tau^{-(r+1)/2}$, at most. Thus

$$\omega=\lambda\tau^{-\frac{1}{2}}+\mathrm{O}(\lambda^2\tau^{-1},\ \tau^{-3/2}), \qquad \tau\gg1.$$

The solution for ω has been carried out numerically for various values of λ. The computations were assisted by transforming the variables as follows

$$\tau=\sigma^2, \qquad \tau'=\sigma'^2,$$

and adjusting the degree of the singularity of the integrand by writing

$$\omega'(\sigma)=-\frac{2\sigma}{(1+\pi\lambda\sigma)}\left\{\omega(\sigma)+\lambda\left[\int_0^\sigma\frac{\{\omega'(\sigma')-\omega'(\sigma)\}}{\sqrt{(\sigma^2-\sigma'^2)}} \cdot \mathrm{d}\sigma'-2F(\sigma)\right]\right\},$$

where

$$F(\sigma)=\exp(-\sigma^2)\int_0^\sigma \exp(\sigma'^2)\mathrm{d}\sigma',$$

and $\omega'(\sigma)=\mathrm{d}\omega/\mathrm{d}\tau$, etc.

The numerical integration was carried out on the C.S.I.R.O. digital computer using a process similar to that due to Huen, and using three stages of approximation for each interval h of σ ($h=1/50$). As the values of ω and ω' were found at the end of each interval they were recorded and ω' stored for use in the computation of all succeeding values. The computation of each value proceeded in three stages. Thus, $\omega'^{(1)}_N$ was found by extrapolation

$$\omega'^{(1)}_N = 2\omega'_{N-1} - \omega'_{N-2},$$

whence by integration,

$$\omega^{(1)}_N = \omega_{N-1} + \tfrac{1}{2}h(\omega'_{N-1} + \omega'^{(1)}_N),$$

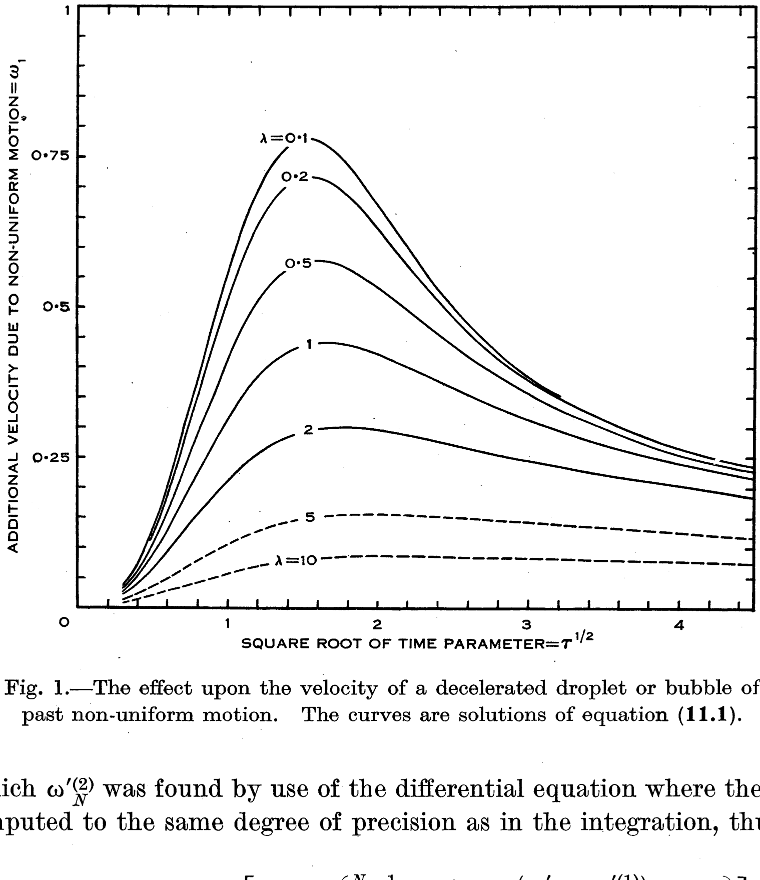

Fig. 1.—The effect upon the velocity of a decelerated droplet or bubble of past non-uniform motion. The curves are solutions of equation (**11.1**).

from which $\omega'^{(2)}_N$ was found by use of the differential equation where the integral was computed to the same degree of precision as in the integration, thus

$$\omega'^{(2)}_N = -\frac{2Nh}{(1+\pi\lambda Nh)}\left[\omega^{(1)}_N + \lambda\left\{\sum_{n=0}^{N-1}\frac{(1+\delta_{0N})}{2}\,\frac{(\omega'_N - \omega'^{(1)}_N)}{\sqrt{(N^2-n^2)}} - 2F_N\right\}\right],$$

in which F_N, taken for values at the interval h, were supplied from punched tape, and δ_{0N} is the Kronecker δ-function. Using $\omega'^{(2)}_N$ a new value of ω_N was found by integration, from which the final value of ω'_N was obtained

$$\omega'_N = \omega'^{(3)}_N = -\frac{2Nh}{(1+\lambda\pi Nh)}\cdot\left[\overline{\omega^{(2)}_N - \omega^{(1)}_N} + \tfrac{1}{2}\pi\lambda\overline{\omega'^{(1)}_N - \omega'^{(2)}_N}\right].$$

A final integration over the interval gave ω_N, truncation errors being of order h^{-3} only. The distance travelled, s_N, at the end of the Nth interval was obtained by integration following the evaluation of ω_N, thus

$$s_N = s_{N-1} + (\tfrac{1}{2}h)^2(2N-1)(\omega_{N-1}+\omega_N).$$

Figure 1 shows the ratio ω/λ, for values of λ exceeding the range of the physically important cases. The curves are plotted against the abscissa σ which is proportional to the square root of the elapsed time.

TABLE 1

THE MAXIMUM SHOWN EFFECT OF PAST HISTORY UPON THE VELOCITY $\omega_{max.}$ OF A DROPLET OR BUBBLE AT ELAPSED TIME τ

λ	$\omega_{max.}$	τ
0·1	0·0780	1·233
0·2	0·1433	1·240
0·5	0·2893	1·259
1·0	0·4406	1·287
2·0	0·6011	1·327
5·0	0·7705	1·389
10·0	0·8692	1·439

Table 1 gives the values of the maximum change in velocity due to the history of the motions. In particular it is seen that the difference may be as much as 0·5 for the maximum physically admissible value of λ. This occurs at an elapsed time $\tau=1\cdot30$.

VI. THE DISTANCE TRAVELLED

In the absence of effects due to history, the distance travelled would be unity. The excess distance travelled, s, is of order $\tau^{5/2}$ for small τ, but for large τ, according to equation (**6**) of order $\tau^{\frac{1}{2}}$. Such a behaviour for large elapsed times does not occur in practice. The actual distance travelled before coming to rest is finite. The theoretical result is due to the neglect of terms of order R^2 in the calculation of the drag.

If allowance be made for terms of this order the integrand of the equation of motion for large τ' would, from equation (**8.2**), become of order $\tau'^{-3/2}R^{-2}$ instead of $\tau'^{-\frac{1}{2}}$. Figure 2 shows the excess distance travelled as the ratio s/λ, on the basis of equation (**6**) for the drag force. The steadily increasing behaviour of s for large elapsed times is evident.

To allow for the inclusion of the drag force for large elapsed times, the integrand of equation (**11**) must be multiplied by a factor $25\pi\lambda^2/2R^2\tau$. Thus ω would decrease as $\tau^{-3/2}$ and s would vary as $\tau^{-\frac{1}{2}}$ and tend to a constant limit. However, the solution obtained here may be considered as a reasonable approximation for $\tau \leqslant 16/R^2$, where R is the Reynolds number of the droplet or bubble at its initial velocity.

A rough estimate of the total distance travelled may be obtained by assuming the motion to be controlled by the weighting factor $\tau^{-\frac{1}{2}}$ up to the time at which

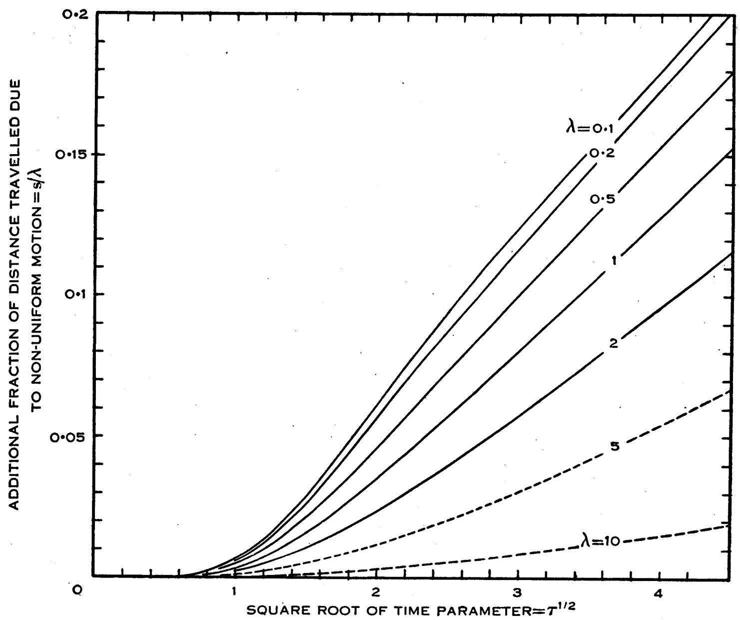

Fig. 2.—The additional distance travelled by a decelerated droplet or bubble due to the effect of non-uniform motion. The curves are solutions of equations (**11.1**) and (**11.2**).

the additional factor entering into the integrand is of the order unity, at which time the motion ceases. This time $\tau_{\text{max.}}$ may be taken as

$$\tau_{\text{max.}} \simeq 36\lambda^2/R^2.$$

TABLE 2

APPROXIMATE MAXIMUM EXCESS DISTANCE $s_{\text{max.}}$ TRAVELLED FOR A DROPLET OR BUBBLE

λ	$s_{\text{max.}}/\lambda$
0·1	$-0{\cdot}0411+0{\cdot}0332/R$*
0·2	$-0{\cdot}0533+0{\cdot}0683/R$
0·5	$-0{\cdot}0597+0{\cdot}160/R$
1·0	$-0{\cdot}0631+0{\cdot}287/R$
2·0	$-0{\cdot}0540+0{\cdot}447/R$

* R=Reynolds number.

Table 2 gives the approximate values of the maximum excess distance assuming the curves of Figure 2 to apply up to $\tau_{\text{max.}}$.

VII. Conclusion

The effect of the finite rate of the diffusion of vorticity in the case of accelerated motion of small droplets and bubbles can be an important factor in determining at any instant their velocity and position. Although the effect is small for liquid droplets immersed within a gas, it is important in the case of vapour bubbles in a liquid or for two immiscible liquids.

Any attempt to carry out laboratory experiments to simulate the case of small droplets or bubbles must preserve a strict dynamical similitude which extends to maintaining the correct value of the parameter λ under the conditions of the experiment.

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