CONDUCTION OF HEAT IN AN INFINITE REGION BOUNDED INTERNALLY BY A CIRCULAR CYLINDER OF A PERFECT CONDUCTOR

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Summary

Numerical information is given for radial flow of heat in an infinite region bounded internally by a circular cylinder of radius a containing a perfect conductor, there being contact resistance $1/H$ per unit area across the cylinder. Problems considered are: (i) the perfect conductor and the surrounding region initially at different temperatures; (ii) heat supply to the perfect conductor. Results are expressed in terms of the dimensionless parameters $\tau=\kappa t/a^2$, $h=K/aH$, $\alpha=2\pi a^2\rho c/S$, where K, ρ, c, κ are the thermal conductivity, density, specific heat, and diffusivity in the region outside the cylinder, S is the thermal capacity of the perfect conductor per unit length of the cylinder, and t is the time. Tables and graphs of the temperature of the perfect conductor are given for $0\cdot 2<\tau<20$; $h=0$, $0\cdot 5$, 1, 2, 3, 4, 5, 7, 10, 20; and $\alpha=0\cdot 5$, 1, $1\cdot 5$, 2. The temperature outside the cylinder, a problem involving fluid motion within the cylinder, and the heating of a buried cable carrying electric current are also discussed.

I. INTRODUCTION

Radial flow of heat in the infinite region surrounding an infinite circular cylinder which contains a perfect conductor or well-stirred fluid has been discussed at some length as an example of an important boundary condition in conduction of heat (Jaeger 1940). Problems of this type are of great importance since they provide good approximations to many important practical problems such as the heating of a buried cable, while the transient heating of a solid by a wire carrying current has frequently been proposed as a method for measuring thermal conductivity (Fischer 1939; van der Held, Hardebol, and Kalshoven 1953). A better representation of experimental conditions is frequently obtained by adding the additional complication of a contact resistance between the perfect conductor and the surrounding solid.

The solutions of all the problems considered below may readily be obtained by the method of the Laplace transformation (cf. e.g. Carslaw and Jaeger 1947). Unfortunately, rather complicated integrals are involved which have to be evaluated numerically. Approximations useful for large or small values of the time may also always be found by the usual methods, and, because of the absence of accurate numerical information, there has been a tendency to use these in the reduction of experimental results. Physically, they frequently correspond to replacing the problem by the simpler one of a line source in an infinite medium.

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In some cases this is satisfactory, but the parameters involved (in particular the times concerned) are often such as to make them quite inapplicable.

Two particular problems occur in many contexts. The first of these is that in which the perfect conductor and the surrounding material are initially at different temperatures. This has been discussed (for the case of no contact resistance) using an approximate method by Whitehead (1944), in connexion with the dissipation of heat from a suddenly heated electrical cable. Jaeger (1940), Tranter (1947), and Bullard (1954) have given the exact solution, and the latter has given some numerical values ; he also points out the possibility of using these results for the measurement of thermal conductivity. This problem, including the case of contact resistance, is discussed in Sections III and IV.

In the second problem, the perfect conductor and its surroundings are initially at zero temperature and the perfect conductor is subsequently heated at a constant rate. This has been discussed approximately in connexion with the heating of electric cables by Whitehead and Hutchings (1938). van der Held, Hardebol, and Kalshoven (1953) use the exact solution with no contact resistance for the measurement of thermal conductivity. Blackwell (1954) has given the solution for the case in which there is contact resistance, and uses the first two terms of its asymptotic expansion in the determination of thermal conductivity. Numerical information about this problem is given in Section V.

The application of these results to the problem of the heating of a buried cable, and the solution of a more general problem of this type, are discussed in Section VII.

The case in which the cylinder contains well-stirred fluid and heat is supplied to it by exchange of fluid has been studied by Jaeger (1940). Some numerical results are given in Section VI.

In most cases numerical results are given only for the surface temperature of the solid. Values for the temperature in the interior of the solid may be obtained from these results by the use of Duhamel's theorem (Carslaw and Jaeger 1947, Section 10) and tables given by Jaeger (1956).

The calculations given here were made in connexion with the measurement of thermal conductivity of rocks from boreholes. This application is discussed in a companion paper (Beck, Jaeger, and Newstead 1956).

II. Notation

In all cases the region outside the circular cylinder $r=a$ will be supposed to contain solid of density ρ, specific heat c, thermal conductivity K, and diffusivity $\kappa=K/\rho c$; v will be written for the temperature in this material at distance r from the axis at time t, the material being always supposed to be initially at zero temperature ; v_s will be written for the surface temperature of this region, that is, the value of v when $r\to a+0$.

The cylinder $r=a$ is supposed to contain a perfect conductor (or well-stirred fluid), its heat capacity per unit length of the cylinder being S. V will be written for its temperature. In some problems heat will be supplied to the perfect conductor at the rate Q per unit time per unit length.

At the surface $r=a$ it will be assumed that there is a thermal contact resistance $1/H$ per unit area between the perfect conductor and the surrounding solid, so that the flux of heat outwards across this surface is

$$H(V-v_s). \quad \ldots\ldots\ldots\ldots \quad (1)$$

The case $H\rightarrow\infty$ gives perfect thermal contact at the surface.

The theoretical formulae may be expressed in terms of three dimensionless parameters, namely,

$$\tau=\kappa t/a^2, \quad \ldots\ldots\ldots\ldots \quad (2)$$

$$\alpha=2\pi a^2\rho c/S, \quad \ldots\ldots\ldots\ldots \quad (3)$$

$$h=K/aH. \quad \ldots\ldots\ldots\ldots \quad (4)$$

The numerical values of these likely to arise in practice will now be discussed. When units are involved they will be c.g.s., calorie, and °C. Only poor conductors need be considered, since if the region $r>a$ contains a good conductor the assumption that $r<a$ contains a perfect conductor will cease to be a reasonable approximation to any likely physical situation.

For an experiment of 10 min duration in a hole of radius $a=2$ cm in material of diffusivity $\kappa=0\cdot01$, (2) gives $\tau=1\cdot5$ so that in such experiments the range $0\cdot2$–10 of τ may be regarded as important. If, however, the radius is reduced to 2 mm, the values of τ become large and simple approximate formulae are available.

The parameter α is twice the ratio of the heat capacity of a cylinder of the solid of radius a and unit length to that of unit length of the cylinder of perfect conductor. If the perfect conductor is well-stirred water and it is surrounded by rock, α is about 1; for solid brass in rock, α is about $1\cdot5$; for the composite probe used by Bullard (1954), $\alpha=2$; for brass in a lighter material such as wood, α falls to $0\cdot3$. If a tube of perfect conductor is used instead of a solid cylinder higher values of α occur. It may be noted here that if $\alpha=2$ the asymptotic expansion for large values of the time takes a slightly simpler form so that there is some reason for approximating to this case when designing apparatus.

The likely values of h may be estimated by considering the case of an air gap 1 mm thick and of conductivity $0\cdot000053$ in a hole of radius $a=2$ cm in rock of conductivity $K=0\cdot0053$. This gives $h=5$. Thus, for a slightly irregular hole in dry rock, values of h at least as large as this must be allowed for. On the other hand, if the hole is filled with well-stirred fluid or there is really good contact, as in the case of Bullard's experiments, it is reasonable to neglect contact resistance and to take $h=0$.

III. The Case of Different Initial Temperatures

Suppose the region $r<a$ to be initially at temperature V_0 and the region $r>a$ at zero temperature, no heat being supplied to the system.

Then the temperature V for $r<a$ at time t is given by

$$V/V_0=F(h,\ \alpha,\ \tau), \quad \ldots\ldots\ldots\ldots \quad (5)$$

where

$$F(h, \alpha, \tau) = \frac{4\alpha}{\pi^2}\int_0^\infty \frac{\exp(-\tau u^2)du}{u\Delta(u)}, \qquad \text{(6)}$$

$$\Delta(u) = [uJ_0(u) - (\alpha - hu^2)J_1(u)]^2 + [uY_0(u) - (\alpha - hu^2)Y_1(u)]^2, \qquad \text{(7)}$$

τ, α, h are defined in (2)–(4), and $J_n(u)$ and $Y_n(u)$ are the Bessel functions of order n of the first and second kinds.

This result is easily obtained by the Laplace transformation method as in Jaeger (1940). For the important case of perfect thermal contact, $h=0$, it reduces to one given therein and also by Bullard* (1954). Values for the case $h=0$ are given in Table 1 for various values of τ and α.

TABLE 1

VALUES OF $F(0, \alpha, \tau)$

τ \ α	0·5	1·0	1·5	2·0	4·0	6·0	8·0
0·2	0·757	0·595	0·482	0·401	0·229	0·156	0·118
0·3	0·707	0·529	0·414	0·335	0·182	0·122	0·091
0·4	0·668	0·480	0·365	0·290	0·152	0·101	0·075
0·5	0·634	0·440	0·328	0·257	0·132	0·087	0·064
0·6	0·604	0·408	0·298	0·231	0·116	0·076	0·056
0·7	0·578	0·380	0·274	0·210	0·104	0·068	0·050
0·8	0·555	0·357	0·253	0·193	0·095	0·062	0·046
0·9	0·534	0·336	0·236	0·179	0·087	0·057	0·042
1·0	0·514	0·317	0·221	0·166	0·080	0·052	0·039
2·0	0·381	0·206	0·135	0·099	0·047	0·030	0·022
3·0	0·302	0·152	0·098	0·071	0·033	0·022	0·016
4·0	0·250	0·120	0·076	0·055	0·026	0·017	0·012
5·0	0·212	0·099	0·062	0·045	0·021	0·014	0·010
6·0	0·184	0·084	0·053	0·038	0·018	0·012	0·009
7·0	0·162	0·073	0·046	0·033	0·016	0·010	0·008
8·0	0·144	0·064	0·040	0·029	0·014	0·009	0·007
9·0	0·129	0·057	0·036	0·026	0·012	0·008	0·006
10·0	0·117	0·052	0·033	0·024	0·011	0·007	0·006
15·0	0·079	0·035	0·022	0·016	0·008	0·005	0·004
20·0	0·059	0·026	0·017	0·012	0·006	0·004	0·003

For the case in which there is contact resistance, three parameters are involved and values for the most important cases $\alpha=1$ and $\alpha=2$ are shown in Figures 1 and 2. These values are sufficient for many practical purposes; for other values of α curves can be constructed by interpolation. The plot of log F against log τ was introduced by Bullard and is most convenient, both for display and practical applications.

* The notation is chosen to correspond with Bullard's; $F(0, \alpha, \tau)$ is his $F(\alpha, \tau)$. He gives values of this function to 4 D for $\alpha=2$, 4, 6, 8, and $\tau=0\cdot2(0\cdot2)3(1)10$.

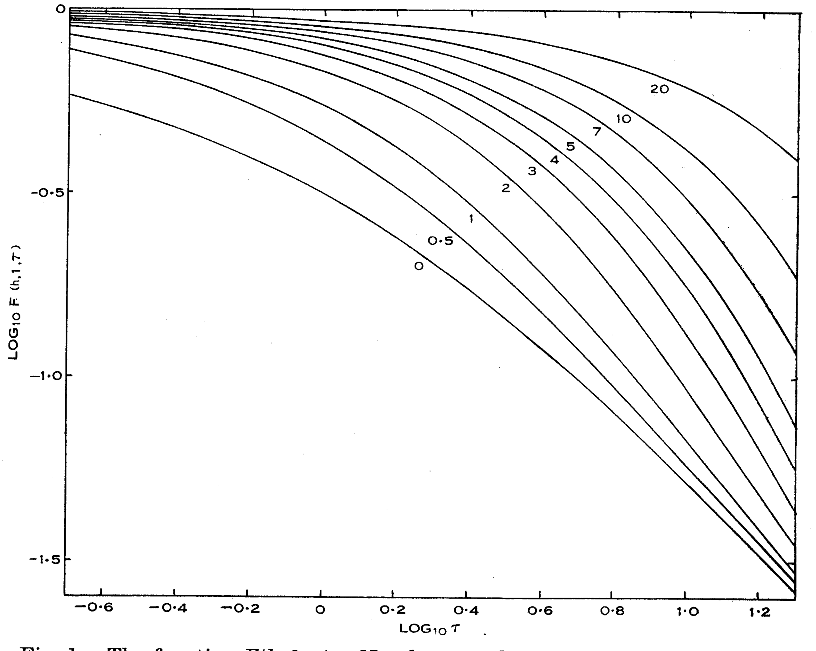

Fig. 1.—The function $F(h, 1, \tau)$. Numbers on the curves are the values of h.

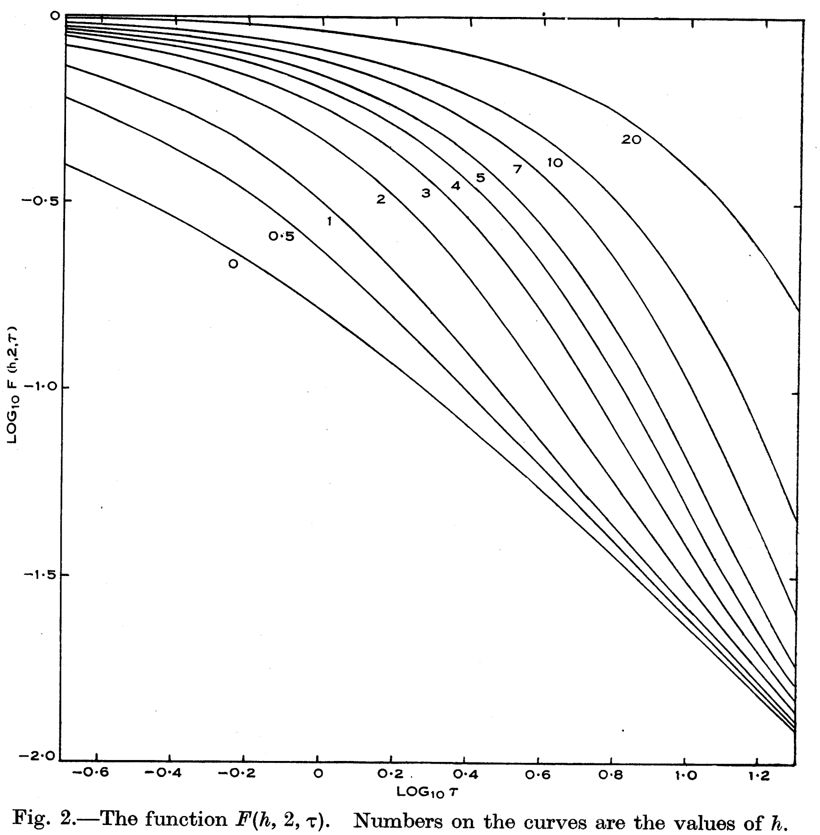

Fig. 2.—The function $F(h, 2, \tau)$. Numbers on the curves are the values of h.

The values of τ in Table 1 and Figures 1 and 2 are those in the region which was shown in Section II to be important for many practical applications. For these values of τ the integral (6) must be evaluated numerically. For larger or smaller values, approximations are available.

If τ is small it may be shown as in Carslaw and Jaeger (1947) that

$$F(h,\ \alpha,\ \tau)=1-(\alpha\tau/h)+O(\tau^{3/2}), \qquad (8)$$

except for $h=0$, in which case it becomes

$$F(h,\ \alpha,\ \tau)=1-\frac{2\alpha}{\pi^{\frac{1}{2}}}\tau^{\frac{1}{2}}+\alpha(\alpha-\tfrac{1}{2})\tau+O(\tau^{3/2}). \qquad (9)$$

For large values of τ, as in Blackwell (1954),

$$F(h,\ \alpha,\ \tau)=\frac{1}{2\alpha\tau}+\frac{(4h-\alpha)}{4\alpha^2\tau^2}-\frac{(\alpha-2)}{4\alpha^2\tau^2}\left(\ln\frac{4\tau}{C}-1\right)+O(\tau^{-3}\ln\tau), \qquad (10)$$

where

$$C=1\cdot7811\ldots=\mathrm{e}^{\gamma}, \qquad (11)$$

and

$$\gamma=0\cdot5772\ldots \text{ is Euler's constant.}$$

IV. The Temperature in the Solid

In Section III only the temperature in the perfect conductor was discussed. The temperature in the solid involves yet another parameter, r/a, except for its limiting value v_s, as $r\to a$, which is given by

$$\frac{v_s}{V_0}=F(h,\ \alpha,\ \tau)+\frac{h}{\alpha}\ \frac{\mathrm{d}F(h,\ \alpha,\ \tau)}{\mathrm{d}\tau}. \qquad (12)$$

As remarked in Section I, values of v for $r>a$ may be found from v_s by using Duhamel's theorem and tabulated values for the temperature in the region $r>a$ with its surface maintained at unit temperature.

For the case of perfect thermal contact, $h=0$, only three parameters are involved, and in this case the temperature v for $r>a$ is given by

$$v/V_0=\Phi(\alpha,\ \tau,\ r/a), \qquad (13)$$

where

$$\Phi(\alpha,\ \tau,\ r/a)=\frac{2}{\pi}\int_0^\infty\frac{\exp(-\tau u^2)\{\mathrm{J}_0(ru/a)[u\mathrm{Y}_0(u)-\alpha\mathrm{Y}_1(u)]-\mathrm{Y}_0(ru/a)[u\mathrm{J}_0(u)-\alpha\mathrm{J}_1(u)]\}\mathrm{d}u}{[u\mathrm{J}_0(u)-\alpha\mathrm{J}_1(u)]^2+[u\mathrm{Y}_0(u)-\alpha\mathrm{Y}_1(u)]^2}. \qquad (14)$$

Some values of $\Phi(\alpha,\ \tau,\ r/a)$ for the most interesting case $\alpha=2$ are shown in Figure 3.

The value of v given by (13) may be regarded as the temperature at r due to the liberation of an amount of heat $Q'=SV_0$ per unit length over the cylindrical surface $r=a$. This is the Green's function of the problem, and, by the symmetry of the Green's function or by a direct calculation as in Jaeger (1940), it follows that the temperature V in $r<a$ due to the liberation of an amount of heat Q' per unit length over the cylindrical surface $r=r'$ is

$$V=\frac{Q'}{S}\Phi(\alpha,\ \tau,\ r'/a). \qquad (15)$$

In view of the complicated nature of the result (**14**) it is of some interest to remark that for the case $\alpha=2$ the values of (**15**) taken from Figure 3 differ very little (except for $r/a=1\cdot5$) from those for the cylindrical surface source (Carslaw and Jaeger 1947, Section 103) in an infinite medium, namely,

$$\frac{Q'}{4\pi Kt}\exp\{-(r'^2+a^2)/4\varkappa t\}I_0\left(\frac{ra}{2\varkappa t}\right). \qquad (16)$$

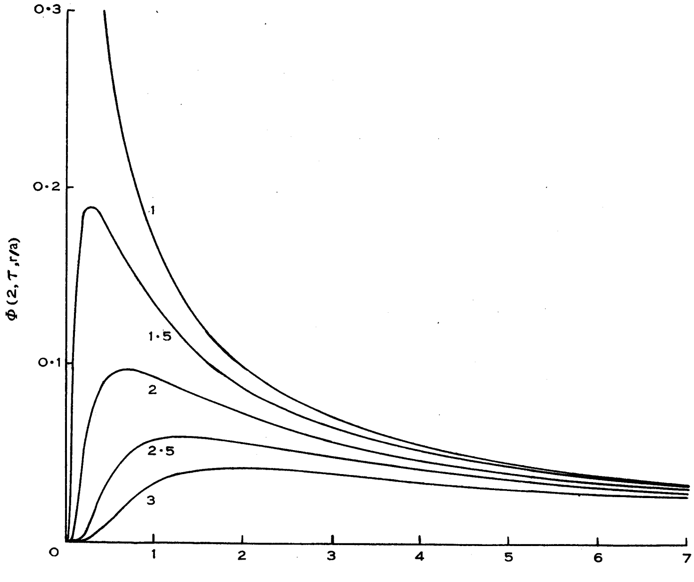

Fig. 3.—Values of $\Phi(2, \tau, r/a)$. The numbers on the curves are the values of r/a.

V. Constant Supply of Heat in the Perfect Conductor

In this case it is assumed that both the regions $r<a$ and $r>a$ are initially at zero temperature and that heat is supplied at the constant rate Q per unit time per unit length to the perfect conductor. The temperature V of the perfect conductor is given by

$$KV/Q=G(h,\ \alpha,\ \tau), \qquad (17)$$

where

$$G(h,\ \alpha,\ \tau)=\frac{2\alpha^2}{\pi^3}\int_0^\infty\frac{\{1-\exp(-\tau u^2)\}du}{u^3\Delta(u)}, \qquad (18)$$

and $\Delta(u)$ is given by (**7**).

The quantity

$$SV/Qt=(2\pi/\alpha\tau)G(h, \alpha, \tau) \quad \text{.............. (19)}$$

is the ratio of the actual temperature to that which would be attained in the absence of loss by conduction and is useful in the reduction of experimental data.

The temperature for $r>a$ may be found as before. The most interesting quantity is the temperature drop over the contact resistance at $r=a$; this is given by

$$\frac{Qh}{2K\pi}\{1-F(h, \alpha, \tau)\}, \quad \text{................ (20)}$$

where $F(h, \alpha, \tau)$ is defined by **(6)** and was discussed in Section III.

TABLE 2

VALUES OF $G(0, \alpha, \tau)$

τ \ α	0·5	1·0	1·5	2·0	4·0	6·0	8·0	∞
0·2	0·013	0·023	0·030	0·035	0·047	0·052	0·056	0·068
0·3	0·019	0·032	0·040	0·046	0·060	0·066	0·069	0·080
0·4	0·025	0·040	0·050	0·056	0·070	0·076	0·079	0·090
0·5	0·030	0·047	0·058	0·065	0·079	0·085	0·088	0·098
0·6	0·035	0·054	0·065	0·073	0·087	0·093	0·096	0·105
0·7	0·039	0·060	0·072	0·080	0·094	0·100	0·102	0·112
0·8	0·044	0·066	0·078	0·086	0·100	0·106	0·109	0·118
0·9	0·048	0·071	0·084	0·092	0·106	0·112	0·115	0·123
1·0	0·052	0·077	0·090	0·098	0·112	0·117	0·119	0·128
2·0	0·087	0·117	0·130	0·138	0·150	0·154	0·156	0·162
3·0	0·114	0·145	0·158	0·165	0·175	0·178	0·180	0·185
4·0	0·136	0·166	0·178	0·184	0·194	0·197	0·198	0·202
5·0	0·155	0·184	0·195	0·200	0·209	0·211	0·213	0·216
6·0	0·170	0·198	0·208	0·214	0·221	0·224	0·225	0·228
7·0	0·184	0·211	0·220	0·225	0·232	0·234	0·235	0·238
8·0	0·196	0·222	0·230	0·235	0·241	0·243	0·244	0·247
9·0	0·207	0·231	0·240	0·244	0·250	0·252	0·252	0·255
10·0	0·217	0·240	0·248	0·252	0·257	0·259	0·260	0·262
15·0	0·255	0·274	0·280	0·283	0·287	0·288	0·289	0·290
20·0	0·282	0·297	0·302	0·305	0·308	0·309	0·310	0·311

For the case of perfect thermal contact at $r=a$, $h=0$,values of $G(0, \alpha, \tau)$ are given in Table 2. It may be noted that the case $\alpha \to \infty$ is that of prescribed flux Q at $r=a$ into the region $r>a$ which is treated in Carslaw and Jaeger (1947, Section 127).

For the general case, values of $G(h, \alpha, \tau)$ are shown in Figures 4 and 5 for $\alpha=1$ and $\alpha=2$, respectively, G being plotted against $\log \tau$.

If τ is small,

$$G(h, \alpha, \tau)=\frac{\alpha}{2\pi}\left\{\tau-\frac{\alpha\tau^2}{2h}+O(\tau^{5/2})\right\}, \quad \text{............ (21)}$$

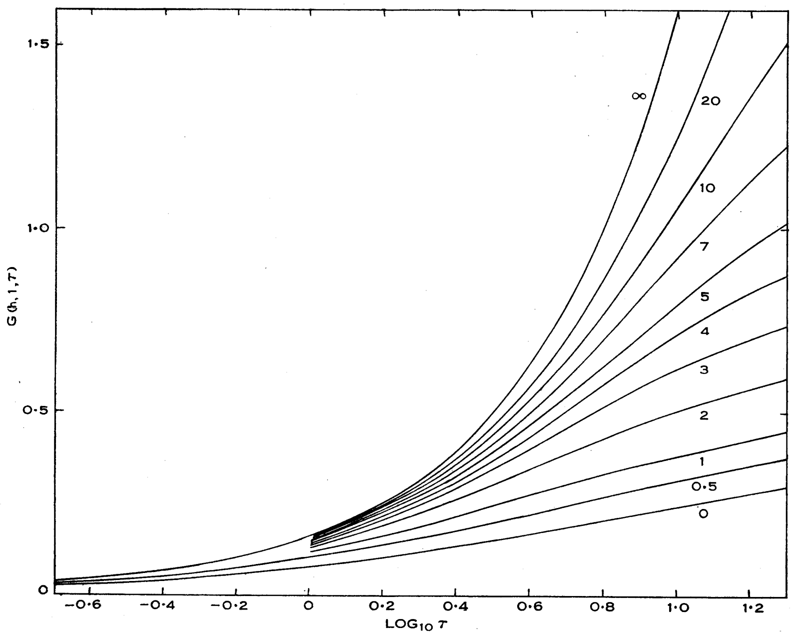

Fig. 4.—The function $G(h, 1, \tau)$. The numbers on the curves are the values of h.

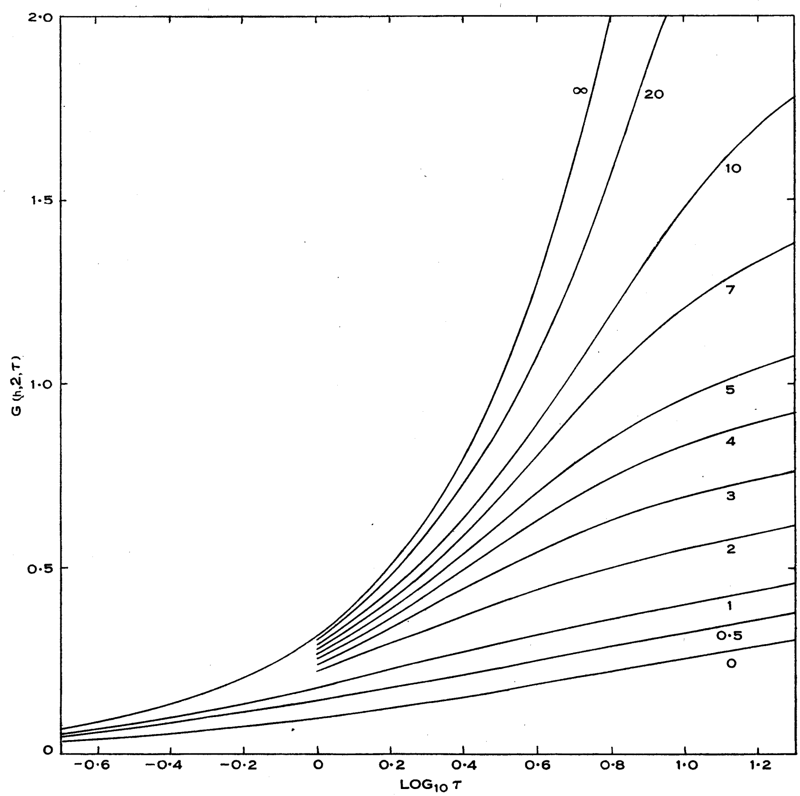

Fig. 5.—The function $G(h, 2, \tau)$. The numbers on the curves are the values of h.

if $h \neq 0$, while if $h=0$,

$$G(0, \alpha, \tau)=\frac{\alpha}{2\pi}\left\{\tau-\frac{4\alpha}{3\pi^{\frac{1}{2}}}\tau^{3/2}+O(\tau^2)\right\}. \quad \ldots\ldots\ldots\ldots (22)$$

For large values of τ,

$$G(h, \alpha, \tau)=\frac{1}{4\pi}\left\{2h+\ln\frac{4\tau}{C}-\frac{(4h-\alpha)}{2\alpha\tau}+\frac{\alpha-2}{2\alpha\tau}\ln\frac{4\tau}{C}+\ldots\right\}, \quad \ldots (23)$$

where C is defined in (**11**). The fact that for large values of τ, G varies linearly with $\ln\tau$ has been much used for the determination of thermal conductivity. It appears from Figures 4 and 5 that for $h>1$ the curve of G against $\ln\tau$ has an inflexion between $\tau=2$ and $\tau=10$ so that, if an experimental curve terminates in this region, this might be regarded as the asymptote (**23**) and a completely wrong value of K found.

It may be remarked that the functions $F(h, \alpha, \tau)$ and $G(h, \alpha, \tau)$ are connected by the relation

$$\frac{d}{d\tau}G(h, \alpha, \tau)=\frac{\alpha}{2\pi}F(h, \alpha, \tau). \quad \ldots\ldots\ldots\ldots (24)$$

The following limiting relations are also useful.

$$G(\infty, \alpha, \tau)=\alpha\tau/2\pi, \quad \ldots\ldots\ldots\ldots (25)$$

$$G(h, \infty, \tau)=h/2\pi+G(0, \infty, \tau). \quad \ldots\ldots\ldots\ldots (26)$$

VI. The Case of Moving Fluid

Another problem which may be solved in the same way is that in which the region $r<a$ contains mass M_1 per unit length of well-stirred fluid of specific heat c_1. Mass m_1 of fluid is removed from the region per unit time per unit length and replaced by fluid at temperature V_0. If both the fluid in the region $r<a$ and the solid in the region $r>a$ are initially at zero temperature, and heat is supplied in the region $r<a$ at the rate Q per unit time per unit length, the temperature V in this region at time t is given by

$$V=\left(V_0+\frac{Q}{m_1c_1}\right)H(\mu, \alpha, \tau), \quad \ldots\ldots\ldots\ldots (27)$$

where

$$H(\mu, \alpha, \tau)=1-\frac{4\alpha\mu}{\pi^2}\int_0^\infty\frac{\exp(-\tau u^2)du}{u\Delta(\mu, \alpha, u)}, \quad \ldots\ldots\ldots (28)$$

$$\Delta(\mu, \alpha, u)=[(\mu-u^2)J_0(u)+\alpha uJ_1(u)]^2+[(\mu-u^2)Y_0(u)+\alpha uY_1(u)]^2, \ldots (29)$$

μ is the dimensionless parameter

$$\mu=m_1a^2/\kappa M_1, \quad \ldots\ldots\ldots\ldots (30)$$

and τ and α are defined in (**2**) and (**3**).

Some values of $H(\mu, \alpha, \tau)$ for the case $\alpha=1$ are shown in Figure 6. This theory has been used for the measurement of water movement in or across boreholes; an account of the method will be given elsewhere.

For large values of the time a first approximation to the value of $H(\mu, \alpha, \tau)$ is

$$1-\frac{4\alpha}{\mu\pi^2}I(0, 1, \tau), \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{31})$$

where $I(0, 1, \tau)$ is the function specifying the flux into the region $r>a$ for constant surface temperature which has been tabulated by Jaeger and Clarke (1942).

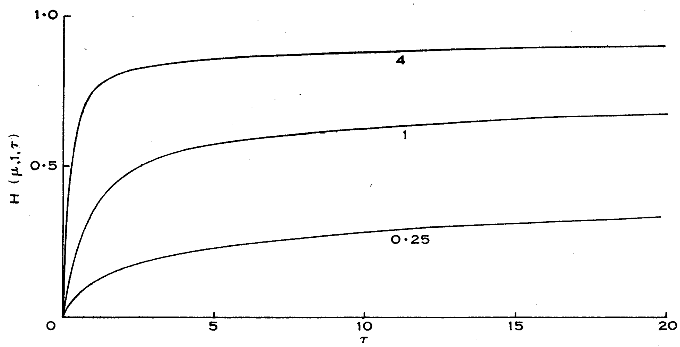

Fig. 6.—The function $H(\mu, 1, \tau)$. The numbers on the curves are the values of μ.

VII. The Heating of Buried Electric Cable

The fundamental situation in the heating of buried cable is stated by Whitehead and Hutchings (1938) as follows: the current-carrying core of the cable consists of a perfect conductor of thermal capacity S_1 per unit length; this is separated by insulating material from a protecting sheath which consists of perfect conductor of thermal capacity S_2 per unit length; the external radius of the sheath is a and it is surrounded by earth of thermal constants K, ρ, c, $\varkappa$, there being no contact resistance between the sheath and the surrounding earth. The thermal capacity of the cable insulation is neglected, and the thermal resistance between the core and the sheath will be written in the form $1/(2\pi aH)$ per unit length to conform with the notation used earlier.

The essential problem is that of heat supply in the core at the rate Q per unit time per unit length starting at time $t=0$. The most important practical cases are: (i) when a peak load is suddenly added to steady conditions due to a base load, in which case the time of heating is of the order of an hour, and (ii) the case of a short circuit in which the core is heated almost instantaneously and it is required to find the way in which this heat is dissipated.

Whitehead and Hutchings (1938) discuss the first of these problems, making a number of approximations so that, while their result is valid for very large values of the time, it is quite impossible to estimate its error at moderate times. In connexion with the second problem, Whitehead (1944) gives an approximate solution of the problem of Section III for $h=0$, and Tranter (1947) has pointed out that it is more satisfactory to evaluate the integrals directly than to use such approximations.

The work of the preceding sections applies immediately to the case of the unsheathed cable, $S_2=0$, which, however, is not of any great engineering importance. It also applies to the case in which the thermal capacity of the core is negligible, so that, if heat is supplied at the rate Q per unit time per unit length to the core, it may be regarded as being supplied at the same rate to the sheath through the thermal resistance $1/(2\pi aH)$. The temperature V_2 in the sheath is then given by

$$V_2=\frac{Q}{K}G(0,\ \alpha_2,\ \tau), \qquad \text{(32)}$$

where

$$\alpha_2=2\pi a^2\rho c/S_2, \qquad \text{(33)}$$

and the temperature V_1 of the core is given by

$$V_1=V_2+\frac{Q}{2\pi aH}. \qquad \text{(34)}$$

The assumption leading to (**32**) is that made by Whitehead and Hutchings (1938) in their Appendix III, so that Table 2 provides exact values for their quantity β.

Accurate expressions for the complete system are easily obtained. If the core is initially at temperature V_0 and the rest of the system at zero temperature, no heat being supplied to the core, its subsequent temperature V_1 is given by

$$\frac{V_1}{V_0}=\frac{4\alpha_1\alpha_2{}^2}{\pi^2}\int_0^\infty \frac{\exp\,(-u^2\tau)\mathrm{d}u}{u\Delta_1(u)}, \qquad \text{(35)}$$

where h and τ are defined in (**4**) and (**2**),

$$\alpha_1=\frac{2\pi a^2\rho c}{S_1}, \qquad \alpha_2=\frac{2\pi a^2\rho c}{S_2}, \qquad \text{(36)}$$

and

$$\begin{aligned}\Delta_1(u)=&[u(\alpha_1+\alpha_2-hu^2)\mathrm{J}_0(u)-\alpha_2(\alpha_1-hu^2)\mathrm{J}_1(u)]^2\\ &+[u(\alpha_1+\alpha_2-hu^2)\mathrm{Y}_0(u)-\alpha_2(\alpha_1-hu^2)\mathrm{Y}_1(u)]^2. \qquad \text{(37)}\end{aligned}$$

For the case in which the whole system is initially at zero temperature and heat is supplied at the rate Q per unit time per unit length to the core, its temperature V_1 at time t is given by

$$V_1=\frac{2Q\alpha_1{}^2\alpha_2{}^2}{K\pi^3}\int_0^\infty \frac{\{1-\exp\,(-u^2\tau)\}\mathrm{d}u}{u^3\Delta_1(u)}, \qquad \text{(38)}$$

where the symbols are those defined above. The application of these results to engineering practice will be discussed more fully elsewhere.

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IX. References

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