APPLICATION OF MATHEMATICAL THEORIES OF APPROXIMATION TO AERIAL SMOOTHING IN RADIO ASTRONOMY

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[*Manuscript received February* 21, 1956]

Summary

An aerial rarely provides a perfect image of a radio brightness distribution. If we consider an array as a filter of " spatial harmonics ", the image function is a trigonometric sum approximating the object function. An application of mathematical theories shows the influence of the length and the shape of the array on the difference between object and image. Whatever the array, the image contrasts are bounded. The results provided by various arrays of the same length may be reduced by linear transforms. Inaccuracies of measurement, especially those due to the receiver noise, add to the systematic error due to the finite length of the antenna. We may try to get a compromise between these various causes of uncertainty.

I. Introduction

The most general effect of an aerial used to survey a radio brightness distribution (that we call " object " by analogy with optics) is to provide information on the object. All properly designed aerials having the same length provide the same quantity of information, but in a way more or less convenient to use.

If we want the information to be directly available, we use a narrow beam antenna. In fact, however, its design results from considerations about its radiation pattern. If we adopt criteria about the difference between object and image, we may design quite different antennae, better fitted to the problem of radio astronomy (Arsac 1955*c*).

If we have an imperfect antenna, to get the best from it, we may wish to extract from the given image all the information implicitly contained in it. It can be shown that all the images of the same object provided by aerials of the same length may be transformed one into the other by linear transformations (Arsac 1955*b*, 1955*c*). The choice of a criterion for the object-image difference allows determination of a transformation so that the obtained image agrees with this criterion.

Interferometry, used as far as a distance L, gives the same information as an antenna of length L, but piece by piece. The combination of this information piece by piece according to a suitable process allows an image of the object to be formed. Again, the choice of a criterion for the object-image difference determines the process to be used.

The most frequently used criterion is that of the least mean square error (Bracewell and Roberts 1954 ; Arsac 1955*b*, 1955*c*). It may be slightly defective,

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for a large error is not excluded, if limited to a narrow interval. The mathematical theories of approximation of functions by trigonometric sums give us some precision on the maximum of error, in connexion with the bounds of image contrasts.

II. Particular Feature of Aerial Smoothing

The study of the observation of brightness distribution by an antenna shows that it behaves in the same way as the formation of an image by an optical instrument :

$$P(\theta)=(1/2\pi)\int_{-\pi}^{+\pi}f(u)g(\theta-u)\mathrm{d}u,$$

where $f(u)$ is the brightness distribution of the object (assuming the problem reduced to one dimension), $g(\theta)$ is the antenna gain, $P(\theta)$ the image (received power) ; u and θ are angular coordinates on a great circle of the celestial sphere ; P, f, and g are merely positive functions (Bracewell and Roberts 1954 ; Arsac 1955*b*, 1955*c*).

The interpretation of such an equation by Fourier transforms is well known (Duffieux 1946). However, it seems to us imperfectly fitted to radio astronomy. The essential characteristic of the functions P, f, g is that they are functions of an angular coordinate on a great circle of the celestial sphere ; so they are periodic with period 2π. It is only with care that we can define their Fourier transform, for it vanishes everywhere, except for an infinite set of equidistant points where it is infinite. We must introduce the theory of " distributions " (Schwartz 1950). We may resolve such a difficulty by assuming that the function f vanishes identically outside the interval $-\pi$, $+\pi$. Its transform is then

$$F(x)=\int_{-\pi}^{+\pi}f(\theta)\mathrm{e}^{-\mathrm{i}x\theta}\mathrm{d}\theta.$$

It seems more convenient to represent these functions by trigonometric series :

$$f(\theta)=\sum_{-\infty}^{+\infty}c'_p\mathrm{e}^{\mathrm{i}p\theta},$$

$$g(\theta)=a'_0+2\sum_{1}^{n}a'_p\cos p\theta.$$

$g(\theta)$ includes only a bounded number of terms owing to the finite length of the antenna (Arsac 1955*b*, 1955*c*). Without loss of generality, we can assume g even. Then $a'_{-p}=a'_p$ and

$$P(\theta)=\sum_{-n}^{+n}a'_pc'_p\mathrm{e}^{\mathrm{i}p\theta}.$$

In some cases (for instance, investigation of the Sun or some very bright radio stars) there are sources on only a limited region of the sky, so that the function f nearly vanishes outside a certain interval of length $2b_0$. We can represent f by a new series

$$f(\theta)=\sum_{-\infty}^{+\infty}c_p\mathrm{e}^{\mathrm{i}p\pi\theta/b}$$

in the interval $-b$, $+b$, outside which f vanishes. b may be any number greater than b_0. So there is an infinity of possible representations. To connect them, we have only to introduce the Fourier transform as defined above

$$F(x)=\int_{-b_0}^{+b_0} f(\theta)\mathrm{e}^{-\mathrm{i}x\theta}\mathrm{d}\theta,$$

limiting the interval of integration to the interval $-b_0$, $+b_0$, outside which $f=0$. Then

$$c_p=(1/2b)F(p\pi/2b).$$

It happens that the most general array, constituted by a line of antennae whose abscissae are integer multiples of the same length a (we call it the *fundamental length* of the array) has a gain given by

$$g(\theta)=a_0+2\sum_1^n a_p \cos\,(2p\pi a\theta/\lambda).$$

The sum is always limited to n because of the finite length of the array. Under certain assumptions (a of the order of many λ) $na=L$. If we take $a/\lambda=1/2b<1/2b_0$ the array passes the necessary harmonics to define f on $-b$, $+b$. We get then

$$P(\theta)=\sum_{-n}^{+n} a_p c_p \cos\,(p\pi\theta/b).$$

Such relations imply that:

(i) g is a trigonometric sum,
(ii) P is a trigonometric sum.

III. General Properties of Array Patterns

It does not seem that the importance of the first conclusion had been pointed out before a recent article by Simon (1955). A trigonometric sum of nth degree has these properties:

(i) it has at most $2n$ zeros,
(ii) its derivatives are bounded. By Bernstein's (1926) theorem, if $Q(\theta)$ is a trigonometric sum of nth degree and $|\,Q(\theta)\,|\leqslant M$ then

$$|\,Q^{(p)}(\theta)\,|\leqslant n^p M.$$

When arrays are concerned, g is in fact a function of $2\pi a\theta/\lambda$. Noticing that $g(\theta)\geqslant 0$ and g_0 being the maximum of g,

$$|\,g-\tfrac{1}{2}g_0\,|<\tfrac{1}{2}g_0,$$
$$|\,g^{(p)}(\theta)\,|\leqslant(2\pi L/\lambda)^p\cdot\tfrac{1}{2}g_0.$$

Especially for the small values of θ,

$$g\sim g_0-\tfrac{1}{2}\theta^2\,|\,g_0''\,|>g_0-\tfrac{1}{2}\theta^2(2\pi L/\lambda)^2\cdot\tfrac{1}{2}g_0.$$

The value $g=\frac{1}{2}g_0$ cannot be reached for values of θ smaller than $\sqrt{2}\lambda/2\pi L$. This limit is smaller than the usual value λ/L.

A comparison of the gains of any array and Dolph's array allows us to get either a limit of the width of the main lobe as a function of the level of the side lobes, or a gain limit of the array over an interval not including the centre of the main lobe. It may be proved that the gain of an array cannot remain 4 times smaller than the gain product of the extreme antennae of the array for sufficiently great values of θ. This result confirms the optical theorem showing that diffracted energy in the far part of a diffraction figure is chiefly determined by brightness on the pupil edge (Dossier 1954).

IV. Image Qualities

The second conclusion we pointed out seems still more important in radio astronomy. $P(\theta)$ is not any function, but a trigonometric sum. Experience substitutes for the unknown function $f(\theta)$ the polynomial P.

If we want image P to reproduce object f, $\mathrm{P}(\theta)$ *must approximate* $\mathrm{f}(\theta)$. We are led to the problem of the approximation of a function by a trigonometric sum.

Let us sum up the results of mathematical analysis (Jackson 1930).

Let $f(\theta)$ be a periodic function of period 2π: we define its Fourier components by

$$c_p=(1/2\pi)\int_{-\pi}^{+\pi} f(\theta)e^{-ip\theta}d\theta.$$

We can get polynomials approximating f, i.e. such as for all values of θ, $|f-P_n|\leqslant r_n$, r_n being the realized approximation. They are built from Fourier components of f by a summatory process (Hardy and Rogosinski 1938)

$$P_n^m=\sum_{p=-n}^{p=+n} m_n^p c_p e^{ip\theta}.$$

The values m_n^p are most frequently that of a function $m(t)$ for $t=p/n+1$; $m(t)$ vanishes if $|t|>1$. $m(t)$ is the summatory function. In very general conditions $P_n^m \to f$ if $n\to\infty$ ($r_n\to 0$ if $n\to\infty$).

n being fixed, the choice of the summatory function allows us to modify the possible values of r_n. Among all the processes $m(t)$, there is one and only one minimizing r_n. The corresponding polynomial P_n, called " of best approximation ", is such that the difference P_n-f takes, over the interval $-\pi$ to π (one of the limits being excluded), $n+2$ extreme values of the same size and opposite sign. If $f(\theta)$ has a bounded derivative of pth order $|f^{(p)}(\theta)|\leqslant M_p$, the minimum of r_n, that we write $E_n(f)$ and call the best approximation of nth order of f, is such that

$$E_n(f)<C_p(M_p/n^p).$$

The first values of C_p are:

$$C_1=\frac{\pi}{2},\ C_2=\frac{\pi^2}{8},\ C_3=\frac{\pi^3}{24},\ \text{etc.}$$

and

$$C_2<C_4<\ldots<4/\pi<\ldots<C_3<C_1=\tfrac{1}{2}\pi.$$

The summatory process that minimizes r_n varies with the properties of $f(\theta)$ (Favard 1937).

Most summatory processes saturate, i.e. the approximation attainable cannot be smaller than a given limit, only attainable for a certain class of functions.

Thus the Fejer process, defined by $m(t)=1-|t|$,

$$\sigma_n(\theta)=\sum_{-n}^{+n}[1-|p/(n+1)|]c_p e^{ip\theta}$$

(writing σ_n for the corresponding polynomial $P_n(\theta)$) saturates with the degree of approximation $1/n$, i.e. for every function there exists at least one value of θ such that $|\sigma_n-f|>a/n$, whereas for certain functions $|\sigma_n-f|<b/n$. The Fourier process

$$S_n(\theta)=\sum_{-n}^{+n}c_p e^{ip\theta},$$

$m(t)=1$ if $|t|\leqslant 1$ and $m(t)=0$ if $|t|>1$, does not saturate. If possible, and if $E_n(f)$ has the same meaning as above,

$$|S_n-f|<E_n(f)\log(2\pi n).$$

It is one of the advantages of the Fourier process. Though in most cases it does not provide the best approximation, there is no lower limit for the one it gives ; which is easily understood, for only the Fourier process reproduces a trigonometric polynomial without error. Furthermore, the best summatory process of an indefinitely differentiable function is Fourier's (Favard 1937).

Nevertheless, the sum S_n may not converge to f if $n\to\infty$, at least for certain values of θ. This is the Gibbs phenomenon. In these cases, other more powerful summatory processes make the sums converge to f in every point. Thus the processes $m(t)=1-|t|^k$ are more powerful than Fourier's.

In the following, we assume the gain of arrays to be normalized so that

$$(1/2b)\int_{-b}^{+b}g(\theta)d\theta=1 \quad (a_0=1).$$

From the formulae, as stated above, it appears that an array acts for the object f as a summatory process, and provides the approximation

$$P(\theta)=\sum_{-n}^{+n}a_p c_p e^{ip\theta}.$$

The approximation obtained depends on n and the set of values a_p that characterizes the antenna completely and that we have called the " spatial pass band " (Arsac 1955*b*, 1955*c*). For an array of length L and of fundamental length a, $na=L$. n may be increased, the length L being a constant, by decreasing a. It does not necessarily follow a better approximation.

If $f(\theta)$ has a pth derivative bounded by M_p, the function $f(\pi\theta/b)$, with period 2π, has a pth derivative bounded by $(b/\pi)^p M_p$. Thus

$$E_n(f)<C_p(b/\pi)^p(M_p/n^p)=C_p(\lambda/2\pi L)^p M_p.$$

The best approximation depends only on the total length of the array.

The saturation problem is more complex. From the definition of P and f it follows that:

$$(1/2b)\int_{-b}^{+b} P(\theta)\mathrm{e}^{-\mathrm{i}p\pi\theta/b}\mathrm{d}\theta = a_p c_p,$$

$$(1/2b)\int_{-b}^{+b} f(\theta)\mathrm{e}^{-\mathrm{i}p\pi\theta/b}\mathrm{d}\theta = c_p.$$

Thus

$$(1/2b)\int_{-b}^{+b} (P-f)\mathrm{e}^{-\mathrm{i}p\pi\theta/b}\mathrm{d}\theta = (a_p-1)c_p,$$

$$|\,(a_p-1)c_p\,| \leqslant (1/2b)\int_{-b}^{+b} |\,P-f\,|\mathrm{d}\theta, \quad |\,(a_p-1)\,|\,c_p \leqslant r_n.$$

The difference $P-f$ cannot be smaller everywhere than the greatest value of

$$|\,(a_p-1)c_p\,| = |\,(a_p-1)(a/\lambda)F(2p\pi a/\lambda)\,|.$$

For instance, if

$$a_p = 1-(p/n)^k,$$

then

$$r_n > (\lambda/2\pi L)^k(1/2b)[(2p\pi a/\lambda)^k F(2p\pi a/\lambda)].$$

Let N_k be the upper bound of $x^k F(x)$ on the set of points $x=2p\pi a/\lambda$, where

$$r_n > N_k/4b\pi L.$$

At this point the process saturates. If n increases, L being constant, the denominator increases, for b increases if a decreases. But simultaneously the distance between two consecutive points $x=2p\pi a/\lambda$ decreases, and so N_k may be increased. For instance, if $x^k F(x)$ has its maximum value for $x=x_0$, N_k may be smaller than this maximum if x_0 and its neighbourhood do not belong to the set of points $x=2p\pi a/\lambda$. By decreasing a, one of the set will tend to x_0, and N_k increases. We cannot draw a conclusion in the general case.

We have considered interferometry elsewhere (Arsac 1955*b*). There is no advantage in doing interferometric measurements for antenna spacings other than integer multiples of a same length a (defined as above, i.e. smaller than the inverse of the apparent diameter of the source, when a is measured with wavelength as unity). Besides, there is an exact equivalence of problems, interferometric measures for distances a; $2a$; . . .; $na=L$ giving the same harmonics as an array of length L and fundamental length a.

An array designed to give the Fourier sum of f induces an error smaller than $E_n(f)\log(2\pi n)$. There is advantage in keeping $E_n(f)$ constant (L constant) and decreasing n. Which is easily understood: it often happens that from a certain value of x, $F(x)$ has equidistant zeros of abscissae px_1 (for instance, a circular source of uniform brightness). If we take $2\pi a/\lambda = x_1$ all the harmonics not passed by the antenna vanish. There is no error. If we decrease a (for instance, if we reduce it by half) the rejected harmonics are of order $\frac{1}{2}px_1$, and one harmonic out of two is different from 0. There is inevitably an error. Finally :

(i) A uniform array gives a Fejer sum of f (nth order if there are $n+1$ antennae). *It can never give an image identical with the object.* In any case there is at least one point where the difference $P-f$ exceeds the maximum of $xF(x)$ on the set of points $2p\pi a/\lambda$; for a circular source of uniform brightness and apparent diameter $2b_0$, for example,

$$F(x)=2\pi b_0 BJ_1(b_0x)/x.$$

For $b=2b_0$, $N_1 \sim 0 \cdot 6(2\pi b_0 B)$, $r/B > 0 \cdot 3b_0\lambda/bL = 0 \cdot 15\lambda/L$. Moreover, the error can be bigger than this value. If we use the results provided by the array directly, it is useless to measure with more accuracy than about $\lambda/10L$.

(ii) An array designed to give the Fourier sum of f may, in some cases, give an image identical with the object. For very regular functions (f has no rapid variations), it will generally provide good results. For very irregular objects (very rapid variations of f) the Fourier series may not converge to f and the error may be quite large. This is connected with the Gibbs phenomenon (Hardy and Rogosinski 1938). The uniform array does not show the Gibbs phenomenon.

(iii) A Dolph array is only optimum for practically point sources.

(iv) Arrays such as $a_p=1-|p/(n+1)|^k$ may be interesting to build, for their order of saturation is smaller than that of the uniform array. The higher the order of the zero of $m(t)$ for $t=1$, the more powerful the summatory process. On the other hand, it provides an even more accurate approximation than the order of the zero of $1-m(t)$, for $t=0$ is higher. If $E_n(f) < C/n^r$, $m(t)$ sums to f with an approximation of the same order if the zero of $1-m(t)$ for $t=0$ is of a higher order than r. It is the same for $m(t)=1-|t|^k$ if $k>r$. In practice k values as small as 2, 4, and 6 will be enough. We have pointed out elsewhere (Arsac 1955*b*) how one could design an array of a given spatial pass band.

(v) Interferometry, which separately gives the different harmonics of f over a certain interval, allows one to get such sums of f as wished. The above remarks will help in doing so.

(vi) If we wish to build an antenna able to provide an image comparing various theories (for instance, investigation of solar limb brightening) we will choose the type of antenna best suited for the function studied, even if this function is not very well known.

(vii) Usually it is not possible to calculate the systematic error deriving from the finite length of the antenna, for the limits reached by f or its derivatives are unknown. If one of these limits is known *a priori*, or a limit of the coefficients c_p from a certain rank, an upper bound of the error can be estimated. Nevertheless, Bernstein's theorem gives us a better understanding of the nature of errors due to the finite length of the antenna. If $P(\theta)$ is bounded by M,

$$|P'(\theta)| < (2\pi L/\lambda)\tfrac{1}{2}M, \quad |P''(\theta)| < (2\pi L/\lambda)^2 \cdot \tfrac{1}{2}M.$$

Assume that we observe an object of the shape given in Figure 1, and let the shape of the image be given in Figure 2. Let D be the width of arc AC; we will certainly get

$$M/D<(2\pi L/\lambda)\tfrac{1}{2}M, \quad D>\lambda/\pi L.$$

This bound cannot be reached because of the continuity of $P'(\theta)$. If we represent arc ABC by an arc of a sinusoid

$$P=\tfrac{1}{2}M[1-\sin\{\pi(\theta-\theta_0)/D\}],$$

then

$$M^2\pi^2/2D^2<(2\pi L/\lambda)^2\cdot\tfrac{1}{2}M, \quad D>\lambda/2L.$$

If we want to represent such an object accurately (for instance, if we want D to be slightly different from d) L must exceed $\lambda/2d$. The regions of maximum error of P are those of rapid variation of f.

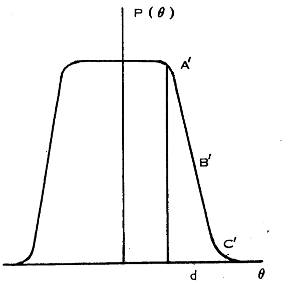

Fig. 1.—Object distribution.

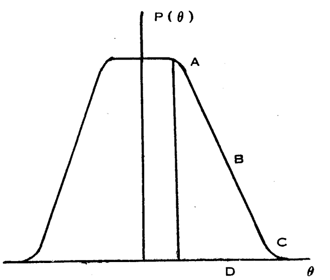

Fig. 2.—Image distribution.

In the preceding theory, we have not taken into account the restriction $g>0$. It implies that certain sums cannot be directly obtained. For instance, with the Fourier sum we have

$$g(\theta)=\frac{\sin\{(2n+1)\pi a\theta/\lambda\}}{\sin\{\pi a\theta/\lambda\}}.$$

As this expression takes negative values, an array cannot provide a Fourier sum. We remove the difficulty by adding a constant to $P(\theta)$.

V. Image Restoration

The knowledge of array gain allows the various images of the same object given by various antennae of the same length to be reduced to one another. For we have

$$P(\theta)=\sum_{-n}^{+n} a_p c_p e^{ip\pi\theta/b}.$$

A harmonic analysis of P gives the coefficients a_pc_p and, if a_p is known and different from 0, c_p may be deduced from it. It is then possible to build any sum of f. Let us write

$$k(\theta)=\frac{1}{2n+1}\left[\frac{a_0'}{a_0}+2\sum_{1}^{n}\frac{a_p'}{a_p}\cos\frac{p\pi\theta}{b}\right],$$

$$\sum_{-n}^{+n} a_p' c_p e^{ip\pi\theta/b}=\sum_{-n}^{+n} P\left(\frac{2qb}{2n+1}\right)k\left(\theta-\frac{2qb}{2n+1}\right).$$

These equations entirely resolve the problem of the restoration of function f from function P. Any other polynomial may be built from the approximating polynomial provided by the antenna. But we cannot get anything else than a polynomial of nth degree. As the result of the above study, the Fourier sum S_n, called by Bracewell and Roberts (1954) the "principal solution", is not of necessity the best image we can get with the information given by the antenna. Sums corresponding to the $1-|t|^k$ process will be often more suitable. Besides, this result has been pointed out by Bracewell and Roberts, who notice that when the function f is very irregular the defects of the Fourier sum S_n may be

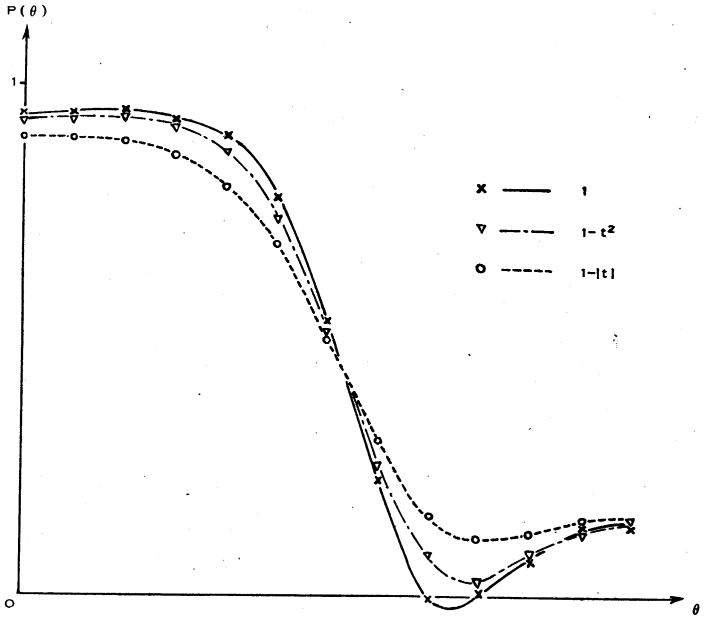

Fig. 3.—An image distribution (- - -), the first stage of restoration (–·–·–·), and the Fourier sum S_6 (——).

smoothed by reducing the Fourier components by "some suitable weighting function". The method of Van Cittert's restorations, quoted by Bracewell and Roberts, is particularly interesting when the antenna is a uniform array. By writing $P=f*g$ for the operation

$$P(\theta)=(1/2b)\int_{-b}^{+b} f(u)g(\theta-u)\mathrm{d}u,$$

we calculate :

$$P_1=P+(P-P*g),$$
$$P_2=P_1+(P-P_1*g), \text{ etc.}$$

If P is the Fejer sum of the nth order of f, P_{k-1} is the corresponding sum by the $1-|t|^k$ process. One operation will be very often enough ; it will be rarely useful to perform more than three or five. Figure 3 shows the σ_6 sum corresponding to the Sun for a wavelength of $3\cdot 2$ cm. It is the image of the

Sun that would be given by a uniform array of seven antennae, of length $L=348\lambda$ (11·10 m), $a=58\lambda$. Such a sum is built from results of interferometric measures (Alon, Arsac, and Steinberg 1955). We note the result of a first restoration $m(t)=1-t^2$ and the Fourier sum S_6. We have been able to get such a sum directly (Arsac 1955*a*, 1955*c*). There is only a little difference between these last polynomials.

Let us recall that we may realize the restoration by an optical process. Such a technique recalls Gabor's methods in electronic microscopy. It is enough to have the array passing n harmonics and its spatial pass band to be known to get the best possible image (of nth order naturally).

Let us finally remark that, the restored function being still a polynomial of nth degree, the limits of Bernstein's theorem hold in this case. This may constitute a test for the validity of certain operations of approximate restoration (Bracewell 1955).

VI. Noisy Image

We have just considered the systematic error due to finite length of the antenna. To this error is added the inevitable error of measurement, chiefly due to noise originating in the receiver. To the function P is added an error DP. This one is only to be considered in the same interval as P. If we know many periods of P, the error may be reduced by calculating their mean. Let us consider only one period.

The function DP, being defined in the same interval $-b$ to $+b$, may be represented by Fourier series with the coefficients given by

$$|z_p|=|(1/2b)\int_{-b}^{+b}DP(\theta)e^{-ip\theta}d\theta|<Z, \quad \text{if } |DP|<Z.$$

Their modulus is bounded by Z. In other words the experiment does not provide the exact values of a_pc_p; they may be inaccurate by an error equal to Z at most. This may be a limitation on the resolving power of the antenna.

The known part of the spectrum of f is not limited to the rank of the last passed harmonic, but to the rank of the last harmonic whose amplitude is higher than Z. As for every continuous function, c_p decreases as $1/p$, it is necessary that the last passed harmonics should not be transmitted with too small an amplitude. Here lies the principal defect of the uniform array.

We have established that for a linear array built with antennae of abscissae integer multiples of the same length a, two of the antennae of the array transmit a harmonic of rank proportional to their distance with an amplitude equal to the product of their gain (Arsac 1955*b*, 1955*c*). Thus it is possible with a given number K of antennae to transmit a number of harmonics very much higher than K. We must manage to form the set of integers from 1 to N (N being the highest possible one) with the distances between the antennae of the array taken two by two in every possible manner. With four antennae at abscissae 0, 1, 4, 6, six harmonics are transmitted all with the same amplitude, the antennae of the array being identical. Such an array provides the Fourier sum S_6 (Arsac

1955a). Arrays having a greater number of antennae cannot provide the Fourier sum, some harmonics having doubled (or more) amplitude. Nevertheless, it is possible to transmit 13 harmonics with six antennae, 23 with eight antennae, . . ., etc. We may transmit a number of harmonics of order $K^2/4$. Such a possibility is rather convenient, for it allows us to get the maximum information with a given number of antennae. Besides, such arrays add a continuous background to the function P, which diminishes the accuracy of measurements.

Let us compare the 0, 1, 4, 6 array with a uniform array 0, 1, 2, 3, 4, 5, 6 whose axial gain would be the same. Figure 4 shows a record obtained for $\lambda=3\cdot2$ cm with a 0, 1, 4, 6 array of parabolic antennae of diameter $1\cdot10$ m having a 2° beam in the meridian plane and 7° in the perpendicular plane ($a=58\lambda$,

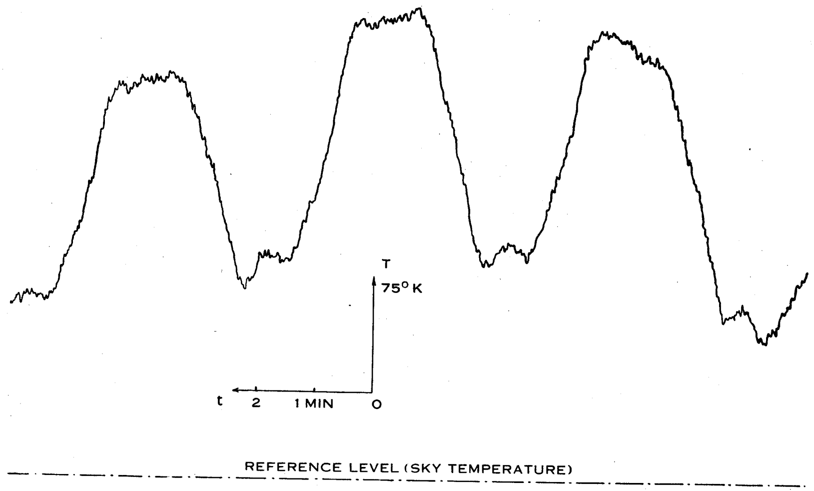

Fig. 4.—Solar scan.

$L=348\lambda$). The result obtained with the uniform array of the same axial gain is presented in Figure 5. The error of measurement is smaller for the uniform array. But this latter transmits six harmonics with the amplitudes:

$$a_1=1\cdot96,\ a_2=1\cdot65,\ a_3=1\cdot30,\ a_4=0\cdot98,\ a_5=0\cdot63,\ a_6=0\cdot33,$$

instead of $a_p=1$ ($p=1, \ldots, 6$) for the 0, 1, 4, 6 array. Consequently, if the error of measurement is bigger for the incomplete array, the systematic error is smaller, for the last harmonics are known with a higher accuracy. If we had compared two arrays built with the same antennae, we should have found that the two pass the last harmonic with the same amplitude. The limitation of the resolving power by noise is then the same for the two arrays.

Generally speaking, for an array built with identical antennae, each harmonic is transmitted with an amplitude at least equal to the product of the gain of two antennae of the array. If an interferometric study of the object is possible as far as a distance L with two antennae of the array, the resolving power is not limited by noise.

If we have K antennae, suitable disposition allows us to obtain up to $K^2/4$ harmonics. The more we have pass harmonics, the less they will be determined with accuracy. For :

$$g(\theta)=a_0+2\sum_1^n a_p \cos(p\pi a\theta/\lambda),$$

$$g(\theta)=a_0+2\sum_1^n a_p=K^2.$$

If n increases, the a_p, or at least some of them, must decrease. Then we may try to get a compromise between the systematic error due to the number of harmonics and the errors of measurement due to amplitudes of harmonics. Only a special study of each problem to be resolved will yield the most suitable disposition of antennae.

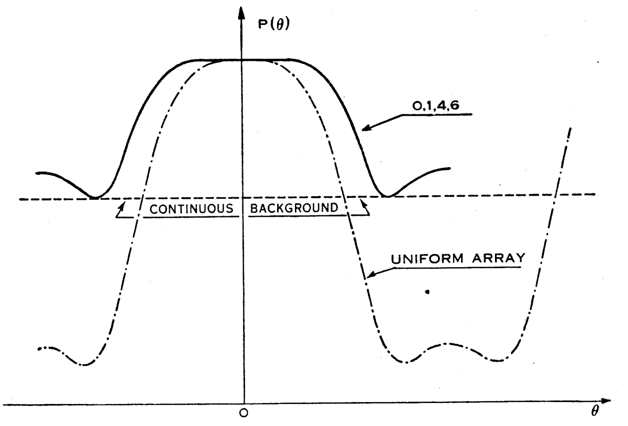

Fig. 5.—Theoretical images for the 0, 1, 4, 6 and the uniform array.

If we have K antennae and a given length L, a judicious disposition allows one to get a greater number of harmonics in the same band of the spectrum of f, and then to explore wider regions of the celestial sphere. Again, this operation leads to a greater inaccuracy in determining some harmonics. Once again we have to establish a compromise between the different qualities to be found in the antenna.

VII. References

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